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COMPLEX DELAY-DIFFERENTIAL EQUATIONS OF MALMQUIST TYPE

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Abstract. In this paper, we investigate some results on complex delay-differential equations of the classical Malmquist theorem. A classic illustrations of their results states us that if a complex delay equation

$$w(t+1)+w(t-1)=R(t,w)$$

with $R(t,w)$ rational in both arguments admits (concede) a transcendental meromorphic solution of finite order, then $deg_w R(t,w) \leq 2$. Development and upgrade of such results are presented in this paper. In addition, Borel exceptional zeros and poles seem to appear in special situations.

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1. Introduction

Existence of large classes of solutions that are meromorphic in the whole complex plane is a rare property for differential equations. According to a classical result due to Malmquist, if the first order differential equation

$$w' = R(t,w) \tag{1}$$

where $R(t,w)$ is a rational in both arguments, has a transcendental meromorphic solution, then (1) reduces into the Riccati Equation

$$w' = a_2 w^2 + a_1 w + a_0 \tag{2}$$

with rational co-efficients. For more details concerning the equations (1) and (2) as well as for generalization of the malmquist theorem, see [8].

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We first recall some existence results for solutions meromorphic in the complex plane. An example of a complex delay equations combining existence and growth restriction has been offered by S. Bank and R. Kaufman [2].

Theorem 1.1. *For any rational function $R(t)$ the delay Equation*

$$w(t+1) - w(t) = R(t)$$

always has a meromorphic solution w such that $T(r,w) = O(r)$.

Ablowtz, Halburd and Herbst [1] studied complex delay equations related to (1) and (2) namely the equations,

$$w(t+1) + w(t-1) = \frac{\tilde{a_0}(t) + \tilde{a_1}(t)w + \cdots + \tilde{a_p}(t)w^p}{\tilde{b_0}(t) + \tilde{b_1}(t)w + \cdots + \tilde{b_q}(t)w^q} \tag{3}$$

and

$$w(t+1) + w(t-1) = a(t) + b(t)w + c(t)w^2, \tag{4}$$

where the co-efficients are meromorphic functions to be specified later on. Also the equation

$$w(t+1) + w(t-1) = \frac{\tilde{a_0}(t) + \tilde{a_1}(t)w + \cdots + \tilde{a_p}(t)w^p}{\tilde{b_0}(t) + \tilde{b_1}(t)w + \cdots + \tilde{b_q}(t)w^q}, \tag{5}$$

which is similar to (3), was studied in [1]. The following these results, reminiscent of the classical malmquist theorem, were proved in [1].

Theorem 1.2. [1] *If the difference equation* (3), *with polynomial co-efficients $\tilde{a_i}(t), \tilde{b_i}(t)$ admits a transcendental meromorphic solution of finite order, then $d = max\{p,q\} \leq 2$.*

Theorem 1.3. [1] *Suppose that the coefficients $a(t), b(t)$ in the difference equation* (4) *are polynomials and that $c(t)$ is a non-zero complex constant. Then any transcendental entire solution of* (4) *is of infinite order.*

Theorem 1.4. [1] *If the difference equation* (5) *with polynomial coefficients $\tilde{a_i}(t), \tilde{b_i}(t)$ admits a transcendental meromorphic solution of finite order, then $d = max\{p,q\} \leq 2$.*

This paper has been organized as follows. Here the essential growth problem for meromorphic solution of complex difference equations is to find a lower bound for their characteristic function. Theorem 1.6 is a generalization of Theorem 1.3 and Theorem 1.8 is devoted to considering a generalized form of the delay equation (5). More precisely we show that in special cases only, it may happen that zeros and poles are Borel exceptional value of a meromorphic solutions.

Proposition 1.1. [1] *Let $C_1 \ldots C_n \in \mathbb{C} \setminus \{0\}$. If the difference equation*

$$\sum_{i=1}^{k} w(t+c_i) = \frac{\tilde{a_0}(t) + \tilde{a_1}(t)w + \cdots + \tilde{a_p}(t)w^p}{\tilde{b_0}(t) + \tilde{b_1}(t)w + \cdots + \tilde{b_q}(t)w^q} \tag{6}$$

with rational coefficients $\tilde{a_i}(t), \tilde{b_i}(t)$ admits a transcendental meromorphic solution of finite order, then $d \leq k$.

Proposition 1.2. [1] *Let $C_1, \dots C_n \in \mathbb{C} \setminus \{0\}$. If the difference equations*

$$\prod_{i=1}^{k} w(t+c_i) = \frac{\tilde{a_0}(t) + \tilde{a_1}(t)w + \cdots + \tilde{a_p}(t)w^p}{\tilde{b_0}(t) + \tilde{b_1}(t)w + \cdots + \tilde{b_q}(t)w^q} \tag{7}$$

with rational coefficients $\tilde{a_i}(t), \tilde{b_i}(t)$ admits a transcendental meromorphic solution of finite order, then $d \leq k$.

