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INVARIANT SUBMANIFOLDS OF $N(k)$-CONTACT METRIC MANIFOLDS WITH GENERALIZED TANAKA WEBSTER CONNECTION

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Abstract. The object of the present paper is to study some geometric properties of invariant submanifolds of $N(k)$-contact metric manifold admitting generalized Tanaka-Webster connection.

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1. Introduction

The Tanaka-Webster connection is canonical affine connection defined on a non-degenerate pseudo-Hermition CR-manifold [7, 10]. The generalized Tanaka-Webster connection for contact metric manifolds by the canonical connection was first studied by Tanno [9]. This connection coincides with the Tanaka-Webster connection if the associated CR-structure is integrable. For a real hypersurface in a Kahler manifold with almost contact structure (ϕ, ξ, η, g), Cho [4, 5] adapted Tanno's generalized Tanaka-Webster connection for a non-zero real number k. Using the generalized Tanaka-Webster connection, Some geometers have studied characterizations of real hypersurfaces in complex space forms [6].
In the present paper we have calculated the curvature tensor of the ambient manifold admitting generalized Tanaka-Webster connection and got the relation between curvature tensors of an $N(k)$-contact metric manifold M and an invariant submanifold of M. And also studied geometric properties of invariant submanifolds satisfying curvature conditions with respect to curvature tensor W_2.

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2. Preliminaries

Let M be an almost contact metric manifold with the structure tensors (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ a vector field, η a 1-form and g is a Riemannian metric on M [2]. Then

$$\begin{aligned} &\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \ \ \eta \cdot \phi = 0, \\ &g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \end{aligned} \tag{1}$$

for any $X, Y \in \Gamma(TM)$.
Let Φ denote the 2-form in M given by $\Phi(X,Y) = g(X, \phi Y)$. The k-nullity distribution on a contact metric manifold M for a real number k is a distribution [8]

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\} \tag{2}$$

for any $X, Y \in T_pM$, where R denotes the Riemannian curvature tensor of M and T_pM denotes the tangent vector space of M at any point $p \in M$.
If the characteristic vector field ξ of a contact metric manifold belongs to the k-nullity distribution, then

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y). \tag{3}$$

A contact metric manifold with $\xi \in N(k)$ is called a $N(k)$-contact metric manifold. In an $N(k)$ contact metric manifold the following relations hold:

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{4}$$

$$h\xi = 0, \tag{5}$$

$$h^2 = (k-1)\phi^2, \tag{6}$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{7}$$

$$(\nabla_X \eta)Y = g(X + hX, \phi Y), \tag{8}$$

$$\begin{aligned} (\nabla_Y h)X - (\nabla_X h)Y = {} & 2(\kappa - 1)g(Y, \phi X)\xi + (1-\kappa)[\eta(Y)\phi X \\ & - \eta(X)\phi Y] + \eta(Y)\phi h X - \eta(X)\phi h Y \end{aligned} \tag{9}$$

for all $X, Y \in \Gamma(TM)$, where h is a symmetric tensor and ∇ is the Levi-Civita connection on the manifold M.

The generalized Tanaka-Webster connection $\nabla^\star$ for a contact manifold M is defined by

$$\nabla^\star_X Y = \nabla_X Y + (\nabla_X \eta)Y \ \xi - \eta(Y)\nabla_X \xi - \eta(X)\phi, \tag{10}$$

where $\nabla^\star$ and ∇ denote the generalized Tanaka-Webster connection and Levi-Civita connection respectively on the ambient manifold M.

Let N be an $(2n+1)$- dimensional immersed submanifold of M. Then the Gauss and Weingarten formulas are respectively given by

$$\nabla_X Y = \tilde{\nabla}_X Y + \sigma(X,Y) \tag{11}$$

and

$$\nabla_X V = -A_V X + \tilde{\nabla}^\perp_X V \tag{12}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$, where σ, $\tilde{\nabla}$, $\tilde{\nabla}^{\perp}$ and A denote respectively the second fundamental form, Levi-civita connection, the normal connection and the shape operator on the submanifold N. The second fundamental form and shape operator are related by

$$g(A_V X, Y) = g(\sigma(X, Y), V), \tag{13}$$

where g denotes the induced metric on N as well as the Riemannian metric g on M.
The covariant derivative of σ is given by

