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HIGHER DERIVATIVE VERSIONS ON THEOREMS OF S. BERNSTEIN

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Abstract. Let $p(z) = \sum\limits_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n and $p'(z)$ its derivative. If $\max\limits_{|z|=r} |p(z)|$ is denoted by $M(p,r)$. If $p(z)$ has all its zeros on $|z| = k$, $k \leq 1$, then it was shown by Govil [3] that

$$M(p',1) \leq \frac{n}{k^n + k^{n-1}} M(p,1)\,.$$

In this paper, we first prove a result concerning the sth derivative where $1 \leq s < n$ of the polynomial involving some of the co-efficients of the polynomial. Our result not only improves and generalizes the above inequality, but also gives a generalization to higher derivative of a result due to Dewan and Mir [2] in this direction. Further, a direct generalization of the above inequality for the sth derivative where $1 \leq s < n$ is also proved.

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1. Introduction

Let $p(z)=\sum\limits_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n in the complex plane and $p'(z)$ its derivative.

Let us denote $\max\limits_{|z|=r}|p(z)|$ by $M(p,r)$. Concerning the estimate of $|p'(z)|$ on the unit disc $|z|\le 1$, we have the following famous result due to Bernstein [9]

$$M(p',1)\le nM(p,1). \tag{1.1}$$

The result is best possible and equality in (1.1) holds for polynomials $p(z)=\alpha z^n$ with $|\alpha|=1$.

If we restrict ourselves to the class of polynomials having no zero inside the unit disc $|z|=1$, then Erdös conjectured and later Lax [5] proved

$$M(p',1)\le \frac{n}{2}M(p,1). \tag{1.2}$$

Inequality (1.2) is best possible and the extremal polynomial is $p(z)=\alpha+\beta z^n$, where $|\alpha|=|\beta|$.

As a generalization of inequality (1.2), Malik [6] proved the following.

Theorem 1.1. *If $p(z)=\sum\limits_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n not vanishing in $|z|<k$, $k\ge 1$, then*

$$M(p',1)\le \frac{n}{1+k}M(p,1). \tag{1.3}$$

In the literature, there is no inequality analogous to (1.3) in the case $p(z)\neq 0$ in $|z|<k$, $k\le 1$. While trying to obtain an analogous inequality to (1.3) for this class of polynomials, Govil [3] was, in particular, only able to prove the following.

Theorem 1.2. *If $p(z)=\sum\limits_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros on $|z|=k$, $k\le 1$, then*

$$M(p',1)\le \frac{n}{k^n+k^{n-1}}M(p,1). \tag{1.4}$$

Inequality (1.4) was improved by Dewan and Mir [2] by involving some co-efficients of $p(z)$.

Theorem 1.3. *If $p(z)=\sum\limits_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros on $|z|=k$, $k\le 1$, then*

$$M(p',1)\le \frac{n}{k^n}\left\{\frac{n|a_n|k^2+|a_{n-1}|}{n|a_n|(1+k^2)+2|a_{n-1}|}\right\}M(p,1). \tag{1.5}$$

2. Lemmas

The following lemmas are needed for the proofs of the theorem and the corollary.

Lemma 2.1. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros in* $|z| \le k$, $k \ge 1$, *then for* $0 \le \theta < 2\pi$ *and* $1 \le s < n$,

$$|p^{(s)}(k^2 e^{i\theta})| \ge k^{n-2s} |q^{(s)}(e^{i\theta})|, 1 \le s < n,$$

where, $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

Proof. The polynomial $P(z) = p(kz)$ has all its zeros in $|z| \le 1$ and hence the polynomial $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)} = z^n \overline{p\left(\frac{k}{\bar{z}}\right)} = k^n \overline{q\left(\frac{z}{k}\right)}$ has all its zeros in $|z| \ge 1$. Since $|P(z)| = |Q(z)|$ for $|z| = 1$, it follows that $|Q(z)| \le |P(z)|$ on $|z| \ge 1$. Hence $Q(z) - \lambda P(z)$ has all its zeros in $|z| < 1$ if $|\lambda| > 1$. By Gauss-Lucas Theorem the sth derivative where $1 \le s < n$, of the polynomial $Q(z) - \lambda P(z)$ that is, $Q^{(s)}(z) - \lambda P^{(s)}(z)$ also has all its zeros in $|z| < 1$, which implies that

$$|Q^{(s)}(z)| \le |P^{(s)}(z)|, \ |z| \ge 1.$$

In particular, for $|z| = k$, since $k \ge 1$,

$$k^{n-s} |q^{(s)}(e^{i\theta})| \le k^s \left| p^{(s)}\left(k^2 e^{i\theta}\right)\right|$$

and the lemma follows. □

Lemma 2.2. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree at most* n, *then it is well known* [8, p. 346 or 7, Vol. 1, Problem III 269, p. 137] *that*

$$M(p, R) \le R^n M(p, 1), \quad R \ge 1. \tag{2.1}$$

Lemma 2.3. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros in* $|z| \le k$, $k \ge 1$, *then for* $1 \le s < n$,

