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LOCAL SPECTRAL THEORY AND QUASINILPOTENT OPERATORS

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Abstract. In this paper we show that if $A \in L(X)$ and $R \in L(X)$ is a quasinilpotent operator commuting with A then $X_A(F) = X_{A+R}(F)$ for all subset $F \subseteq \mathbb{C}$ and $\sigma_{loc}(A) = \sigma_{loc}(A+R)$. Moreover, we show that A and $A + R$ share many common local spectral properties such as SVEP, property (C), property (δ), property (β) and decomposability. Finally, we show that quasisimility preserves local spectrum.

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1. Introduction

The single valued extension property dates back to the early days of local spectral theory and appeared in the work of Dunford [11], [12], Dunford-Schwartz [13] and Colojoarvă and Foiás [9]. The following localized version of single valued extension property was introduced by Finch [16]. The single valued extension property has now developed into one of the major tools in the local spectral theory and Fredholm theory for operators on Banach spaces, see [1], [20].

Throughout this paper, $L(X, Y)$ denotes the set of all bounded linear operators from Banach space X to Banach space Y, and $L(X) := L(X, X)$. For $A \in L(X)$, let $ker(A)$ denote the kernel of A and $R(A)$ denote the range of A. We use $\sigma(A)$, $\sigma_{ap}(A)$, $\sigma_{sur}(A)$ and $\rho(A)$ to denote the spectrum, the approximate point spectrum, the surjectivity spectrum and the resolvent set of A, respectively.

The *local resolvent set* $\rho_A(x)$ of A at $x \in X$ is defined as the union of all open subsets U of $\mathbb{C}$ such that there exists an analytic function $f : U \to X$ which satisfies

$$(\lambda I - A)f(\lambda) = x \quad \text{for all } \lambda \in U.$$

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The *local spectrum* $\sigma_A(x)$ of A at x is the set defined by $\sigma_A(x) := \mathbb{C} \setminus \rho_A(x)$. Note that $\sigma_A(x)$ is a closed subset of $\sigma(A)$ and it may be empty. For every subset F of $\mathbb{C}$, the *local spectral subspace* of A associated with F is the set

$$X_A(F) := \{x \in X : \sigma_A(x) \subseteq F\}.$$

It is clear from the definition that $X_A(F)$ is a linear subspace of X and in general, $X_A(F)$ is not closed. Moreover, for every closed $F \subseteq \mathbb{C}$ we have

$$(\lambda I - A)X_A(F) = X_A(F) \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus F,$$

see, Proposition 1.2.16 [20]. Note that $X_{A+\lambda I}(F) = X_A(F - \lambda)$ for every subset $F \subseteq \mathbb{C}$ and all $\lambda \in \mathbb{C}$. For every closed subset $F \subseteq \mathbb{C}$, the *glocal spectral subspace* $\mathcal{X}_A(F)$ is defined as the set of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to X$ which satisfies

$$(\lambda I - A)f(\lambda) = x \quad \text{for each } \lambda \in \mathbb{C} \setminus F.$$

In general, $\mathcal{X}_A(F) \subseteq X_A(F)$ for every closed $F \subseteq \mathbb{C}$. Note that $X_A(F)$ as well as $\mathcal{X}_A(F)$, may not closed. But the two concepts of local spectral subspace and glocal spectral subspace coincide if A has SVEP, see, Proposition 3.3.2 [20].

An operatpr $A \in L(X)$ is said to have *Dunford's property (C)* (*property* (C) for brevity) if the local spectral subspace $X_A(F)$ is closed for every closed subset F of $\mathbb{C}$. We say that $A \in L(X)$ is said to have *property* (Q) if $\mathcal{X}_A\{\lambda\}$ is closed for every $\lambda \in \mathbb{C}$. Note that property (Q) is strictly weaker than property (C), see more details [1], [9], [15] and [20]. Recall that an operator $A \in L(X)$ is said to be *decomposable* if, for every open cover $\{U, V\}$ of $\mathbb{C}$, there exist $A-$invariant closed linear subspaces Y and Z of X for which

$$X = Y + Z, \quad \sigma(A|Y) \subseteq U \text{ and } \quad \sigma(A|Z) \subseteq V.$$

The class of decomposable operators contains all normal operators and more generally all spectral operators. Operators with totally disconnected spectrum are decomposable by the Riesz functional calculus. In particular, compact and algebraic operators are decomposable. An operator $A \in L(X)$ is said to have the *decomposition property* (δ) (abbreviated *property* (δ)) if, $X = \mathcal{X}_A(\overline{U}) + \mathcal{X}_A(\overline{V})$ for every open cover $\{U, V\}$ of $\mathbb{C}$. It is clear that every decomposable operator has property (δ).