$$(\nabla_X \sigma)(Y, Z) = \tilde{\nabla}^{\perp}_X \sigma(Y, Z) - \sigma(\tilde{\nabla}_X Y, Z) - \sigma(Y, \tilde{\nabla}_X Z), \tag{14}$$

for any $X, Y, Z \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$.
Let $R_N(X, Y)Z$ and $R(X, Y)Z$ denote the Riemannian curvature tensors of the submanifold N and the ambient manifold M respectively. Then we have

$$\begin{aligned} R(X, Y)Z = R_N(X, Y)Z + (\nabla_X \sigma)(Y, Z) - (\nabla_Y \sigma)(X, Z) + A_{\sigma(X,Z)} Y \\ - A_{\sigma(Y,Z)} X \end{aligned} \tag{15}$$

for $X, Y, Z \in \Gamma(TN)$ [3].
A submanifold N of an almost contact metric manifold is said to be invariant [1] if the structure vector field ξ is tangent to N at every point of N and ϕX is tangent to N for any vector field X tangent to N at every point of N, that is, if $X \in \Gamma(TN)$ then $\phi X \in \Gamma(TN)$ at every point of N.

3. Invariant Submanifold of N(k)-contact metric manifold admitting generalized Tanaka-Webster connection

Let M be an $N(k)$-contact metric manifold admitting generalized Tanaka Webster connection and N be an invariant submanifold of M.
Now by taking $Y = \xi$ in (10) and using (1), (7) and (8), we get

$$\nabla^{\star}_X \xi = 0. \tag{16}$$

Next we calculate

$$\nabla^{\star}_X \phi Y = \nabla_X \phi Y - g(X + hX, Y)\xi + \eta(X)Y \tag{17}$$

and

$$\nabla^{\star}_X \phi hY = \nabla_X \phi hY - g(X + hX, hY)\xi + \eta(X)hY. \tag{18}$$

Using (2) and (8) in (10), we get

$$\nabla^{\star}_Y Z = \nabla_Y Z + g(Y + hY, \phi Z)\xi + \{\eta(Z)\phi Y - \eta(Y)\phi Z\} + \eta(Z)\phi hY. \tag{19}$$

Taking covariant derivative $\nabla^{\star}$ with respect to X of (19), we obtain

$$\begin{aligned} \nabla^{\star}_X \nabla^{\star}_Y Z = {} & \nabla_X \nabla_Y Z + (\nabla_X \eta)\nabla_Y Z \cdot \xi - \eta(\nabla_Y Z)\nabla_X \xi - \eta(X)\phi \nabla_Y Z \\ & + Xg(Y + hY, \phi Z)\xi + g(Y + hY, \phi Z)\nabla^{\star}_X \xi + X\eta(Z)\phi Y \\ & + \eta(Z)\nabla^{\star}_X \phi Y - X\eta(Y)\phi Z - \eta(Y)\nabla^{\star}_X \phi Z - X\eta(Z)\phi hY \\ & + \eta(Z)\nabla^{\star}_X \phi hY. \end{aligned} \tag{20}$$

Using (16), (17) and (18) in (20), to get

$$\begin{aligned}\nabla^\star_X\nabla^\star_Y Z &= \nabla_X\nabla_Y Z + g(X+hX,\nabla_Y\phi Z)\xi - g(Y+hY,Z)\eta(X)\xi\\ &+\eta(Z)g(X+hX,Y+hY)\xi - \eta(\nabla_Y Z)\nabla_X\xi - \eta(X)\phi\nabla_Y Z\\ &+Xg(Y+hY,\phi Z)\xi + X\eta(Z)\phi Y + \eta(Z)\nabla_X\phi Y\\ &-\eta(Z)g(X,Y)\xi - 2\eta(Z)g(hX,Y)\xi + \eta(X)\eta(Z)Y\\ &-X\eta(Y)\phi Z - \eta(Y)\nabla_X\phi Z + \eta(Y)g(X+hX,Z)\xi\\ &-\eta(X)\eta(Y)Z - X\eta(Z)\phi hY + \eta(Z)\nabla_X\phi hY\\ &-\eta(Z)g(hX,hY)\xi + \eta(X)\eta(Z)hY.\end{aligned} \tag{21}$$