$$M(q^{(s)}, 1) \le k^n M(p^{(s)}, 1). \tag{2.2}$$

Proof. By Lemma 2.1,

$$M(q^{(s)}, 1) \le \frac{1}{k^{n-2s}} M(p^{(s)}, k^2)\,. \tag{2.3}$$

Here $p^{(s)}(z)$ is a polynomial of degree at most $(n - s)$, and applying Lemma 2.2 with $R = k^2 \ge 1$, we have

$$M(p^{(s)}, k^2) \le (k^2)^{n-s} M(p^{(s)}, 1)\,. \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$M(q^{(s)},1) \leq \frac{k^{2n-2s}}{k^{n-2s}} M(p^{(s)},1),$$

which proves Lemma 2.3. □

Lemma 2.4. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, then for $1 \leq s < n$,*

$$k^n M(p^{(s)},1) \leq M(q^{(s)},1)\,. \tag{2.5}$$

Proof. Since $p(z)$ has no zero in $|z| < k$, $k \leq 1$, $q(z)$ has all its zeros in $|z| \leq \frac{1}{k}$, $\frac{1}{k} \geq 1$. Thus applying Lemma 2.3 to the polynomial $q(z)$, we get

$$M(p^{(s)},1) \leq \left(\frac{1}{k}\right)^n M(q^{(s)},1),$$

and hence the lemma follows readily. □

Lemma 2.5. ([1]) *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then on $|z| = 1$,*

$$\frac{C(n,s)|a_0|k^{s+1} + |a_s|k^{2s}}{C(n,s)|a_0| + |a_s|k^{s+1}} |p^{(s)}(z)| \leq |q^{(s)}(z)|, \tag{2.6}$$

where $C(n,s) = \frac{n!}{s!(n-s)!}$ and

$$\frac{1}{C(n,s)} \left|\frac{a_s}{a_0}\right| k^s \leq 1\,. \tag{2.7}$$

Lemma 2.6. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then on $|z| = 1$,*

$$\frac{C(n,s)|a_n|k^{s-1} + |a_{n-s}|}{C(n,s)|a_n|k^{2s} + |a_{n-s}|k^{s-1}} |q^{(s)}(z)| \leq |p^{(s)}(z)|, \tag{2.8}$$

where $C(n,s) = \frac{n!}{s!(n-s)!}$ and

$$\frac{1}{C(n,s)} \left|\frac{a_{n-s}}{a_n}\right| \leq k^s\,. \tag{2.9}$$

Proof. Since $p(z)$ has all its zeros on $|z| = k$, $k \leq 1$, $q(z)$ has all its zeros on $|z| = \frac{1}{k}$, $\frac{1}{k} \geq 1$. Applying Lemma 2.5 to the polynomial $q(z)$, the two results follow. □

Lemma 2.7. ([4]) *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n*, then on* $|z| = 1$*,*

$$|p^{(s)}(z)| + |q^{(s)}(z)| \le n(n-1)(n-2)\cdots(n-s+1)M(p,1), \tag{2.10}$$

where $1 \le s < n$*.*

Lemma 2.8. ([4]) *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < k$*,* $k \ge 1$*, then on* $|z| = 1$*,*

$$k^s|p^{(s)}(z)| \le |q^{(s)}(z)|\,. \tag{2.11}$$

Lemma 2.9. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros on* $|z| = k$*,* $k \le 1$*, then on* $|z| = 1$*,*

$$\frac{1}{k^s}|q^{(s)}(z)| \le |p^{(s)}(z)|\,. \tag{2.12}$$

Proof. The proof follows immediately on applying Lemma 2.7 to $q(z)$ as in the proof of (2.8) of Lemma 2.6. □

3. Main results

It is clearly of interest to find a bound for the sth derivative, $1 \le s < n$, of the polynomial $p(z)$, which gives the corresponding generalization of Theorem 1.3 for higher derivative. More precisely, we prove

Theorem 3.1. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros on* $|z| = k$*,* $k \le 1$*, then for* $1 \le s < n$*,*

$$\begin{aligned} M(p^{(s)},1) \quad \le \quad & \frac{n(n-1)\cdots(n-s+1)}{k^n} \\ & \times \frac{C(n,s)|a_n|k^{2s} + |a_{n-s}|k^{s-1}}{C(n,s)|a_n|(k^{s-1}+k^{2s}) + |a_{n-s}|(1+k^{s-1})} M(p,1), \end{aligned} \tag{3.1}$$

where $C(n,s) = \frac{n!}{s!(n-s)!}$*. When* $s = 1$*, the bound of* (3.1) *reduces to that of* (1.5) *of Theorem* 1.3*.*

Proof. Let z_0 be a point on $|z| = 1$ such that

$$|q^{(s)}(z_0)| = M(q^{(s)},1).$$

Then by Lemma 2.7, it follows that

$$|p^{(s)}(z_0)| + |q^{(s)}(z_0)| \le n(n-1)(n-2)\cdots(n-s+1)M(p,1)\,. \tag{3.2}$$

