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A COMMON FIXED POINT THEOREM IN AN M^*-METRIC SPACE AND AN APPLICATION

Gharib M. Gharib[1], Abed Al-Rahman M. Malkawi[2], Ayat M. Rabaiah[3], Wasfi A. Shatanawi[4], Maha S. Alsauodi[5]

[1]Department of Mathematics, Faculty of Science
Zarqa University, Zarqa, Jordan
e-mail: `ggharib@zu.edu.jo`

[2]Department of Mathematics, Faculty of Science
The University of Jordan, Amman, Jordan
Department of Mathematics, Faculty of Sciences and Literature
Jordan University of Science and Technology, Irbid, Jordan
e-mail: `math.malkawi@gmail.com; Abd9180065@ju.edu.jo; ammalkawi@just.edu.jo`

[3]Department of Mathematics, Faculty of Science
The University of Jordan, Amman, Jordan
e-mail: `ayatrabaiah@yahoo.com; Aya916322@ju.edu.jo`

[4]Department of General Sciences
Prince Sultan University, Riyadh, Saudi Arabia
Department of Mathematics Hashemite University, Zarqa, Jordan
e-mail: `wshatanawi@psu.edu.sa; wshatanawi@yahoo.com; swasfi@hu.edu.jo`

[5]Department of Mathematics, Faculty of Science
The University of Jordan, Amman, Jordan
e-mail: `alsoudimaha@gmail.com`

Abstract. In this paper, we introduce the concept of M^*-metric spaces and how much the M^*-metric and the b-metric spaces are related. Moreover, we introduce some ways of generating M^*-metric spaces. Also, we investigate some types of convergence associated with M^*-metric spaces. Some common fixed point for contraction and generalized contraction mappings in M^*-metric spaces. Our work has been supported by many examples and an application.

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[0]Corresponding author: Abed Al-Rahman M. Malkawi(`math.malkawi@gmail.com`).

1. Introduction

In 1994, the concept of D^*-metric space is defined by Dhage [12] which is a generalized metric space.

Definition 1.1. ([12]) Let $\mathbb{X} \neq \emptyset$ be a set. A function $D^* : \mathbb{X}\times\mathbb{X}\times\mathbb{X} \to [0,\infty)$ is called a D^*-metric, if the following properties are satisfied for each $x, y, z \in \mathbb{X}$.
(D^*1) : $D^*(x, y, z) \geq 0$.
(D^*2) : $D^*(x, y, z) = 0$ iff $x = y = z$.
(D^*3) : $D^*(x, y, z) = D^*(p(x, y, z))$; for any permutation $p(x, y, z)$ of x, y, z.
(D^*4) : $D^*(x, y, z) \leq D^*(x, y, \ell_2) + D^*(\ell_2, z, z)$.
A pair $(\mathbb{X}, D^*)$ is called a D^*-metric space.

In the following, the notion of the $b-$metric space is defined by Bakhtin [6] and Czerwik [11], there are many fixed point theorems in a b-metric space for more information. I refer to the reader to look at $[1 - 11]$, $[15 - 39]$.

Definition 1.2. ([6, 11]) Let $\mathbb{X} \neq \emptyset$ be a set and $S \geq 1$ be a real number. A function $d : \mathbb{X} \times \mathbb{X} \to [0,\infty)$ is called a b-metric [6, 12], if it satisfies the following properties for each $x, y, z \in \mathbb{X}$.
(b1) : $d(x, y) = 0$ iff $x = y$;
(b2) : $d(x, y) = d(y, x)$;
(b3) : $d(x, z) \leq S\left[d(x, y) + d(y, z)\right]$.

Now, we define the notion of the M^*-metric space which is a generalization of a b-metric space and an M^*-metric space the tetrahedral inequality axiom is weaker than for a D^*-metric space.

Definition 1.3. Let $\mathbb{X}$ be a non empty set and $R \geq 1$ be a real number. A function $M^* : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0,\infty)$ is called a M^*-metric, if the followings are satisfied the properties: for each $x, y, z \in \mathbb{X}$.
(M^*1) : $M^*(x, y, z) \geq 0$.
(M^*2) : $M^*(x, y, z) = 0$ iff $x = y = z$.
(M^*3) : $M^*(x, y, z) = M^*(p(x, y, z))$; for any permutation $p(x, y, z)$ of x, y, z.
(M^*4) : $M^*(x, y, z) \leq RM^*(x, y, u) + M^*(u, z, z)$.
A pair $(\mathbb{X}, M^*)$ is called an M^*-metric space.

Now, we introduce two examples that satisfy the four axioms for M^*-metric.

Example 1.4. For $x, y, z \in\mathbb{R}$, define
(1) $M_1^*(x, y, z) = \frac{1}{R}\left[|x - y| + |y - z| + |z - x|\right]$.
(2) $M_\infty^*(x, y, z) = \frac{1}{R}\max\left\{|x - y|, |y - z|, |z - x|\right\}$.
Then we can say that $(\mathbb{R}, M_1^*)$ and $(\mathbb{R}, M_\infty^*)$ are M^*-metric spaces.

Example 1.5. Define a function M^* on $\mathbb{X}\times\mathbb{X}\times\mathbb{X}$ by

$$M^*(x,y,z)=\begin{cases} 0, & \text{if } x=y=z,\\ 1, & \text{otherwise.}\end{cases}$$

Then M^* is the discreet M^*-metric on $\mathbb{X}$.

Note: In the following, we will present very important characteristics that are always realized in the M^*-metric space, the importance of which lies in the theories presented in this paper. It is worth noting that these characteristics need not be satisfied in MR-metric space defined by Malkawi et. al. [23].

(M^*5) : $M^*(x,x,y)=M^*(x,y,y)$.

(M^*6) : $M^*(x,y,y)\leq RM^*(y,y,z)+M^*(z,x,x)$.

Since

$$\begin{aligned} M^*(x,x,y) &\leq RM^*(x,x,x)+M^*(x,y,y)\\ &= M^*(x,y,y)\end{aligned}$$

and

$$\begin{aligned} M^*(x,y,y) &\leq RM^*(y,y,y)+M^*(y,x,x)\\ &= M^*(x,x,y).\end{aligned}$$

Thus, we have

$$M^*(x,x,y)=M^*(x,y,y).$$

Next, also we have from (M^*5)

$$\begin{aligned} M^*(x,y,y) &= M^*(y,y,x)\\ &\leq RM^*(y,y,z)+M^*(z,x,x)\\ &= RM^*(y,y,z)+M^*(x,z,z).\end{aligned}$$

Remark 1.6. The M^*-metrics in examples 1.4, 1.5 are satisfied the following properties: For all x,y,z,ℓ_1,ℓ_2 in $\mathbb{X}$, we have

(M^*7) : $M^*(x,y,y)\leq RM^*(x,y,z)$.