Similarly one can find

$$\begin{aligned}\nabla^\star_Y\nabla^\star_X Z &= \nabla_Y\nabla_X Z + g(Y+hY,\nabla_X\phi Z)\xi - g(X+hX,Z)\eta(Y)\xi\\ &+\eta(Z)g(Y+hY,X+hX)\xi - \eta(\nabla_X Z)\nabla_Y\xi - \eta(Y)\phi\nabla_X Z\\ &+Yg(X+hX,\phi Z)\xi + Y\eta(Z)\phi X + \eta(Z)\nabla_Y\phi X\\ &-\eta(Z)g(Y,X)\xi - 2\eta(Z)g(hY,X)\xi + \eta(Y)\eta(Z)X\\ &-Y\eta(X)\phi Z - \eta(X)\nabla_Y\phi Z + \eta(X)g(Y+hY,Z)\xi\\ &-\eta(Y)\eta(X)Z - Y\eta(Z)\phi hX + \eta(Z)\nabla_Y\phi hX\\ &-\eta(Z)g(hY,hX)\xi + \eta(Y)\eta(Z)hX.\end{aligned} \tag{22}$$

With a simple calculation we have

$$\begin{aligned}\nabla^\star_{[X,Y]}Z &= \nabla_{[X,Y]}Z + g(\nabla_X Y - \nabla_Y X,\phi Z)\xi + g(h\nabla_X Y\\ &-h\nabla_Y X,\phi Z)\xi + \eta(Z)(\phi\nabla_X Y - \phi\nabla_Y X + \phi h\nabla_X Y\\ &-\phi h\nabla_Y X) - \eta(\nabla_X Y - \nabla_Y X)\phi Z.\end{aligned} \tag{23}$$

Using (21), (22) and (23), to get

$$\begin{aligned}R^\star(X,Y)Z &= R(X,Y)Z + k[-g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi\\ &+\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX - \eta(Y)\eta(Z)X\\ &+\eta(X)\eta(Z)Y] + 2Y\eta(Z)\phi hX - 2X\eta(Z)\phi hY\\ &+2g(\phi X,Y)\phi Z + g(Z,\phi Y+\phi hY)(\phi X+\phi hX)\\ &-g(Z,\phi X+\phi hX)(\phi Y+\phi hY).\end{aligned} \tag{24}$$

Taking $Y=\xi$ in equation (11), we have

$$\nabla_X\xi = \tilde{\nabla}_X\xi + \sigma(X,\xi).$$

Using (7) in the above equation and equating the tangential and normal components we get

$$\tilde{\nabla}_X\xi = -\phi X - \phi hX \tag{25}$$

and

$$\sigma(X,\xi) = 0. \tag{26}$$

From (15) and (24), taking $Z = \xi$ and using (3) and (26), we obtain

$$R_N(X,Y)\xi = k\{\eta(Y)X - \eta(X)Y\}. \tag{27}$$

From the above equation (27), we obtain

$$S_N(X,\xi) = 2nk\eta(X). \tag{28}$$

Also a simple calculation gives

$$R_N(\xi,Y)Z = A_{\sigma(Y,Z)}\xi - (\nabla_\xi\sigma)(Y,Z) + k\{g(Y,Z)\xi - \eta(Z)Y\}. \tag{29}$$

Definition 3.1. The curvature tensor W_2 is defined by

$$W_2(X,Y,U,V) = R(X,Y,U,V) + \frac{1}{2n}[g(X,U)S(Y,V) - g(Y,U)S(X,V)]. \tag{30}$$

Theorem 3.2. *Let N be an invariant submanifold of N(k)-contact metric manifold M admitting generalized Tanaka Webster connection. If the curvature tensor W_2 vanishes then N becomes an Einstein manifold.*

Proof. Suppose the submanifold N of N(k)-contact metric manifold M with generalized Tanaka connection satisfies $W_2 = 0$, then we have

$$R_N(X,Y,U,V) = \frac{1}{2n}[g(Y,U)S_N(X,V) - g(X,U)S_N(Y,V)].$$

Putting $X = U = \xi$ in the above equation and using (27) and (28), we get

$$S_N(Y,V) = -2nkg(Y,V) \tag{31}$$

□

Corollary 3.3. *Let N be an invariant submanifold of N(k)−contact metric manifold M admitting generalized Tanaka Webster connection. If W_2 curvature tensor vanishes then the submanifold N is an Einstein manifold. Further for $k = 1$, the submanifold N is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$.*

Proof. Let the submanifold N satisfies $W_2 = 0$ and N be an Einstein manifold. Then from (30) and (31), we get

$$R_N(X,Y,U,V) = k[g(X,U)g(Y,V) - g(Y,U)g(X,V)]. \tag{32}$$

□

Definition 3.4. A contact metric manifold is called W_2-semisymmetric if it satisfies $R(X,Y)\cdot W_2 = 0$, where $R(X,Y)$ is to be considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y.