Definition 1.1. An operator $A \in L(X)$ is said to have the *single-valued extension property* An operator $A \in L(X)$ is said to have the *single-valued extension property* at $\lambda_0 \in \mathbb{C}$ (SVEP at λ_0 for brevity), if for every open disc U centered at λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - A)f(\lambda) = 0$ for all $\lambda \in U$ is the constant function $f \equiv 0$. An operator $A \in L(X)$ is said to have the SVEP if A has the SVEP at every point $\lambda \in \mathbb{C}$.

In this case, $\sigma_A(x) = \phi$ if and only if $x = 0$, and we have $X_A(F) = \mathcal{X}_A(F)$ for every closed subset $F \subseteq \mathbb{C}$. Obviously, SVEP at a point is inherited by restrictions to closed invariant subspaces. It is clear that $A \in L(X)$ has SVEP at every point of the resolvent set $\rho(A)$. Moreover, from the identity theorem for analytic function it is easily seen that $A \in L(X)$ has SVEP at every point of the boundary $\partial\sigma(A)$ of the spectrum $\sigma(A)$. In particular, $A \in L(X)$ has SVEP at every isolated point of $\sigma(A)$. Also, it should be noted that, by Propsition 1.2.16 [20],

$$A \text{ has SVEP } \iff X_A(\phi) = \{0\} \iff X_A(\phi) \text{ is closed.}$$

It is well known that both SVEP and property (C) are preserved under the Riesz functional calculus, see [1] and [20].

Proposition 1.2. *Let $A \in L(X)$ and $\mu \in \mathbb{C}$. If $\sigma_{ap}(A)$ does not cluster at μ then A has the SVEP at μ.*

Proof. Suppose that $\sigma_{ap}(A)$ does not cluster at μ. Then there exists an open neighborhood U of μ such that $\lambda I - A$ is injective for every $\lambda \in U \setminus \{\mu\}$. Let $f : V \to X$ be an analytic function defined on an open neighborhood V of μ for which

$$(\lambda I - A)f(\lambda) = 0 \ \text{ for all } \lambda \in V.$$

Thus $(\lambda I - A)f(\lambda) = 0$ for all $\lambda \in U \cap V$, and hence $f(\lambda) \in ker(\lambda I - A)$ for all $\lambda \in (U \cap V) \setminus \{\mu\}$. It follows that $f(\lambda) = 0$ for all $\lambda \in (U \cap V) \setminus \{\mu\}$. By the continuity of f, we have $f(\lambda) = 0$ for all $\lambda \in U \cap V$. It follows from the identity theorem that $f(\lambda) = 0$ for all $\lambda \in V$. We conclude that $f \equiv 0$ in V. Hence A has the SVEP at μ. □

It is well known that $\sigma_{sur}(A) = \sigma_{ap}(A^*)$. Obviously, by Proposition 1.2, if $\sigma_{sur}(A)$ does not cluster at μ then A^* has the SVEP at μ.

Let $O(U, X)$ denote the Fréchet algebra of all $X-$valued analytic functions on the open subset $U \subseteq \mathbb{C}$ endowed with uniform convergence on compact subsets of U. An operator $A \in L(X)$ is said to have *Bishop's property* (β) (abbreviated *property* (β)) if for every open subset U of $\mathbb{C}$ and any sequence $\{f_n\}_{n=1}^{\infty} \subseteq O(U, X)$, $\lim_{n\to\infty}(\lambda I - A)f_n(\lambda) = 0$ in $O(U, X)$ implies $\lim_{n\to\infty} f_n(\lambda) = 0$ in $O(U, X)$.