(M^*8) : $M^*(x,y,z)\leq \frac{1}{R}\left[M^*(x,\ell_1,\ell_1)+M^*(z.\ell_1,\ell_2)\right]$.

We well use the following example to show that (M^*6) does not implies (M^*7).

Example 1.7. Suppose $\mathbb{X}$ has at least three elements. Define M^* on $\mathbb{X}\times\mathbb{X}\times\mathbb{X}$ by

$$M^*(x,y,z)=\begin{cases} 0, & \text{if } x=y=z,\\ \frac{1}{2R}, & \text{if } x,y,z \text{ are distinct,}\\ 1, & \text{otherwise.}\end{cases}$$

Then $(\mathbb{X},M^*)$ is an M^*-metric space but (M^*7) is not satisfied.

By adding some conditions and properties, we will presented some of the interconnections between M^*-metric space and b-metric space.

Proposition 1.8. *If the M^*-metric space $(\mathbb{X}, M^*)$ satisfies (M^*5) and (M^*6), then $d(x,y) = M^*(x,y,y)$ is a b-metric on $\mathbb{X}$.*

Proof. Let $x, y \in \mathbb{X}$, we want to show $(\mathbb{X}, d)$ is a b-metric space.

(i) By (M^*1), $d(x,y) = M^*(x,y,y) \geq 0$.
(ii) By (M^*2), $d(x,y) = M^*(x,y,y) = 0$ iff $x = y$.
(iii) By $(M^*5), (M^*3)$,

$$d(x,y) = M^*(x,y,y) = M^*(x,x,y) = M^*(y,x,x) = d(y,x).$$

(iv) By (M^*6),

$$\begin{aligned} d(x,y) = M^*(x,y,y) &\leq RM^*(x,z,z) + M^*(z,y,y) \\ &= Rd(x,z) + d(z,y) \\ &\leq R\left[d(x,z) + d(z,y)\right]. \end{aligned}$$

Thus, $(\mathbb{X}, d)$ is a b-metric space. □

Example 1.9. Let $\mathbb{X} := l_p(\mathbb{R})$ with $0 < p < 1$, where $l_p(\mathbb{R}) := \{\{x_n\} \subset \mathbb{R} : \sum\limits_{n=1}^{\infty} |x_n|^p < \infty\}$. Define $M^* : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to \mathbb{R}^+$ by

$$M^*(x,y,z) = \begin{cases} 0, & \text{iff} \quad x = y = z, \\ 1, & \text{iff} \quad x, y, z \text{ are distinct}, \\ \left(\sum\limits_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}}, & \text{iff} \quad x \neq y = z \text{ or } x = z \neq y, \\ \left(\sum\limits_{n=1}^{\infty} |y_n - x_n|^p\right)^{\frac{1}{p}}, & \text{iff} \quad y \neq z = x \text{ or } x = y \neq z, \\ \left(\sum\limits_{n=1}^{\infty} |z_n - x_n|^p\right)^{\frac{1}{p}}, & \text{iff} \quad x \neq z = y \text{ or } y = x \neq z, \end{cases}$$

where $x = \{x_n\}$, $y = \{y_n\}$ and $z = \{z_n\}$. Then $(\mathbb{X}, M^*)$ is a M^*-metric space with coefficient $R > 1$.

To show M^* is an M^*-metric, we have to show that only (M^*4) is hold, since $(M^*1), (M^*2)$ and (M^*3) are obvious.

Case 1: If x, y, z are distinct, then we have two cases:

(1) If $u \notin \{x, y, z\}$, then

$$\begin{aligned} 1 &= M^*(x,y,z) \\ &\le RM^*(x,y,u) + M^*(u,z,z) \\ &= R\cdot 1 + \left(\sum_{n=1}^{\infty} |z_n - u_n|^p\right)^{\frac{1}{p}}. \end{aligned}$$

(2) If $u = x$, then

$$\begin{aligned} 1 &= M^*(x,y,z) \\ &\le RM^*(x,y,u) + M^*(u,z,z) \\ &= R\cdot 1 + \left(\sum_{n=1}^{\infty} |u_n - z_n|^p\right)^{\frac{1}{p}}. \end{aligned}$$

It is similar if $u = y$ or $u = z$.

Case 2 : If $x = y \neq z$, then we have three cases:

(1) If $u \notin \{y, z\}$, then

$$\begin{aligned} M^*(x,y,z) &= \left(\sum_{n=1}^{\infty} |y_n - z_n|^p\right)^{\frac{1}{p}} \\ &\le RM^*(x,y,u) + M^*(u,z,z) \\ &= R\left(\sum_{n=1}^{\infty} |y_n - u_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |u_n - z_n|^p\right)^{\frac{1}{p}}. \end{aligned}$$

(2) If $u = x = y \neq z$, then

$$\begin{aligned} M^*(x,y,z) &= \left(\sum_{n=1}^{\infty} |y_n - z_n|^p\right)^{\frac{1}{p}} \\ &\le RM^*(x,y,u) + M^*(u,z,z) \\ &= 0 + \left(\sum_{n=1}^{\infty} |u_n - z_n|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{n=1}^{\infty} |y_n - z_n|^p\right)^{\frac{1}{p}}. \end{aligned}$$

(3) If $x = y \neq z = u$, then

$$\begin{aligned} M^*(x,y,z) &= \left(\sum_{n=1}^{\infty} |y_n - z_n|^p\right)^{\frac{1}{p}} \\ &\leq RM^*(x,y,u) + M^*(u,z,z) \\ &= R\left(\sum_{n=1}^{\infty} |y_n - u_n|^p\right)^{\frac{1}{p}} + 0 \\ &= R\left(\sum_{n=1}^{\infty} |y_n - z_n|^p\right)^{\frac{1}{p}} \end{aligned}$$

and it is similar if $x \neq y = z$.

Moreover, $(\mathbb{X}, M^*)$ is not D^*-metric space.