Proposition 3.5. *Let N be an invariant submanifold of N(k)-contact metric manifold M admitting generalized Tanaka Webster connection. Then the curvature tensor W_2 on N satisfies the condition $W_2(X,Y,U,\xi) = 0$.*

Proof. Taking $V = \xi$ in (30), we get

$$W_2(X, Y, U, \xi) = R_N(X, Y, U, \xi) + \frac{1}{2n}\{g(X, U)S_N(Y, \xi) - g(Y, U)S_N(X, \xi).$$

The above equation can be rewritten as

$$W_2(X, Y, U, \xi) = -R_N(X, Y, \xi, U) + \frac{1}{2n}\{g(X, U)S_N(Y, \xi) - g(Y, U)S_N(X, \xi)\}. \tag{33}$$

Using (27) and (28) in (33), we obtain

$$W_2(X, Y, U, \xi) = 0.$$

Hence the proof. □

Theorem 3.6. *A W_2-semisymmetric invariant submanifold N of N(k)-contact metric manifold M admitting generalized Tanaka Webster connection is an Einstein manifold.*

Proof. Let the submanifold N be W_2-semisymmetric, i.e.

$$(R(X, Y) \cdot W_2)(Z, U, V) = 0.$$

The above equation can be written as

$$\begin{aligned} &R_N(X, Y)W_2(Z, U)V - W_2(R_N(X, Y)Z, U)V \\ &- W_2(Z, R_N(X, Y)U)V - W_2(Z, U)R_N(X, Y)V = 0. \end{aligned} \tag{34}$$

Taking $X = \xi$ in (34) and using (29), we get

$$\begin{aligned} &A_{\sigma(Y, W_2(Z,U)V)}\xi - (\nabla_\xi \sigma)(Y, W_2(Z, U)V) + k\{g(Y, W_2(Z, U)V)\xi \\ &- \eta(W_2(Z, U)V)Y\} - W_2(A_{\sigma(Y,Z)}\xi - (\nabla_\xi \sigma)(Y, Z) + k\{g(Y, Z)\xi - \\ &\eta(Z)Y\}, U)V - W_2(Z, A_{\sigma(Y,U)}\xi - (\nabla_\xi \sigma)(Y, U) + k\{g(Y, U)\xi - \\ &\eta(U)Y\})V - W_2(Z, U)(A_{\sigma(Y,V)}\xi - (\nabla_\xi \sigma)(Y, V) + k\{g(Y, V)\xi \\ &- \eta(V)Y\}) = 0. \end{aligned}$$

Taking inner product with ξ in the above equation and using the property of invariant submanifold and Proposition 3.1, we obtain

$$k\ W_2(Z, U, V, Y) = 0. \tag{35}$$

From (30) and (35), we write

$$R_N(Z, U, V, Y) = \frac{1}{2n}\{g(U, V)S_N(Y, Z) - g(Z, V)S_N(U, Y)\}. \tag{36}$$

Taking $U = V = \xi$ in (36) and using (27), we get

$$S_N(Y, Z) = 2ng(Y, Z). \tag{37}$$

Hence the proof. □

Corollary 3.7. *A W_2-semisymmetric invariant submanifold N of N(k)-contact metric manifold M admitting generalized Tanaka Webster connection is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$.*

Proof. Substituting (37) in (36), we get

$$R_N(Z,U,V,Y) = g(U,V)g(Y,Z) - g(Z,V)g(U,Y).$$

Hence the submanifold N is of constant curvature -1. Then N is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$. □

Definition 3.8. The projective curvature tensor P and the concircular curvature tensor $\tilde{Z}$ in a Riemannian manifold (M^n, g) are defined by

$$P(X,Y)W = R(X,Y)W - \frac{1}{n-1}(X \wedge_S Y)W, \tag{38}$$

$$\tilde{Z}(X,Y)W = R(X,Y)W - \frac{r}{n(n-1)}(X \wedge_g Y)W, \tag{39}$$

respectively.