In [6], Albrecht and Eschmeier proved that $A \in L(X)$ has property (β) if and only if its adjoint $A^* \in L(X^*)$ on the topological dual space X^* has property (δ), and the same equivalence holds when the roles of (β) and (δ) are interchanged. As observed in [7], an operator is decomposable if and only if it has both properties (β) and (δ). It is well known that

$$\text{property } (\beta) \ \Rightarrow \ \text{property } (C) \ \Rightarrow \ \text{property } (Q) \Rightarrow \text{SVEP}.$$

In general, the converse implications do not hold, see [1], [7], [9] and [20].

Proposition 1.3. *If $A \in L(X)$ and $N \in L(X)$ is a nilpotent operator commuting with A then A has SVEP if and only if $A + N$ has SVEP. Moreover, A has property (β) if and only if $A + N$ has property (β).*

Proof. Let $N^p = 0$ for some $p \in \mathbb{N}$. Suppose that A has SVEP at λ_0. To establish SVEP for $A+N$, it suffices to show that for every open $U \subseteq \mathbb{C}$ and every analytic function $f : U \to X$ for which $(\lambda I - (A + N))f(\lambda) = 0$ for all $\lambda \in U$, it follows that $f \equiv 0$ on U. Let $\lambda_0 \in \mathbb{C}$, and let $f : U \to X$ be an analytic function on an open neighborhood U of λ_0 such that

$$(\lambda I - (A+N))f(\lambda) = 0 \quad \text{for all} \quad \lambda \in U.$$

Thus $(\lambda I - A)f(\lambda) = Nf(\lambda)$ for all $\lambda \in U$, and hence

$$(\lambda I - A)N^{p-1}f(\lambda) = N^{p-1}(\lambda I - A)f(\lambda) = N^p f(\lambda) = 0 \quad \text{for all} \quad \lambda \in U.$$

It is clear that $N^{p-1}f(\lambda)$ is analytic. It follows from the definition of SVEP that $N^{p-1}f(\lambda) = 0$ for all $\lambda \in U$. Also, we obtain

$$(\lambda I - A)N^{p-2}f(\lambda) = N^{p-2}(\lambda I - A)f(\lambda) = N^{p-1}f(\lambda) = 0 \quad \text{for all} \quad \lambda \in U.$$

Since A has SVEP, we have $N^{p-2}f(\lambda) = 0$ for all $\lambda \in U$. Because $A^p = 0$, by induction, we can show that

$$f(\lambda) = 0 \quad \text{for all } \lambda \in U,$$

and hence $f \equiv 0$. It follows that $A + N$ has SVEP. The converse implication is similar.

Finally, suppose that $A + N$ has property (β). To establish property (β) for A, it suffices to show that if for every open subset U of $\mathbb{C}$ and any sequence $\{f_n\}_{n=1}^{\infty} \subseteq O(U,X)$, $\lim_{n\to\infty}(\lambda I - A)f_n(\lambda) = 0$ in $O(U,X)$ implies $\lim_{n\to\infty} f_n(\lambda) = 0$ in $O(U,X)$. Let $\{f_n\}_{n=1}^{\infty} \subseteq O(U,X)$ such that $\lim_{n\to\infty}(\lambda I - A)f_n(\lambda) = 0$ in $O(U,X)$. Then

$$\lim_{n\to\infty}[(\lambda I - (A+N))f_n(\lambda) + Nf_n(\lambda)] = \lim_{n\to\infty}(\lambda I - A)f_n(\lambda) = 0$$

in $O(U,X)$. Thus we have

$$\lim_{n\to\infty} N^{p-1}((\lambda I - (A+N))f_n(\lambda) + Nf_n(\lambda)) = \lim_{n\to\infty}(\lambda I - (A+N))N^{p-1}f_n(\lambda) = 0$$

in $O(U,X)$. Since $A+N$ has property (β),

$$\lim_{n\to\infty} N^{p-1}f_n(\lambda) = 0$$

in $O(U,X)$. Clearly, we have

$$\lim_{n\to\infty}(\lambda I - (A+N))N^{p-2}f_n(\lambda) = \lim_{n\to\infty} N^{p-2}((\lambda I - (A+N))f_n(\lambda) + Nf_n(\lambda)) = 0$$

in $O(U,X)$. Since $A+N$ has property (β),

$$\lim_{n\to\infty} N^{p-2}f_n(\lambda) = 0$$

in $O(U,X)$. By induction, $\lim_{n\to\infty} f_n(\lambda) = 0$ in $O(U,X)$. Hence A has property (β). The converse implication is similar. □

It is well known that (β) and (δ) are completely dual, and $A \in L(X)$ is decomposable if and only if A has both properties (β) and (δ). We have the following.