Let $x = (1, 1, ..., 1, 0, 0, ...), y = (-1, -1, ..., -1, 0, 0, ...)$ and $u = (1, -1, ..., -1, 0, 0, ...)$, where the number of nonzero element of x, y, u is $2n$. So,

$$\begin{aligned} M^*(x,x,y) &= \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{2n} 2^p\right)^{\frac{1}{p}} = 2\cdot(2n)^{\frac{1}{p}}, \\ M^*(x,x,u) &= \left(\sum_{i=1}^{\infty} |x_i - u_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n} 2^p\right)^{\frac{1}{p}} = 2\cdot(n)^{\frac{1}{p}}, \\ M^*(u,y,y) &= \left(\sum_{i=1}^{\infty} |x_i - u_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n} 2^p\right)^{\frac{1}{p}} = 2\cdot(n)^{\frac{1}{p}}. \end{aligned}$$

But,

$$32 = M^*(x,x,y) \nleq M^*(x,x,u) + M^*(u,y,y) = 8 + 8 = 16,$$

when $n = 2$ and $p = \frac{1}{2}$.

Theorem 1.10. *If $(\mathbb{X}, M^*)$ is an M^*-metric space, then any function $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^+$ defined by*

(i) *for $1 \leq q < \infty$, $d(x,y) = \{M^{*q}(x,y,y) + M^{*q}(x,x,y)\}^{\frac{1}{q}}$ is a b-metric on $\mathbb{X}$.*

(ii) *$d(x,y) = \max\{M^*(x,y,y), M^*(x,x,y)\}$ for all $x, y \in \mathbb{X}$, is a b-metric on $\mathbb{X}$.*

Proof. It suffices to prove (i), since (ii) are the same.

Obviously, $d(x,y) \geq 0$ for all $x, y \in \mathbb{X}$ and $d(x,y) = 0$ if and only it $x = y$.

Now, let $x, y, z \in \mathbb{X}$. Then, for $1 \leq q < \infty$,

$$\begin{aligned} d(x,y) &= \{M^{*q}(x,y,y) + M^{*q}(x,x,y)\}^{\frac{1}{q}} \\ &= \{M^{*q}(y,y,x) + M^{*q}(y,x,x)\}^{\frac{1}{q}} \\ &= \{M^{*q}(y,x,x) + M^{*q}(y,y,x)\}^{\frac{1}{q}} \\ &= d(y,x). \end{aligned}$$

$$\begin{aligned} d(x,y) &= \{M^{*q}(x,y,y) + M^{*q}(x,x,y)\}^{\frac{1}{q}} \\ &\leq \{(RM^*(y,y,z) + M^*(z,x,x))^q \\ &\quad + (RM^*(y,y,z) + M^*(z,x,x))^q\}^{\frac{1}{q}} \\ &\leq R\{(M^*(y,y,z) + M^*(z,x,x))^q \\ &\quad + (M^*(y,y,z) + M^*(z,x,x))^q\}^{\frac{1}{q}} \\ &\leq R[\{M^{*q}(y,y,z) + M^{*q}(x,x,z)\}^{\frac{1}{q}} \\ &\quad + \{M^{*q}(z,y,y) + M^{*q}(z,z,y)\}^{\frac{1}{q}}]. \\ &\leq R\left[d(x,z) + d(z,y)\right]. \end{aligned}$$

Hence d is a b-M^*-metric on $\mathbb{X}$. □

Theorem 1.11. *Let $M^* : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$ be a function satisfying (M^*1), (M^*2), (M^*3), (M^*7) and (M^*8). Then M^* is an M^*-metric on $\mathbb{X}$.*

Proof. In order to show that M^* is an M^*-metric on $\mathbb{X}$ it is enough to show that (M^*4) is satisfied. Let $x, y, z \in \mathbb{X}$,

$$\begin{aligned} M^*(x,y,z) &\leq \frac{1}{R} M^*(x,\ell_1,\ell_1) + M^*(z,\ell_1,\ell_2) \\ &\leq \frac{R}{R} M^*(x,\ell_1,y) + M^*(\ell_1,\ell_1,z) \\ &\leq RM^*(x,y,\ell_1) + M^*(\ell_1,z,z). \end{aligned}$$

Thus (M^*4) holds and hence M^* is an M^*-metric on $\mathbb{X}$. □

2. Ways of Generating M^*-metrics

In this section, we present some ways of generating M^*-metric spaces. Let $\aleph = \left\{(\flat_1, \flat_2, \flat_3) \in (\mathbb{R}^+)^3 : \flat_1 \leq \frac{1}{R}(\flat_2 + \flat_3), \flat_2 \leq \frac{1}{R}(\flat_1 + \flat_3), \flat_3 \leq \frac{1}{R}(\flat_1 + \flat_2)\right\}$.

Theorem 2.1. *Suppose that the function $\Psi : \aleph \to \mathbb{R}^+$ satisfies*

(i) $\Psi(\Im_1, \Im_2, \Im_3) = \Psi(p(\Im_1, \Im_2, \Im_3))$, *for any permutation* $p(\Im_1, \Im_2, \Im_3)$ *of* $\Im_1, \Im_2, \Im_3$.

(ii) $\Psi(\Im_1, \Im_2, \Im_3) = 0$ *iff* $\Im_1 = \Im_2 = \Im_3 = 0$,

(iii) $\Psi(t, t, 0) \leq \Psi(\Im, \Im_1, \Im_2)$ *for every* $(\Im, \Im_1, \Im_2) \in \aleph$,

(iv) $\Psi(\Im_1, \Im_2, \Im_3) \leq \frac{1}{R}\left[\Psi(\Im_1', \Im, \Im_1'') + \Psi(\Im_2', \Im, \Im_2'')\right]$

for all $(\Im_1, \Im_2, \Im_3), (\Im_1', \Im, \Im_1''), (\Im_2', \Im, \Im_2'')$ *and* $(\Im_3', \Im, \Im_3'')$ *in* $\aleph$, *where* $(\Im_1, \Im_1', \Im_2'), (\Im_1, \Im_1'', \Im_2''), (\Im_2, \Im_2', \Im_3'), (\Im_2, \Im_2'', \Im_3''), (\Im_3, \Im_3', \Im_1'), (\Im_3, \Im_3'', \Im_1'')$ $\in \aleph$. *Let* $(\mathbb{X}, d)$ *be a* $b-$*metric space. Define a function* $M^* : \mathbb{X}\times\mathbb{X}\times\mathbb{X} \to [0, \infty)$ *by*

$$M^*(x, y, z) = R\Psi\left(d(x, y), d(y, z), d(z, x)\right).$$

Then M^* *is an* M^**-metric on* $\mathbb{X}$.

Proof. Since M^* satisfies $(M^*1), (M^*2)$ and (M^*3), it is enough to show that (M^*7) and (M^*8) are satisfied.