Example 1.5. *Let $c \in \mathbb{C}$ be a constant such that $c \neq \frac{\pi}{2}h$, where $h \in t$. Since,*

$$tan\,(t+c) = \frac{tan\,t + tan\,c}{1 - tan\,t.\,tan\,c}$$

we see that $w(z) = tan\,t$ solves

$$w(t+c) = \frac{1}{C}\frac{w(z) - C}{w(z) + \frac{1}{c}}$$

where $C := -tan\,c \neq 0, \infty$.

Theorem 1.6. *Let $C_1, \dots, C_n \in \mathbb{C} \setminus \{0\}$ and let $l \geq 2$. Suppose w is a transcendental meromorphic solution of the difference equation*

$$\sum_{i=1}^{k} \tilde{a_i}(t) w(t+c_i) = \sum_{i=0}^{l} \tilde{b_i}(t) w(t)^i \tag{8}$$

with rational co-efficient $\tilde{a_i}(t)\,\tilde{b_i}(t)$. Denote $C := max\{|c_1|.......|c_n|\}$. If w has infinitely many poles, then there exists constants $S > 0$ and $r_0 > 0$ such that $n(r, w) \geq Sl^{\frac{r}{c}}$ holds for all $r \geq r_0$.

Proof. *We multiply out the denominators of the coefficients $\tilde{a_i}(t)$, $\tilde{b_i}(t)$ in (8) to obtain*

$$\sum_{i=1}^{k} P_i(t) w(t+c_i) = \sum_{i=0}^{l} Q_i(t) w(t)^i, \tag{9}$$

where the coefficients $P_i(t), Q_i(t)$ are polynomials. We suppose that w, the solution of (8) and (9), is meromorphic with infinitely many poles.

Choose a pole t_0 of w having multiplicity $\tau \geq 1$ such that t_0 is not a zero of $Q_i(t)$. Then the right hand side of (9) has a pole of multiplicity $l\tau$ at t_0. Hence, there exists at least one index $lr_1 \in \{1, 2, 3, \dots, k\}$ such that $t_0 + C_{h_1}$ is a pole of w of multiplicity $\nu_1 \geq l\tau$. Substitute $t_0 + c_{h_1}$ for w in (9) we obtain

$$\sum_{i=1}^{k} P_i(t_0 + C_{h_1}) w(t_0 + C_{h_1} + c_i) = \sum_{i=0}^{k} Q_i(t_0 + C_{h_1}) w(t_0 + C_{h_1})^i \tag{10}$$

we now have two possibilities.

(i) If $t_0 + C_{h_i}$ is a zero of $Q_l(t)$, this process will be terminated and we have to choose another pole t_0 of w in the way we did above .

(ii) If $t_0 + C_{h_i}$ *is not a zero of* $Q_l(t)$*, then we see that the right-hand side of* (10) *has a pole of multiplicity* $l\nu_1$ *at* $t_0 + C_{h_1}$*. Hence, there exists at least one index* $h_2 \in \{1, 2, 3, \ldots, k\}$ *such that* $t_0 + c_{h_1} + c_{h_2}$ *is a pole of* w *of multiplicity* $\nu_2 \geq l\nu_1 \geq l^2\tau$*. At this point we note that, as a polynomial, the coefficient* $Q_l(t)$ *has finitely many zeros, all being inside of a finite dice* $|t| < R$*.*

We proceed to follow the steps (*i*) *and* (*ii*) *above, since there are infinitely many poles of* w*, we will find a pole* t_0 *of* w *such that*

$$t_0 + C_{h_1} + ... + C_{h_j} =: \xi_j$$

is a pole of w *of multiplicity* ν_j *for all* $j \in N$*. Since* $\nu_j \geq l^j\tau \to \infty$*; as* $j \to \infty$*, and since* w *does not have essential singularities in the finite plane, we must have* $|\xi_j| \to \infty$ *as* $j \to \infty$*. It is clear that for* j *large enough, say* $j \geq j_0$*,*

$$\tau l^j \leq \tau(1 + l + ... + l^j) \leq n(|\xi_j|, w)$$
$$\leq n(|w_0| + jC, w) \leq n(\nu + jC, w),$$

where $v \in (|t_0|, |t_0| + C)$ *can be chosen arbitrarily. Letting* $j \to \infty$ *for each choice of* v*, we see that*

$$n(r, w) \geq Sl^{\frac{r}{c}}$$

holds for all

$$r \geq r_0 := (j_0 + 1)C + |w_0|,$$

where

$$S := \tau l^{-(|w_0|+c)/c}.$$

The fact that r_0 *and* S *both depend on* $|w_0|$ *is not a problem, since* w_0 *is fixed.*

Example 1.7. *Fix* $k = l \in \mathbb{N} \setminus \{1\}$*. Let* $c_i \in \mathbb{C}$ *be constants such that* $e^{c_i} = i$ *for all* $i = 1, 2, 3, \ldots, k$*. Then* $w(t) = e^{e^t}/t$ *solves*

$$\sum_{i=1}^{k} (t + c_i) w(t + c_i) = \sum_{i=1}^{k} t^i w(t)^i.$$

we now proceed to consider the value distribution of zeros and poles of solutions of equation (7)*.*

The following results tells us that solutions having Borel exceptional zero and poles appear in special situations only .