Corollary 1.4. *Let $A \in L(X)$ and $N \in L(X)$ be a nilpotent operator commuting with A. Then A has property (δ) if and only if $A+N$ has property (δ). Moreover, A is decomposable if and only if $A+N$ is decomposable.*

Definition 1.5. Let X be a Banach space and $A \in L(X)$ be a bounded operator. Then the localizable spectrum $\sigma_{loc}(A)$ of A will be defined as the set of all $\lambda \in \mathbb{C}$ such that $X_A(\overline{V}) \neq \{0\}$ for each open neighborhood V of λ.

It is well known that $\sigma_{loc}(A)$ is a closed subset of $\sigma(A)$ and that $\sigma_{loc}(A)$ contains the point spectrum and is included in the aproximate point spectrum of A, see [21]. As shown by Eschmeier and Prunaru [17], the localizable spectrum plays an important role in the theory of invariant subspaces; see also [8] and [21].

2. Main results

We say that an operator $R \in L(X)$ is called *quasinilpotent* if $\|R^n\|^{\frac{1}{n}} \to 0$ as $n \to \infty$, equivalently, if $\sigma(R) = \{0\}$. It is clear that $R \in L(X)$ is quasinilpotent if and only if $\sigma_R(x) = \{0\}$ for every $x \in X \setminus \{0\}$.

Theorem 2.1. *Let $A \in L(X)$ and $\lambda \in \mathbb{C}$. Then A has SVEP at λ if and only if $X_A(\phi) \cap ker(\lambda I - A) = \{0\}$. Moreover, A has SVEP if and only if $X_A(\phi) = \{0\}$.*

Proof. Theorem 2.22 and Corollary 2.41 of [1]. ☐

It is clear that if $\mathcal{X}_A(\{\lambda\}) \cap X_A(\phi) = \{0\}$ then we have

$$ker(\lambda I - A) \cap X_A(\phi) \subseteq X_A(\{\lambda\}) \cap X_A(\phi) = \{0\},$$

and hence, by Theorem 2.1, A has SVEP at λ.

Theorem 2.2. *Let $A \in L(X)$ and $R \in L(X)$ be a quasinilpotent operator commuting with A. Then $\sigma_A(x) = \sigma_{A+R}(x)$ for all $x \in X$. Moreover, $X_A(F) = X_{A+R}(F)$ for every subset $F \subseteq \mathbb{C}$. Furthermore, $\mathcal{X}_A(G) = \mathcal{X}_{A+R}(G)$ for every closed subset $G \subseteq \mathbb{C}$.*

Proof. Let $\lambda_0 \notin \sigma_A(x)$. Then there exists an open neighborhood U of λ_0 and an analytic function $f : U \to X$ such that $(\lambda I - A)f(\lambda) = x$ for all $\lambda \in U$. Let $0 < a < b$, and let $W := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq a\}$ and let $V := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq b\}$ with $W \subseteq V \subseteq U$. Because f is analytic on V, there exists a real number $m > 0$ such that

$$\|f(\lambda)\| \leq m \quad \text{for all} \quad \lambda \in V.$$

Let $\epsilon := (b - a)/2$. Since R is quasinilpotent, $\|Q^n\|^{\frac{1}{n}} \to 0$ as $n \to \infty$. Therefore there exists $c > 0$ such that

$$\|R^n\| \leq c\epsilon^n \quad \text{for all} \quad n \in \mathbb{N}.$$

Since f is analytic on W, it follows from Cauchy's integral formula that for each $\lambda \in W$,

$$f^{(n)}(\lambda) = \frac{n!}{2\pi i} \int_{\partial V} \frac{f(\mu)}{(\mu - \lambda)^{n+1}} d\mu \quad \text{for all } n \geq 0.$$