Let $x, y, z, \ell_1, \ell_2 \in \mathbb{X}$. Then it follows from (iii) and (iv) that

$$\begin{aligned} M^*(x, y, y) &= R\Psi(d(x, y), 0, d(y, x)) \\ &\leq R\Psi(d(x, y), d(y, z), d(z, x)) \\ &= RM^*(x, y, z) \end{aligned}$$

and

$$\begin{aligned} M^*(x, y, z) &= R\Psi(d(x, y), d(y, z), d(z, x)) \\ &\leq \Psi\left(d(x, \ell_1), d(\ell_2, x)\right) \\ &\quad + \Psi\left(d(zz, \ell_1), d(\ell_1, \ell_2), d(\ell_2, z)\right) \\ &\leq RM^*(x, \ell_1, \ell_1) + M^*(\ell_1, z, \ell_2). \end{aligned}$$

Thus, the hypothesis of Theorem 1.11 are satisfied for M^* and hence M^* is an M^*-metric on $\mathbb{X}$. □

Theorem 2.2. *Suppose that* $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfies the following properties:*

(i) $\Phi(\flat) = 0$ *iff* $\flat = 0$,

(ii) Φ *is monotone increasing,*

(iii) $\Phi(s + t) \leq \frac{1}{R}[\Phi(s) + \Phi(t)]$ *for all* $s, t \in \mathbb{R}^+$.

Then $\Psi(\flat_1, \flat_2, \flat_3) = \Phi(\flat_1) + \Phi(\flat_2)$ *has all the properties that identified in Theorem* 2.1.

Proof. It is clear that Ψ satisfy (i) of Theorem 2.1. Note that

$$\Psi(\flat_1, \flat_2, \flat_3) = 0 \Longleftrightarrow \Phi(\flat_1) + \Phi(\flat_2) + \Phi(\flat_3) = 0 \Longleftrightarrow \flat_1 = \flat_2 = \flat_3 = 0.$$

Let $(\flat_1, \flat_2, \flat_3) \in \aleph$. Consider the following triples in $\aleph$:
$(\flat_1, \flat_2, \flat_3)$, $(\flat_1', \flat, \flat_1'')$, $(\flat_2', \flat, \flat_2'')$, $(\flat_3', \flat, \flat_3'')$, $(\flat_1, \flat_1', \flat_2')$, $(\flat_1, \flat_1'', \flat_2'')$, $(\flat_2, \flat_2', \flat_3')$,

$(\flat_2, \flat_2'', \flat_3'')$, $(\flat_3, \flat_3', \flat_1') \in \aleph$. Then $(\flat_3, \flat_3'', \flat_1'') \in \aleph$, where $\aleph \subset (\mathbb{R}^+)^3$. So,

$$\begin{aligned}\Psi(\Im_1, \Im_2, \Im_3) &= \Phi(\Im_1) + \Phi(\Im_2) \\ &\leq \Phi(\Im_1' + \Im + \Im_2'') + \Phi(\Im_2' + \Im + \Im_3'') \\ &\leq \frac{1}{R}\Phi(\Im_1') + \frac{1}{R^2}\Phi(\Im) + \frac{1}{R^2}\Phi(\Im_2'') \\ &\quad + \frac{1}{R}\Phi(\Im_2') + \frac{1}{R^2}\Phi(\Im) + \frac{1}{R^2}\Phi(\Im_3'') \\ &\leq \frac{1}{R}[\Psi(\Im_1', \Im, \Im_1'') + \Psi(\Im_2', \Im, \Im_2'')].\end{aligned}$$

It is clear that $\Psi(\flat, \flat, 0) \leq \Psi(\flat, \flat_1, \flat_2)$ for all $(\flat_1, \flat, \flat_2) \in \aleph$. Hence Ψ satisfies all conditions specified in Theorem 2.1. □

Now, in order to show that the two conditions (ii) and (iii) are independent in previous theorem, we give the following example.

Example 2.3. Define a function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\Phi(\flat) = 2\flat$ for all $\flat \in \mathbb{R}^+$ satisfies the hypothesis of Theorem 2.2.

Also, the conditions (ii) and (iii) are independent. For example the function $2\flat^2$, (ii) holds but (iii) does not hold. While $\Phi(\flat) = 0$ if $\flat = 0$ and $\Phi(\flat) = \flat + \frac{1}{\flat}$ if $\flat > 0$ satisfies (iii) but not (ii).

Theorem 2.4. *Let $(\mathbb{X}, d)$ be a metric space. Define real functions M_1^*, M_∞^*, M_3^*, M_4^* on $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$ by*

$$M_1^*(x, y, z) = d(x, y) + d(y, z) + d(z, x),$$
$$M_\infty^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\},$$

$$M_3^*(x, y, z) = \begin{cases} M_1^*(x, y, z), & \text{if } x, y, z \text{ are distinct}, \\ M_\infty^*(x, y, z), & \text{otherwise}, \end{cases}$$

and

$$M_4^*(x, y, z) = \begin{cases} M_\infty^*(x, y, z), & \text{if } x, y, z \text{ are distinct}, \\ M_1^*(x, y, z), & \text{otherwise}. \end{cases}$$

Then $M_1^*, M_\infty^*, M_3^*, M_4^*$ are M^*-metric on $\mathbb{X}$.

Proof. It is clear that M_1^* and M_∞^* are M^*-metrics and all the proofs of M_3^* and M_4^* are similar, it is enough to show that M_4^* is an M^*-metric. Also, it is enough to show that tetrahedral inequality is satisfied.

Let $x, y, z, \ell_1 \in \mathbb{X}$.

Case 1: x, y, z are distinct. While preserving the generality, we assume that

$$d(x, y) \leq d(y, z) \leq d(z, x).$$

(1) If $\ell_1 \notin \{x, y, z\}$, then

$$\begin{aligned} M_4^*(x, y, z) &= d(x, z) \\ &\leq R\left[d(x, \ell_1) + d(\ell_1, z)\right] \\ &\leq R\left[M_4^*(x, y, \ell_1) + M_4^*(\ell_1, z, z)\right]. \end{aligned}$$

(2) If $\ell_1 = x$, then

$$\begin{aligned} M_4^*(x, y, z) &= M_4^*(\ell_1, y, z) = d(\ell_1, z) \\ &\leq RM_4^*(x, y, \ell_1) + M_4^*(\ell_1, z, z). \end{aligned}$$

If $\ell_1 = y$ or $\ell_1 = z$, then the proof is similar.