Theorem 3.9. *Let N be an invariant submanifold of N(k)-contact metric manifold M admitting generalized Tanaka Webster connection. If N satisfies the condition $\tilde{Z}(X,Y)\cdot W_2 = 0$ then either $r = 2n(2n+1)k$ or N is Einstein and hence is locally to the Riemannian product $E^{n+1}(0) \times S^n(4)$.*

Proof. Let the condition $\tilde{Z}(X,Y)\cdot W_2 = 0$ holds in N. Then we have

$$\begin{aligned}&\tilde{Z}(X,Y)\cdot W_2(Z,U)V - W_2(\tilde{Z}(X,Y)Z,U)V - W_2(Z,\tilde{Z}(X,Y)U)V\\ &- W_2(Z,U)\tilde{Z}(X,Y)V = 0.\end{aligned}$$

Putting $X = \xi$ in the above equation, we get

$$\begin{aligned}&\tilde{Z}(\xi,Y)\cdot W_2(Z,U)V - W_2(\tilde{Z}(\xi,Y)Z,U)V - W_2(Z,\tilde{Z}(\xi,Y)U)V\\ &- W_2(Z,U)\tilde{Z}(\xi,Y)V = 0.\end{aligned} \tag{40}$$

Using (39) and (29) in (40), we obtain

$$\begin{aligned}&A_{\sigma(Y,W_2(Z,U)V)}\xi - (\nabla_\xi\sigma)(Y,W_2(Z,U)V) + (k - \frac{r}{2n(2n+1)})\{g(Y,W_2(Z,U)V)\xi\\ &- \eta(W_2(Z,U)V)Y\} - W_2(A_{\sigma(Y,Z)}\xi - (\nabla_\xi\sigma)(Y,Z) + (k - \frac{r}{2n(2n+1)})\{g(Y,Z)\xi\\ &- \eta(Z)Y\},U)V - W_2(Z,A_{\sigma(Y,U)}\xi - (\nabla_\xi\sigma)(Y,U) + (k - \frac{r}{2n(2n+1)})\{g(Y,U)\xi\\ &- \eta(U)Y\})V - W_2(Z,U)[A_{\sigma(Y,V)}\xi - (\nabla_\xi\sigma)(Y,V) + (k - \frac{r}{2n(2n+1)})\{g(Y,V\xi\\ &- \eta(V)Y\}] = 0\end{aligned}$$

Taking inner product with ξ in the preceding equation, we get

$$(k - \frac{r}{2n(2n+1)})W_2(Z,U,V,Y) = 0.$$

i.e. either $r = 2n(2n+1)k$ or N is Einstein. □

Theorem 3.10. *If in an invariant submanifold N of an N(k)-contact metric manifold M admitting generalized Tanaka Webster connection, $P(X,Y)\cdot W_2 = 0$ holds then S_N^2 has the form $S_N(QY,U) = -2(n-1)S_N(U,Y) - (n-1)^2 g(Y,U)$.*

Proof. Let the submanifold satisfy the condition $P(X,Y)\cdot W_2 = 0$. Then we have

$$\begin{aligned}&P(X,Y)W_2(Z,U)V - W_2(P(X,Y)Z,U)V - W_2(Z,P(X,Y)U)V\\&- W_2(Z,U)P(X,Y)V = 0.\end{aligned}$$

Putting $X = \xi$ in the above equation, we get

$$\begin{aligned}&P(\xi,Y)W_2(Z,U)V - W_2(P(\xi,Y)Z,U)V - W_2(Z,P(\xi,Y)U)V\\&- W_2(Z,U)P(\xi,Y)V = 0.\end{aligned} \tag{41}$$