Thus we obtain

$$\left\| \frac{f^{(n)}(\lambda)}{n!} \right\| \leq \frac{mb}{(b - a)^{n+1}} \text{ for all } \quad n \geq 0,$$

and hence

$$\left\| R^n \frac{f^{(n)}(\lambda)}{n!} \right\| \leq \frac{mbc}{2^n(b-a)} \quad \text{for all} \quad \lambda \in W \text{ and } n \geq 0.$$

It follows that for all $\lambda \in W$,

$$\left\| R^n \frac{f^{(n)}(\lambda)}{n!} \right\| \to 0 \quad \text{as } n \to \infty.$$

We define $g : U \to X$ by

$$g(\lambda) := \sum_{n=0}^{\infty} R^n \frac{f^{(n)}(\lambda)}{n!} \quad \text{for all} \quad \lambda \in U.$$

Then clearly, $g(\lambda)$ converges uniformly on W and locally uniformly on U. It follows that g is analytic. Since $(\lambda I - A)f(\lambda) = x$ for all $\lambda \in U$, we obtain by induction that $(\lambda I - A)f^{(n)}(\lambda) = nf^{(n-1)}(\lambda)$ for all $\lambda \in U$ and $n \in \mathbb{N}$. Finally, we claim that $(\lambda I - A - R)g(\lambda) = x$ for all $\lambda \in U$. Since $AR = RA$ and $(\lambda I - A)f^{(n)}(\lambda) = nf^{(n-1)}(\lambda)$, we have for each $\lambda \in U$,

$$\begin{aligned}\sum_{n=1}^{\infty}(\lambda I - A - R)R^n \frac{f^{(n)}(\lambda)}{n!} &= \sum_{n=1}^{\infty}(R^n(\lambda I - A) - R^{n+1})\frac{f^{(n)}(\lambda)}{n!} \\ &= \sum_{n=1}^{\infty} R^n \frac{f^{(n-1)}(\lambda)}{(n-1)!} - \sum_{n=1}^{\infty} R^{n+1}\frac{f^{(n)}(\lambda)}{n!} \\ &= Rf(\lambda).\end{aligned}$$

It follows that

$$\begin{aligned}(\lambda I - A - R)g(\lambda) &= \sum_{n=0}^{\infty}(\lambda I - A - R)R^n \frac{f^{(n)}(\lambda)}{n!} \\ &= (\lambda I - A - R)f(\lambda) + \sum_{n=1}^{\infty}(\lambda I - A - R)R^n \frac{f^{(n)}(\lambda)}{n!} \\ &= (\lambda I - A - R)f(\lambda) + Rf(\lambda) = (\lambda I - A)f(\lambda) = x.\end{aligned}$$

This implies that $\lambda_0 \notin \sigma_{A+R}(x)$, and hence $\sigma_{A+R}(x) \subseteq \sigma_A(x)$. The inclusion $\sigma_A(x) \subseteq \sigma_{A+R}(x)$ is clear, if just interchanging A and $A+R$ in the argument above, then $\sigma_A(x) \subseteq \sigma_{A+R}(x)$ for all $x \in X$ and hence $\sigma_A(x) = \sigma_{A+R}(x)$ for all $x \in X$, as desired. It follows from $\sigma_A(x) = \sigma_{A+R}(x)$ that $X_A(F) = X_{A+R}(F)$ for all subset $F \subseteq \mathbb{C}$. Also, it is easily seen that $\mathcal{X}_A(F) = \mathcal{X}_{A+R}(F)$ for all closed subset $F \subseteq \mathbb{C}$. □

It is clear that every nilpotent operator is a quasinilpotent operator. As an immediate application of Theorem 2.2, we obtain the following corollary.