Case 2: Assume $x = y \neq z$.

(1) If $\ell_1 \notin \{y, z\}$, then

$$\begin{aligned} M_4^*(x, y, z) &= d(y, z) + d(z, y) \\ &\leq R\left[d(y, \ell_1) + d(\ell_1, z) + d(z, \ell_1) + d(\ell_1, y)\right] \\ &\leq RM_4^*(x, y, \ell_1) + M_4^*(\ell_1, z, z). \end{aligned}$$

(2) If $\ell_1 = y$, then

$$\begin{aligned} M_4^*(x, y, z) &= M_4^*(x, \ell_1, z) \\ &\leq RM_4^*(x, y, \ell_1) + M_4^*(\ell_1, z, z). \end{aligned}$$

(3) If $\ell_1 = z \neq y$, then

$$\begin{aligned} M_4^*(x, y, z) &= M_4^*(x, y, \ell_1) \\ &\leq RM_4^*(x, y, \ell_1) + M_4^*(\ell_1, z, z). \end{aligned}$$

Hence M_4^* is an M^*-metric on $\mathbb{X}$. □

3. Types of convergence associated with an M^*-metric

In light of the definition of a D-convergent and a D-Cauchy for a D-metric [13], we define M^*-convergent and M^*-Cauchy for M^*-metric.

Definition 3.1. A sequence $\{x_n\}$ in an M^*-metric space $(\mathbb{X}, M^*)$ is called M^*-convergent if there exists x in $\mathbb{X}$ such that for $\epsilon > 0$, there exists a $N > 0$ integer number such that $M^*(x_n, x_m, x) < \epsilon$ for all $m \geq N$, $n \geq N$. Then we called that $\{x_n\}$ is M^*-convergent to x and x is a limit of $\{x_n\}$.

Definition 3.2. A sequence $\{x_n\}$ in an M^*-metric space $(\mathbb{X}, M^*)$ is called M^*-Cauchy if for a given $\epsilon > 0$, there exists a positive integer N such that $M^*(x_n, x_m, x_p) < \epsilon$ for all $m, n, p \geq N$.

In the following, we introduce the concept of an M^*-strongly convergent and a very M^*-strongly convergent. Take into can a sequence $\{x_n\}$ in an M^*-metric in the following two definitions.

Definition 3.3. Let $(\mathbb{X}, M^*)$ be an M^*-metric space and $\{x_n\}$ be a sequence in $\mathbb{X}$, we say that $\{x_n\}$ is M^*-strongly convergent to an element x in $\mathbb{X}$ if

(i) $M^*(x_n, x_m, x) \to 0$ as $m, n \to \infty$,
(ii) $\{M^*(y, y, x_n)\}$ converges to $M^*(y, y, x)$ for all $y \in \mathbb{X}$.

Definition 3.4. Let $(\mathbb{X}, M^*)$ be an M^*-metric space and $\{x_n\}$ be a sequence in $\mathbb{X}$, we call that $\{x_n\}$ is very M^*-strongly convergent to an element x in $\mathbb{X}$ if

(i) $M^*(x_n, x_m, x) \to 0$ as $m, n \to \infty$,
(ii) $\{M^*(y, z, x_n)\}$ converges to $M^*(y, z, x)$ for all $y, z \in \mathbb{X}$.

By using some properties of Remark 1.1, we present some results on M^*-convergence, M^*-Cauchy, M^*-strongly convergent and very M^*-strongly convergent.

Theorem 3.5. *Let $(\mathbb{X}, M^*)$ be an M^*-metric space. Then $\{x_n\}$ converges to x in $(\mathbb{X}, M^*)$ strongly if and only if $\{x_n\}$ converges to x in $(\mathbb{X}, M^*)$ and* $\lim_{n\to\infty} M^*(x, x, x_n) = 0$.

Proof. Let $\{x_n\}$ be an M^*-convergent sequence in $\mathbb{X}$ with limit x, that is, $\lim_{n\to\infty} M^*(x, x, x_n) = 0$ and $\epsilon > 0$. Then there is a positive integer N such that $M^*(x, x, x_n) < \epsilon$ for all $n \geq N$. Let $y \in \mathbb{X}$. Then for $n \geq N$,

$$\begin{aligned} M^*(y, y, x_n) &\leq RM^*(y, y, x) + M^*(x_n, x, x) \\ &\leq R\left[M^*(y, y, x) + M(x_n, x, x)\right]. \end{aligned}$$

This produces that

$$|M^*(y, y, x_n) - RM^*(y, y, x)| \leq RM^*(x, x, x_n) < R\epsilon = \epsilon_1 \quad \text{for all } n \geq N.$$

Consequently, we get

$$|M^*(y, y, x_n) - M^*(y, y, x)| \leq RM^*(x, x, x_n) < R\epsilon = \epsilon_1.$$

Hence $\{M^*(y, y, x_n)\}$ converges to $M^*(y, y, x)$ for all $y \in \mathbb{X}$. This means that $\{x_n\}$ converges strongly to x in $\mathbb{X}$. □

We can easily prove the theorem from the definition of the M^*-metric.

Theorem 3.6. *Let $(\mathbb{X}, M^*)$ be an M^*-metric space and let $\{x_n\}$ be a sequence in $\mathbb{X}$ and $x \in \mathbb{X}$. Assume the following implications:*

(1) $M^*(x, x, x_n) \to 0$ *as* $n \to \infty$,
(2) $M^*(x, x_n, x_n) \to 0$ *as* $n \to \infty$,

(3) $M^*(x, x_n, x_m) \to 0$ *as* $n, m \to \infty$,
(4) $M^*(y, y, x_n) \to M^*(y, y, x)$ *as* $n \to \infty$ *for all* $y \in \mathbb{X}$,
(5) $M^*(y, x_n, x_m) \to M^*(y, x, x)$ *as* $n, m \to \infty$ *for all* $y \in \mathbb{X}$,
(6) $M^*(y, x, x_n) \to M^*(y, x, x)$ *as* $n \to \infty$ *for all* $y \in \mathbb{X}$,
(7) $M^*(y, z, x_n) \to M^*(y, z, x)$ *as* $n \to \infty$ *for all* $z, y \in \mathbb{X}$.

Then (7) $\Longrightarrow$ (6) $\Longrightarrow$ (1), (7) $\Longrightarrow$ (4) $\Longrightarrow$ (1) *and* (5) $\Longrightarrow$ (3) $\Longrightarrow$ (2).