Using (38) and (29) in (41), we obtain

$$\begin{aligned}&A_{\sigma(Y,W_2(Z,U)V)}\xi - (\nabla_\xi\sigma)(Y,W_2(Z,U)V) + k\{g(Y,W_2(Z,U)V)\xi\\&- \eta(W_2(Z,U)V)Y\} - \frac{1}{2n}\{S_N(Y,W_2(Z,U)V)\xi - S_N(\xi,W_2(Z,U)V)Y\}\\&- W_2(A_{\sigma(Y,Z)}\xi - (\nabla_\xi\sigma)(Y,Z) + k\{g(Y,Z)\xi - \eta(Z)Y\} - \frac{1}{2n}\{S_N(Y,Z)\xi\\&- S_N(\xi,Z)Y\},U)V - W_2(Z,A_{\sigma(Y,U)}\xi, -(\nabla_\xi\sigma)(Y,U) + k\{g(Y,U)\xi\\&- \eta(U),Y\} - \frac{1}{2n}\{S_N(Y,U)\xi - S_N(\xi,U)Y\})V - W_2(Z,U)[A_{\sigma(Y,V)}\xi\\&- (\nabla_\xi\sigma)(Y,V) + k\{g(Y,V)\xi - \eta(V)Y\} - \frac{1}{2n}\{S_N(Y,V)\xi\\&- S_N(\xi,V)Y\}] = 0.\end{aligned}$$

Taking inner product with ξ in the preceding equation, we get

$$\begin{aligned}&kg(Y,W_2(Z,U)V) - \frac{1}{2n}\{S_N(Y,W_2(Z,U)V) - S_N(\xi,W_2(Z,U)V)\\&\eta(Y)\} = 0.\end{aligned} \tag{42}$$

From (28) and Proposition 3.1, (42) reduces to

$$kW_2(Z,U,V,Y) - \frac{1}{2n}W_2(Z,U,V,QY) = 0. \tag{43}$$

Using equation (30), (43) can be written as

$$\begin{aligned}&k[R_N(Z,U,V,Y) + \frac{1}{2n}\{g(Z,V)S_N(U,Y) - g(U,V)S_N(Y,Z)\}]\\&- \frac{1}{2n}[R_N(Z,U,V,QY) + \frac{1}{2n}\{g(Z,V)S_N(U,QY)\\&- g(U,V)S_N(QY,Z)\}] = 0.\end{aligned}$$

Putting $Z = V = \xi$ in the above equation and using (28) and (29), we get

$$S_N(QY,U) = -2(n-1)S_N(U,Y) - (n-1)^2 g(Y,U). \tag{44}$$

□

Definition 3.11. A quasi-conformal curvature tensor $\tilde{C}$ is defined by

$$\begin{aligned}\tilde{C}(X,Y)Z = a\ R(X,Y)Z + b\ [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ - g(X,Z)QY] - \frac{r}{n}(\frac{a}{n-1} + 2b)[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{45}$$

where a and b are constants and R, S, Q and r are the Riemannian curvature tensor of type (1,3), the Ricci tensor of type (0,2), the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and the scalar curvature of the manifold respectively.

Theorem 3.12. *If in an invariant submanifold N of N(k)-contact metric manifold admitting generalized Tanaka Webster connection, $\tilde{C}(X,Y)\cdot W_2 = 0$ holds. Then*

i) *for $b = 0$, we have either $r = 2n(2n+1)k$ or N is an Einstein manifold.*
ii) *S_N^2 has the form $S_N(QY,U) = \frac{2nk}{b}t\ g(U,Y) - (\frac{t}{b} - 2nk)S_N(U,Y)$, where $t = \{a(k - \frac{r}{2n(2n+1)}) + 2b(nk - \frac{r}{2n+1})\}$.*

Proof. Let the submanifold N satisfy the condition $\tilde{C}(X,Y)\cdot W_2 = 0$. Then we have

$$\begin{aligned}\tilde{C}(X,Y)W_2(Z,U)V - W_2(\tilde{C}(X,Y)Z,U)V - W_2(Z,\tilde{C}(X,Y)U)V \\ - W_2(Z,U)\tilde{C}(X,Y)V = 0.\end{aligned}$$

Putting $X = \xi$ in the above equation, we get

$$\begin{aligned}\tilde{C}(\xi,Y)W_2(Z,U)V - W_2(\tilde{C}(\xi,Y)Z,U)V - W_2(Z,\tilde{C}(\xi,Y)U)V \\ - W_2(Z,U)\tilde{C}(\xi,Y)V = 0.\end{aligned} \tag{46}$$