Corollary 2.3. *Let $A \in L(X)$ and let $N \in L(X)$ be nilpotent operator commuting with A. Then $\sigma_A(x) = \sigma_{A+N}(x)$ for all $x \in X$. Moreover, $X_A(F) = X_{A+N}(F)$ for every subset $F \subseteq \mathbb{C}$ and $\mathcal{X}_A(G) = \mathcal{X}_{A+R}(G)$ for every closed subset $G \subseteq \mathbb{C}$.*

Recall that $A \in L(X)$ is said to be *bounded below* if A is inective and has closed range. Denote by $\sigma_{ap}(A)$ the classical *approximate point spectrum* of $A \in L(X)$ defined by

$$\sigma_{ap}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not bounded below }\}.$$

It is well known that $\sigma_{ap}(A)$ is a compact subset of $\mathbb{C}$ that contains the boundary of $\sigma(A)$. Note that if $\sigma_{sur}(A)$ denotes the *surjectivity spectrum*

$$\sigma_{sur}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not onto}\}.$$

Clearly, $\sigma_{sur}(A)$ is a compact subset of $\sigma(A)$ such that $\partial\sigma(A) \subseteq \sigma_{sur}(A)$. Obviously, $\sigma(A) = \sigma_{sur}(A) \cup \sigma_{ap}(A)$. Furthermore, $\sigma_{sur}(A^*) = \sigma_{ap}(A)$ and $\sigma_{ap}(A^*) = \sigma_{sur}(A)$, where A^* is the dual of A.

Corollary 2.4. *If $A \in L(X)$ and $R \in L(X)$ is a quasinilpotent operator commuting with A then we have the following.*
(a) $\sigma_{loc}(A) = \sigma_{loc}(A + R)$,
(b) $\sigma_{sur}(A) = \sigma_{sur}(A + R)$,
(c) $\sigma_{ap}(A) = \sigma_{ap}(A + R)$,
(d) $\sigma(A) = \sigma(A + R)$.

Proof. (a) Suppose that $\lambda \in \sigma_{loc}(A)$. Then $X_A(\overline{V}) \neq \{0\}$ for each open neighborhood V of λ. By Theorem 2.2, $X_{A+R}(\overline{V}) = X_A(\overline{V}) \neq \{0\}$, and hence $\lambda \in \sigma_{loc}(A + R)$. The converse implication is similar.

(b) It follows from Theorem 1.9 of [14] that

$$\sigma_{sur}(A) = \bigcup_{x\in X} \sigma_A(x) = \bigcup_{x\in X} \sigma_{A+R}(x) = \sigma_{sur}(A + R).$$

(c) $\sigma_{ap}(A) = \sigma_{sur}(A^*) = \sigma_{sur}((A + R)^*) = \sigma_{ap}(A + R)$.

(d) $\sigma(A) = \sigma_{sur}(A) \cup \sigma_{ap}(A) = \sigma_{sur}(A + R) \cup \sigma_{ap}(A + R) = \sigma(A + R)$. □

Corollary 2.5. *Let $A, B \in L(X)$ be decomposable operators with $AB = BA$ and let $R = A - B$. Then R is a quasinilpotent operator if and only if $\sigma_A(x) = \sigma_B(x)$ for all $x \in X$.*

Proof. Suppose that R is a quasinilpotent operator. Then, by Theorem 2.2, we have

$$\sigma_A(x) = \sigma_{B+R}(x) = \sigma_B(x) \text{ for all } x \in X.$$

Conversely, if $\sigma_A(x) = \sigma_B(x)$ for all $x \in X$ then $X_A(F) = X_B(F)$ for all $F \subseteq \mathbb{C}$. It follows from Theorem 3.2 of [18] that R is a quasinilpotent operator. □

By Theorem 2.1 and Theorem 2.2, we have the following.

Corollary 2.6. *Let $A \in L(X)$ and $R \in L(X)$ be a quasinilpotent operator commuting with A and let $\lambda \in \mathbb{C}$. Then A has SVEP at λ if and only if $A+R$ has SVEP at λ. In particular, the SVEP is stable under quasinilpotent commuting perturbations.*

Note that, by Theorem 2.2, $X_A(F) = X_{A+R}(F)$ for all closed subset $F \subseteq \mathbb{C}$. We have the following.

Corollary 2.7. *Let $A \in L(X)$ and $R \in L(X)$ be a quasinilpotent operator commuting with A. Then A has property (C) if and only if $A+R$ has property (C). In particular, A has property (Q) if and only if $A+R$ has property (Q).*

In [6], Albrecht and Eschmeier proved that the properties (β) and (δ) are completely dual: an operator has one if and only if its adjoint has the other. It has been observed in [7] that an operator $A \in L(X)$ is decomposable if and only if it has both properties (β) and (δ). Note that $R \in L(X)$ is quasinilpotent if and only if R^* is quasinilpotent. We have the following.