Proof. We can easily prove the theorem from the definition of the M^*-convergent and M^*-metric. $\square$

Inside the following example, we provide some non implications of Theorem 3.6.

Example 3.7. Either (3) or (4) does not imply (5), (6) or (7).

Let $\mathbb{X} = \mathbb{R}$ with an M^*-metric. Then the function M_3^* is defined in Theorem 2.4 on $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$ reduces to the following.

$$M_3^* = \begin{cases} 0, & \text{if } x = y = z, \\ |x-y| + |y-z| + |z-x|, & \text{if } x, y, z \text{ are distinct}, \\ \max\{|x-y|, |y-z|, |z-x|\}, & \text{otherwise.} \end{cases}$$

Then $(\mathbb{X}, M_3^*)$ is an M^*-metric space in which (3) and (4) are satisfied but (5), (6) and (7) are not satisfied.

Let $x_n = 2^{\frac{1}{n}}$ for $n = 1, 2, 3, \ldots$. Then $\{x_n\}$ converges to 1 as $n \to \infty$ with respect to the M^*-metric. For $m > n$, we have

$$M_3^*(1, 2^{\frac{1}{n}}, 2^{\frac{1}{m}}) = \left|1 - 2^{\frac{1}{n}}\right| + \left|2^{\frac{1}{m}} - 2^{\frac{1}{n}}\right| + \left|2^{\frac{1}{m}} - 1\right| \to 0, \quad \text{as } n, m \to \infty.$$

Therefore, $\left\{2^{\frac{1}{n}}\right\}$ convergent to 1 with respect to M_3^*. Since for $y \in \mathbb{X}$,

$$M_3^*\left(y, y, 2^{\frac{1}{n}}\right) = \max\left\{\left|y - 2^{\frac{1}{n}}\right|, 0, \left|y - 2^{\frac{1}{n}}\right|\right\} = \left|y - 2^{\frac{1}{n}}\right|,$$

we have $\lim_{n\to\infty} M_3^*\left(y, y, 2^{\frac{1}{n}}\right) = M^*(y, y, 1)$ for all $y \in \mathbb{X}$. Thus (3) and (4) hold.

Let $y = 3$ and $x = 1$. Trivially, $\lim_{n\to\infty} M_3^*\left(3, 1, 2^{\frac{1}{n}}\right) \neq M_3^*(3, 1, 1)$, thus (6) and (7) do not hold. Additionally $\lim_{n,m\to\infty} M_3^*\left(3, 2^{\frac{1}{n}}, 2^{\frac{1}{m}}\right) \neq M_3^*(3, 1, 1)$, thus (5) does not hold.

Theorem 3.8. *Let* $(\mathbb{X}, M^*)$ *be an* M^**-metric space satisfying* (M_7^*) *and* (M_8^*). *Then the function* d *on* $\mathbb{X} \times \mathbb{X} \to [0, \infty)$ *is defined by* $d(x, y) = M^*(x, y, y)$ *is* b*-metric on* $\mathbb{X}$ *and the following are equivalent:*

(i) $\lim_{n\to\infty} x_n = x$ *in* $(\mathbb{X}, d)$.
(ii) $\lim_{n\to\infty} x_n = x$ *in* $(\mathbb{X}, M^*)$.

(iii) $\lim_{n\to\infty} x_n = x$ *strongly in* $(\mathbb{X}, M^*)$.

Proof. By Proposition 1.8, it is clear that $(\mathbb{X}, d)$ is a b-metric space.

Assume (i) holds, then $\lim_{n\to\infty} x_n = x$ in $(\mathbb{X}, d)$.

Let $\epsilon > 0$. Then there exists an integer number $N > 0$ such that $d(x, x_n) < \frac{\epsilon}{2}$ for all $n \geq N$. For $n, m \geq N$,

$$\begin{aligned} M^*(x, x_n, x_m) &\leq \frac{1}{R}\left[M^*(x, x, x_m) + M^*(x, x, x_n)\right] \\ &= \frac{1}{R}\left[d(x, x_m) + d(x, x_n)\right] < \epsilon. \end{aligned}$$

Thus (i)$\Longrightarrow$(ii).

Assume (ii) holds, then $\lim_{n\to\infty} x_n = x$ in $(\mathbb{X}, M^*)$.

Let $\epsilon > 0$. Then there exists a positive integer N such that $M^*(x_n, x_m, x) < \epsilon$ for all $m, n \geq N$. For $y \in \mathbb{X}$ and $n \geq N$, by (M^*7),

$$\begin{aligned} M^*(y, y, x_n) &\leq RM^*(x, y, x_n) \\ &\leq \frac{R}{R}[M^*(x, x, x_n) + M^*(y, x, x)] \\ &\leq M^*(x, x_n, x_n) + M^*(y, y, x). \end{aligned}$$

Then, we get

$$|M^*(y, y, x_n) - M^*(y, y, x)| \leq M^*(x, x_n, x_n) < \frac{\epsilon}{2} \text{ for all } n \geq N.$$

Hence $\{M^*(y, y, x_n)\}$ converges to $M^*(y, y, x)$ for all $y \in \mathbb{X}$. Thus, (ii)$\Longrightarrow$(iii) hold.

The implicates (iii)$\Longrightarrow$(ii) is trivial.

Now, we need to prove (ii)$\Longrightarrow$(i).

Assume (ii) holds, then $\lim_{n\to\infty} x_n = x$ in $(\mathbb{X}, M^*)$.

Let $\epsilon > 0$. Then there exists a positive integer N such that $M^*(x_n, x_n, x) < \epsilon$ for all $m, n \geq N$. For $n \geq N$, by (M^*7),

$$d(x, x_n) = M^*(x, x, x_n) \leq RM^*(x, x_m, x_n) < R\epsilon.$$

Hence $\lim_{n\to\infty} x_n = x$ in $(\mathbb{X}, d)$. Thus (ii)$\Longrightarrow$(i). □

4. Common fixed point theorems in M^*-metric space

Theorem 4.1. *Let $(\mathbb{X}, M^*)$ be an complete M^*-complete metric space and let $S : \mathbb{X} \to \mathbb{X}$ be a mapping which satisfies the following condition for all*

$x, y, z \in \mathbb{X}$ with $R \geq 1$,

$$M^*(Sx, Sy, Sz) \leq \frac{1}{R} \max \left\{ \begin{array}{c} aM^*(x, y, z), \\ b[M^*(x, Sx, Sy) + 2M^*(y, Sy, Sy)], \\ b[M^*(x, Sy, Sy) + M^*(y, Sx, Sy), \\ M^*(z, Ty, Tx)] \end{array} \right\}, \quad (4.1)$$

where $0 < a < 1$ and $0 < b < \frac{1}{3}$. Then S has a unique fixed point.