Theorem 1.8. *Let* $C_1, \ldots, C_n \in \mathbb{C} \setminus \{0\}$ *and suppose that* w *is a non-rational meromorphic solution of*

$$\prod_{i=1}^{k} w(t + c_i) = \frac{\tilde{a_0}(t) + \tilde{a_1}(t)w + ... + \tilde{a_p}(t)w^p}{\tilde{b_0}(t) + \tilde{b_1}(t)w + ... + \tilde{b_q}(t)w^q} \tag{11}$$

with meromorphic coefficient $\tilde{a_i}(t), \tilde{b_i}(t)$ *of growth* $S(r, w)$ *such that* $a_p(t), b_q(t) \not\equiv 0$.
If

$$max(\lambda(w), \lambda(\frac{1}{w})) < p(w) \tag{12}$$

then (11) *is of the form*

$$\prod_{i=1}^{k} w(t+c_i) = c(t)w(t)^i \tag{13}$$

where c(z) is meromorphic, T(r,c)=S(r,w) and $j \in \mathbb{Z}$.

Proof. *Denote* $X(t) = \prod_{i=1}^{k} w(t+c_i)$. *Fix constants* β *and* γ *such that* $max(\lambda(w), \lambda(\frac{1}{w})) < \beta < \gamma < \rho(w)$, *using (12) and the lemma of the logarithmic derivative, we get*

$$T(r, \frac{w'}{w}) = \bar{N}(r, w) + \bar{N}(r, \frac{1}{w}) + S(r, w)$$
$$= O(r^{\beta}) + S(r, w).$$

Similarly

$$T(r, \frac{X'}{X}) = N(r, \frac{X'}{X}) + m(\frac{X'}{X})$$
$$\leq k\bar{N}(r + C, w(t)) + k\bar{N}(r + C, \frac{1}{w(t)}) + S(r, X)$$
$$= O(r^{\beta}) + S(r, w),$$

where $C := max\{| C_1 |, | C_2 |, \ldots, | C_n |\}$. *Here we have applied the valiron-Mohon'ko Theorem to the equation* (7) *to conclude that* $T(r, X) = dT(r, w) + S(r, w)$ *and so* $S(r, X) = S(r, w)$. *Since zeros and poles are Borel exceptional by* (12), *we may apply a result due to Whittaker, See[* [7], *Satz 13.4], to deduce that w is of regular growth. Hence there exists* $r_0 > 0$ *such that* $T(r, w) > r^{\gamma}$ *for* $r \geq r_0$.

It follows that

$$T(r, \frac{w'}{w}) = S(r, w)$$

and

$$T(r, \frac{X'}{X}) = S(r, w).$$

Rewriting (11) in the form

$$\frac{\tilde{b_q}(t)}{\tilde{a_p}(t)} X(t) = \frac{P(t, w)}{Q(t, w)} = u(t, w) \tag{14}$$

we may suppose that P and Q are monic polynomials in w with coefficients of growth $S(r, w)$. *Denote* $W := \frac{w'}{w}$, $U := \frac{u'}{u}$ *and observe that* $T(r, U) = S(r, w)$ *by (14). Since*

$$\frac{P'Q - PQ'}{Q^2} = u' = Uu = \frac{UP}{Q},$$

we get

$$P'Q - PQ' = UPQ. \tag{15}$$

writing $w' = Ww$ in (15), regarding then (15) as an algebraic equation in w with coefficients of growth $S(r, w)$ and comparing the leading coefficients, we obtain

$$(p - q)W = U.$$

Therefore, $u(t) = \alpha w(t)^{p-q}$ or some $\alpha \in \mathbb{C}$, and so

$$X(t) = \alpha \frac{\tilde{a_p(t)}}{\tilde{b_q}(t)} w(t)^{p-q}, \tag{16}$$

proving the assertion.

Example 1.9. *We observed that $\prod_{i=1}^{k} tan(t + c_i)$ is rational function in tan t not being of the form (13). Since*

$$\lambda(tant) = \lambda(\frac{1}{tant}) = \rho(tant) = 1,$$

then condition (12) in Theorem 1.8 is necessary.

Example 1.10. *Condition (12) in Theorem 1.8 cannot be replaced by*

$$min(\lambda(w), \lambda(\frac{1}{w})) < \rho(w),$$

since $w(t) = sint$ satisfies

$$w(t + 1)w(t - 1) = w(t)^2 - sin^2 1.$$

Example 1.11. *Let $A \in \mathbb{C} \setminus \{0\}$ and $p \in \mathbb{Z}$. Fix constants $\alpha, \beta \in \mathbb{C}$ satisfying*

$$\alpha^{p+2} = A$$

and

$$\beta + \frac{1}{\beta} = -p$$

Then the delay equation

$$w(t + 1)w(t - 1) = \frac{A}{w(t)^p}$$

which is clearly of the form (13), has an entire solution

$$w(t) = \alpha\, exp(\pi(t)e^{tlog\beta}).$$

Here $\pi(t)$ is any periodic entire function of period 1.

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