If we combine (3.2) with (2.8) of Lemma 2.6, we get

$$\begin{aligned}\frac{C(n,s)|a_n|k^{s-1}+|a_{n-s}|}{C(n,s)|a_n|k^{2s}+|a_{n-s}|k^{s-1}}|q^{(s)}(z_0)|+|q^{(s)}(z_0)|\\ \leq n(n-1)(n-2)\cdots(n-s+1)M(p,1),\end{aligned}$$

which on simplification gives

$$\begin{aligned}\frac{C(n,s)|a_n|(k^{s-1}+k^{2s})+|a_{n-s}|(1+k^{s-1})}{C(n,s)|a_n|k^{2s}+|a_{n-s}|k^{s-1}}M(q^{(s)},1)\\ \leq (n-1)(n-2)\cdots(n-s+1)M(p,1). \qquad (3.3)\end{aligned}$$

Inequality (3.3) when combined with Lemma 2.4 we obtain

$$\begin{aligned}M(p^{(s)},1) \leq{}& \frac{n(n-1)(n-2)\cdots(n-s+1)}{k^n}\\ &\times\frac{C(n,s)|a_n|k^{2s}+|a_{n-s}|k^{s-1}}{C(n,s)|a_n|(k^{s-1}+k^{2s})+|a_{n-s}|(1+k^{s-1})}M(p,1),\end{aligned}$$

which completes the proof of the theorem. □

Remark 3.2. Under the same hypothesis of Theorem 3.1, we claim that

$$\frac{C(n,s)|a_n|k^{2s}+|a_{n-s}|k^{s-1}}{C(n,s)|a_n|(k^{s-1}+k^{2s})+|a_{n-s}|(1+k^{s-1})} \leq \frac{1}{1+k^{-s}}\,, \qquad (3.4)$$

where $C(n,s)$ is as stated in Theorem 3.1.

Which on simplification gives

$$|a_{n-s}| \leq C(n,s)|a_n|k^s,$$

and is true by inequality (2.9) of Lemma 2.6.

If we use inequality (3.4) to (3.1) of Theorem 3.1, we obtain the following direct generalization of inequality (1.4) to the sth derivative, $1 \leq s < n$.

Corollary 3.3. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then for $1 \leq s < n$,*

$$M(p^{(s)},1) \leq \frac{n(n-1)\cdots(n-s+1)}{k^n+k^{n-s}}M(p,1). \qquad (3.5)$$

Proof. The proof of this corollary follows on the same lines as that of the theorem, but instead of applying inequality (2.8) of Lemma 2.6, we apply Lemma 2.9. We omit the details. □

For $s = 1$ Corollary 3.3 reduces to Theorem 1.2. Further, it is clear from inequality (3.4) of Remark 3.2 that, the bound obtained from Theorem 3.1 is better than the bound obtained from Corollary 3.3 by involving some coefficients of $p(z)$.

Remark 3.4. As mentioned earlier, the bound as given by the above theorem is much better than the bound obtained by Corollary 3.3. We illustrate this by means of the following example.

Example 3.5. Let $p(z) = z^4 + \frac{1}{72}z^2 + \left(\frac{1}{144}\right)^2$. Clearly $k = \frac{1}{12}$. Then by the Corollary 3.3 we have for $s = 2$,

$$M(p^{(2)}, 1) \leq 1716.083M(p, 1)\,,$$

while by Theorem 3.1, we get

$$M(p^{(2)}, 1) \leq 425.354M(p, 1)\,.$$

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References

[1] A. Aziz and N.A. Rather, *Some Zygmund type L^q inequalities for polynomials*, J. Math. Anal. Appl., **289** (2004), 14-29.

[2] K.K. Dewan and A. Mir, *Note on a theorem of S. Bernstein,* Southeast Asian Bull. Math., **31** (2007), 691-695.

[3] N.K. Govil, *On a theorem of S. Bernstein*, J. Math. Phy. Sci., **14**(2) (1980), 183-187.

[4] N.K. Govil and Q.I. Rahman, *Functions of exponential type not vanishing in a half-plane and related polynomials*, Trans. Amer. Math. Soc., **137** (1969), 501-517.

[5] P.D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial,* Bull. Amer. Math. Soc., **50** (1969), 509-513.

[6] M.A. Malik, *On the derivative of a polynomial,* J. London Math. Soc., **1** (1969), 57-60.

[7] G. Pólya and G. Szegö, *Aufgaben and Lehrsatze aus der analysis,* Springer, Berlin, 1925.

[8] M. Riesz, *Über einen Satz des Herrn Serge Bernstein,* Acta Math., **40** (1916), 337–347.

[9] A.C. Schaeffer, *Inequalities of A. Markoff and S. Bernstein for polynomials and related functions*, Bull. Amer. Math. Soc., **47** (1941), 565-579.