Proof. Let $x_0 \in \mathbb{X}$ be arbitrary, there exists $x_1 \in \mathbb{X}$ such that $Sx_0 = x_1$ and let $\{x_n\}$ in $\mathbb{X}$ be a sequence with $Sx_{n-1} = x_n$. By using (4.1), we have

$$\begin{aligned}
&M^*(x_n, x_n, x_{n+1}) \\
&= M^*(Sx_{n-1}, Sx_{n-1}, Sx_n) \\
&\leq \frac{1}{R} \max \left\{ \begin{array}{c} aM^*(x_{n-1}, x_{n-1}, x_n), b[M^*(x_{n-1}, Sx_{n-1}, Sx_{n-1}) \\ +2M^*(x_{n-1}, Sx_{n-1}, Sx_{n-1})], \\ b[M^*(x_{n-1}, Sx_{n-1}, Sx_{n-1}) + M^*(x_{n-1}, Sx_{n-1}, Sx_{n-1}) \\ +M^*(x_n, Sx_{n-1}, Sx_{n-1})] \end{array} \right\} \\
&= \frac{1}{R} \max \left\{ \begin{array}{c} aM^*(x_{n-1}, x_n, x_n), b[M^*(x_{n-1}, x_n, x_n) \\ +2M^*(x_{n-1}, x_n, x_n)], \\ b[M^*(x_{n-1}, x_n, x_n) + M^*(x_{n-1}, x_n, x_n) \\ +M^*(x_n, x_n, x_n)] \end{array} \right\} \\
&= \frac{1}{R} \max \left\{ \begin{array}{c} aM^*(x_{n-1}, x_{n-1}, x_n), 3bM^*(x_{n-1}, x_n, x_n), \\ 2bM^*(x_{n-1}, x_n, x_n) \end{array} \right\} \\
&\leq \alpha M^*(x_{n-1}, x_{n-1}, x_n), \qquad (4.2)
\end{aligned}$$

where $\alpha = \max\{a, 3b\}$ and $0 < \alpha < 1$.

By repeating the application of the above inequality and equality (4.2), we have

$$\begin{aligned}
M^*(x_n, x_n, x_m) &\leq RM^*(x_n, x_n, x_{n+1}) + RM^*(x_{n+1}, x_{n+1}, x_{n+2}) \\
&\quad \vdots \\
&\quad + RM^*(x_{m-2}, x_{m-2}, x_{m-1}) + M^*(x_{m-1}, x_{m-1}, x_m) \\
&\leq R[M^*(x_n, x_n, x_{n+1}) + M^*(x_{n+1}, x_{n+1}, x_{n+2}) \\
&\quad \vdots \\
&\quad + M^*(x_{m-2}, x_{m-2}, x_{m-1}) + M^*(x_{m-1}, x_{m-1}, x_m)] \\
&\leq (\alpha^n + \alpha^{n+1} + ... + \alpha^{m-1}) RM^*(x_0, x_0, x_1) \\
&\leq \frac{\alpha^n}{1-\alpha} RM^*(x_0, x_0, x_1).
\end{aligned}$$

Thus, $M^*(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.

Also, for $n, m, l \in \mathbb{N}$,

$$\begin{aligned} M^*(x_n, x_m, x_l) &\leq RM^*(x_n, x_m, x_m) + M^*(x_m, x_l, x_l) \\ &\leq \frac{\alpha^n}{1-\alpha} R^2 M^*(x_0, x_0, x_1) + \frac{\alpha^n}{1-\alpha} RM^*(x_0, x_0, x_1). \end{aligned}$$

Taking $n, m, l \to \infty$, we get $M^*(x_n, x_m, x_l) \to 0$, so $\{x_n\}$ is an M^*-Cauchy sequence. Since $\mathbb{X}$ is an M^*-complete, there exists u such that $x_n \to u$ as $n \to \infty$.

If $S(\mathbb{X}) \subseteq \mathbb{X}$, we have $u \in \mathbb{X}$. Then there exists $p \in \mathbb{X}$ such that $p = u$. We claim that $Sp = u$. From

$$\begin{aligned} &M^*(Sp, u, u) \\ &= M^*(Sp, Sp, u) \\ &\leq RM^*(Sp, Sp, Sx_n) + M^*(Sx_n, u, u) \\ &\leq \max \left\{ \begin{array}{c} aM^*(u, u, x_n), b[M^*(u, p, x_{n+1}) + 2M^*(u, Sp, Sp)], \\ +b[M^*(u, Sp, Sp) + M^*(u, Sp, Sp) + M^*(x_n, Sp, Sp)] \end{array} \right\}, \end{aligned}$$

as $n \to \infty$, we get $M^*(Sp, u, u) = 0$ and $Sp = u$, that is, $Sp = p$. Thus, S has a fixed point.

Next, we need to prove that S has a unique fixed point. Assume there exists q in $\mathbb{X}$ such that $q = Sq$. Then, we have

$$\begin{aligned} &M^*(Sp, Sp, Sq) \\ &\leq \frac{1}{R} \max \left\{ \begin{array}{c} aM^*(p, p, q), b[M^*(p, Sp, Sq) + 2M^*(p, Sp, Sp)], \\ b[M^*(p, Sp, Sp) + M^*(p, Sp, Sp) + M^*(q, Sp, Sp)] \end{array} \right\} \\ &= \frac{1}{R} \max \left\{ \begin{array}{c} aM^*(Sp, Sp, Sq), b[M^*(Sp, Sp, Sq) + 2M^*(Sp, Sp, Sp)], \\ b[M^*(Sp, Sp, Sp) + M^*(Sp, Sp, Sp) + M^*(Sq, Sp, Sp)] \end{array} \right\} \\ &= \frac{1}{R} \max \{aM^*(Sp, Sp, Sq), bM^*(Sq, Sp, Sp)\} \\ &= \frac{\beta}{R} M^*(Sp, Sp, Sq), \text{ where } \beta = \max\{a, b\}. \end{aligned}$$

Hence, we have

$$M^*(Sp, Sp, Sq) \leq \gamma M^*(Sp, Sp, Sq),$$

where $\gamma = \frac{\beta}{R}$, that is, $(\gamma - 1)M^*(Sp, Sp, Sq) \geq 0$. Since $\gamma \in (0, 1)$,

$$(\gamma - 1)M^*(Sp, Sp, Sq) \leq 0.$$

This means that S has a unique fixed point. This completes the proof. $\square$

5. M^*-contraction and an application to system of linear equations

In this section, we seek to present a solution to a system of linear equations. Therefore, we will be to prove the following theorems.