Corollary 2.8. *Let $A \in L(X)$ and $R \in L(X)$ be a quasinilpotent operator commuting with A. Then we have the following assertions.*
(a) A has property (δ) if and only if $A+R$ has property (δ).
(b) A has property (β) if and only if $A+R$ has property (β).
(c) A is decomposable if and only if $A+R$ is decomposable. In particular, A has property (Q) if and only if $A+R$ has property (Q).

Proof. (a) Suppose that A has property (δ). Let $\{U, V\}$ be an open neighborhood of $\mathbb{C}$. Then we have $X = \mathcal{X}_A(\overline{U}) + \mathcal{X}_A(\overline{V})$. By Theorem 2.2, we have $X = \mathcal{X}_{A+R}(\overline{U}) + \mathcal{X}_{A+R}(\overline{V})$, and therefore $A+R$ has property (δ). The converse implication follows by interchanging A and $A+R$.

$(b), (c)$ Noting that (β) and (δ) are dual to each other, and that $A \in L(X)$ is decomposable if and only if A satisfies both (β) and (δ). □

Lemma 2.9. *Let $A \in L(X)$ and $B \in L(Y)$ on complex Banach spaces X and Y and let $T \in L(X,Y)$ and $S \in L(Y,X)$ such that $TA = BT$ and $AS = SB$. Suppose that T and S are injective. Then for each closed subset $F \subseteq \mathbb{C}$, $\mathcal{X}_A(F) = \{0\}$ if and only if $\mathcal{Y}_B(F) = \{0\}$.*

Proof. Let $F \subseteq \mathbb{C}$ be closed. If $y = Tx$ for some $x \in \mathcal{X}_A(F)$, then there exists some analytic function $f : \mathbb{C} \setminus F \to X$ with

$$(\lambda I - A)f(\lambda) = x \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus F.$$

Clearly, $Tf(\lambda)$ is analytic on $\mathbb{C} \setminus F$. Because of $TA = BT$, we obtain

$$(\lambda I - B)Tf(\lambda) = T(\lambda I - A)f(\lambda) = Tx = y \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus F,$$

and hence $y \in \mathcal{Y}_B(F)$. We conclude that $T\mathcal{X}_A(F) \subseteq \mathcal{Y}_S(F)$ for all closed subset $F \subseteq \mathbb{C}$. Also, it is easily seen that $S\mathcal{Y}_B(F) \subseteq \mathcal{X}_A(F)$ for all closed subset $F \subseteq \mathbb{C}$. By the injectivity of T and S, we conclude that $\mathcal{X}_A(F) = \{0\}$ if and only if $\mathcal{Y}_B(F) = \{0\}$ for all closed $F \subseteq \mathbb{C}$. □

We say that $A \in L(X)$ and $B \in L(Y)$ on complex Banach spaces X and Y are *quasisimilar* if there exist $T \in L(X,Y)$ and $S \in L(Y,X)$, each injective and with dense range such that $TA = BT$ and $AS = SB$. As an immediate application of Lemma 2.9, we obtain the following theorem.

Theorem 2.10. *If $A \in L(X)$ and $B \in L(Y)$ on complex Banach spaces X and Y are quasisimilar then $\sigma_{loc}(A) = \sigma_{loc}(B)$.*

Let $A \in L(X)$ and $B \in L(Y)$ be quasisimilar operators on complex Banach spaces X and Y. Then clearly, for each integer $n \geq 1$, A^n and B^n are quasisimilar operators. We set $A_1 := A|\overline{R(A^n)}$ and $B_1 := B|\overline{R(B^n)}$. It follows from Lemma 5.2 of [10] that A_1 and B_1 are quasisimilar operators.

Corollary 2.11. *Let $A \in L(X)$ and $B \in L(Y)$ be quasisimilar operators, and let $A_1 := A|\overline{R(A^n)}$ and $B_1 := B|\overline{R(B^n)}$. Then for each integer $n \in \mathbb{N}$, $\sigma_{loc}(A^n) = \sigma_{loc}(B^n)$ and $\sigma_{loc}(A_1) = \sigma_{loc}(B_1)$.*

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