Definition 5.1. Let M^* be an M^*-metric on a set $\mathbb{X}$ and $T : \mathbb{X} \to \mathbb{X}$ be a mapping. T is said to be an M^*-contraction if for all $x, \ell \in \mathbb{X}$ there exists $\delta \in [0, 1)$ such that

$$M^*(Tx, Tx, T\ell) \leq \delta M^*(x, x, \ell).$$

Theorem 5.2. *Let $\mathbb{X}$ be an M^*-complete metric space and $T : \mathbb{X} \to \mathbb{X}$ be an M^*-contraction with $\delta \in [0, 1)$ and $R \geq 1$. Assume that there exists $x \in \mathbb{X}$ such that $M^*(x, x, Tx) < \infty$. Then there is $\ell \in \mathbb{X}$ such that $x_n \to \ell$ and ℓ is a unique fixed point of T.*

Proof. Let $x_0 \in \mathbb{X}$ and a sequence $\{x_n\}$ in $\mathbb{X}$ defined by $x_n = Tx_{n-1} = T^n x_0$. Then, we get

$$\begin{aligned} M^*(T^2x_0, T^2x_0, T^2x) &\leq \delta M(Tx_0, Tx_0, Tx) \\ &\leq \delta^2 M^*(x_0, x_0, x). \end{aligned}$$

If this process is repeated attain,

$$M^*(T^n x_0, T^n x_0, T^n x) \leq \delta^n M^*(x_0, x_0, x).$$

Now, we have to prove that $\{x_n\}$ is an M^*-Cauchy in $\mathbb{X}$.

$$\begin{aligned} M^*(x_n, x_n, x_m) &\leq RM^*(x_n, x_n, x_{n+1}) + RM^*(x_{n+1}, x_{n+1}, x_{n+2}) \\ &\quad \vdots \\ &\quad + RM^*(x_{m-1}, x_{m-1}, x_m) \\ &\leq R\delta^n M^*(x_0, x_0, x) + R\delta^{k+1} M^*(x_0 x_0, x) \\ &\quad \vdots \\ &\quad + R\delta^{m-1} M^*(x_0, x_0, x) \\ &= M^*(x_0, x_0, x) R\delta^n \left[1 + \delta + (\delta)^2 + ... + (\delta)^{m-n-1}\right], \end{aligned} \tag{5.1}$$

where $n > m > 0$. Letting $n, m \to \infty$ in (5.1), we have

$$\lim_{m,n\to\infty} M^*(x_n, x_n, x_m) = 0.$$

Thus, $\{x_n\}$ is an M^*-Cauchy in $\mathbb{X}$. Since $\mathbb{X}$ is an M^*-complete, so $\{x_n\}$ is M^*-convergent to some ℓ.

From this inequality,

$$\begin{aligned} M^*(T\ell, T\ell, \ell) &\leq RM^*(\ell, \ell, x_n) + M^*(x_n, x_n, \ell)) \\ &\leq R\delta M^*(\ell, \ell, x_{n-1}) + M^*(x_n, x_n, \ell) \\ &= R\delta M^*(x_{n-1}, x_{n-1}, \ell) + M^*(x_n, x_n, \ell) \\ &= 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

we know that ℓ is fixed point of T.

Now, we need to prove that ℓ is unique fixed point of T. Assume ℓ_1 is a fixed point of T such that $\ell \neq \ell_1$. Since

$$M^*(\ell, \ell, \ell_1) = M^*(T\ell, T\ell, T\ell_1) \leq \delta M^*(\ell, \ell, \ell_1),$$

we get

$$(1 - \delta)M^*(\ell, \ell, \ell_1) \leq 0.$$

This implies that $M^*(\ell, \ell, \ell_1) = 0$, that is, $\ell = \ell_1$. This completes the proof. □

To achieve our purpose in this section, we must prove the following theorem by Theorem 5.2.

Theorem 5.3. *Let* $\mathbb{X} = \mathbb{R}^n$ *be an* M^**-metric space with the* M^**-metric:*

$$M^*(\wp, q, \ell) = \sum_{i=1}^{n} \left(|\wp_i - q_i| + |q_i - \ell_i| + |\ell_i - \wp_i|\right).$$

If

$$\sum_{i=1}^{n} |\alpha_{ij}| \leq \alpha < 1 \quad \textit{for all} \quad j = 1, 2, ..., n,$$

then the linear system

$$\begin{cases} \alpha_{11}\wp_1 + \alpha_{12}\wp_2 + ... + \alpha_{1n}\wp_n = \gamma_1 \\ \alpha_{21}\wp_1 + \alpha_{22}\wp_2 + ... + \alpha_{2n}\wp_n = \gamma_2 \\ \qquad\qquad\vdots \\ \alpha_{n1}\wp_1 + \alpha_{n2}\wp_2 + ... + \alpha_{nn}\wp_n = \gamma_n \end{cases} \tag{5.2}$$

has a unique solution.

Proof. Since $\mathbb{X} = \mathbb{R}^n$ is an M^*-complete, we have to show that $T : \mathbb{X} \to \mathbb{X}$ is defined by

$$T(\wp) = A\wp + \gamma,$$

where $\wp = (\wp_1, \wp_2, ..., \wp_n) \in \mathbb{R}^n$ and

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix} \neq 0,$$

is an M^*-contraction. Since

$$\begin{aligned}
M^*(T\wp, Tq, T\ell) &= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} \alpha_{ij} \left((\wp_j - q_j) + (q_j - \ell_j) + (\ell_j - \wp_j) \right) \right| \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_{ij}| \left| (\wp_j - q_j) + (q_j - \ell_j) + (\ell_j - \wp_j) \right| \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} |\alpha_{ij}| \left| (\wp_j - q_j) + (q_j - \ell_j) + (\ell_j - \wp_j) \right| \\
&\leq \alpha \sum_{j=1}^{n} \left| (\wp_j - q_j) + (q_j - \ell_j) + (\ell_j - \wp_j) \right| \\
&= \alpha M^*(\wp, q, \ell),
\end{aligned}$$

T is an M^*-contraction and it is obvious that $M^*(\wp, \wp, \ell) < \infty$. By Theorem 5.2, the linear equation system (5.2) has a unique solution. $\square$

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