Using (45) and (29) in (46), we obtain

$$\begin{aligned}& a\{A_{\sigma(Y,W_2(Z,U)V)}\xi - (\nabla_\xi\sigma)(Y,W_2(Z,U)V)\} + \{ak - \frac{r}{2n+1}(\frac{a}{2n} + 2b)\} \\ & \{g(Y,W_2(Z,U)V)\xi - \eta(W_2(Z,U)V)Y\} + b\{S_N(Y,W_2(Z,U)V)\xi \\ & - S_N(\xi,W_2(Z,U)V)Y + g(Y,W_2(Z,U)V)Q\xi - \eta(W_2(Z,U)V)QY\} \\ & - W_2(a\{A_{\sigma(Y,Z)}\xi - (\nabla_\xi\sigma)(Y,Z)\} + \{ak - \frac{r}{2n+1}(\frac{a}{2n} + 2b)\}\{g(Y,Z)\xi \\ & - \eta(Z)Y\} + b\{S_N(Y,Z)\xi - S_N(\xi,Z)Y + g(Y,Z)Q\xi - \eta(Z)QY)\},U)V \\ & - W_2(Z,a\{A_{\sigma(Y,U)}\xi - (\nabla_\xi\sigma)(Y,U)\} + \{ak - \frac{r}{2n+1}(\frac{a}{2n} + 2b)\}\{g(Y,U)\xi \\ & - \eta(U)Y\} + b\{S_N(Y,U)\xi - S_N(\xi,U)Y + g(Y,U)Q\xi - \eta(U)QY)\})V \\ & W_2(Z,U)[a\{A_{\sigma(Y,V)}\xi - (\nabla_\xi\sigma)(Y,V)\} + \{ak - \frac{r}{2n+1}(\frac{a}{2n} + 2b)\}\{g(Y,V)\xi \\ & - \eta(V)Y\} + b\{S_N(Y,V)\xi - S_N(\xi,V)Y + g(Y,V)Q\xi - \eta(V)QY)\}] = 0.\end{aligned}$$

Taking inner product with ξ in the preceding equation, we get

$$\begin{aligned}\{ak - \frac{r}{2n+1}(\frac{a}{2n} + 2b)\}g(Y,W_2(Z,U)V) + b\{S_N(Y,W_2(Z,U)V) \\ - S_N(\xi,W_2(Z,U)V)\eta(Y) + g(Y,W_2(Z,U)V)\eta(Q\xi\} = 0.\end{aligned} \tag{47}$$

Using (28) and Proposition 3.1, (47) reduces to

$$\begin{aligned}&\{a(k-\frac{r}{2n(2n+1)})+2b(nk-\frac{r}{2n+1})\}W_2(Z,U,V,Y)\\&+b\ S_N(W(Z,U)V,Y)=0.\end{aligned}\tag{48}$$

If $b=0$, then either $a(k-\frac{r}{2n(2n+1)})=0$ or $W_2(Z,U,V,Y)=0$. □

If $b\neq 0$, then from (30) and (48), we have

$$\begin{aligned}&t[R_N(Z,U,V,Y)+\frac{1}{2n}\{g(Z,V)S_N(U,Y)-g(U,V)S_N(Z,Y)\}]\\&+b[R_N(Z,U,V,QY)+\frac{1}{2n}\{g(Z,V)S_N(U,QY)-g(U,V)\\&S_N(Z,QY)\}]=0,\end{aligned}$$

where $t=\{a(k-\frac{r}{2n(2n+1)})+2b(nk-\frac{r}{2n+1})\}$.
Putting $Z=V=\xi$ in the above equation and from (27) and (28), we obtain

$$S_N(QY,U)=\frac{2nk}{b}t\ g(U,Y)-(\frac{t}{b}-2nk)S_N(U,Y),$$

where $t=\{a(k-\frac{r}{2n(2n+1)})+2b(nk-\frac{r}{2n+1})\}$.

4. Conclusion

Not all the submanifolds inherit the geometrical structures of the ambient manifold. But under certain curvature conditions and semi symmetry conditions submanifolds possess the properties of the ambient manifold. The invariant submanifold inherits almost all properties of ambient manifold. We established important characterizations of invariant submanifolds of $N(k)$-contact metric manifolds admitting generalized Tanaka Webster connection. We applied the method of invariant submanifolds which is being used in the study of non-linear autonomous systems. Also We use the Tanaka-Webster connection, a canonical affine connection defined on a nondegenarate, pseudo-Hermitian CR-manifold. Generalized Tanaka Webster connection coincides with the Tanaka-Webster connection if the associated CR-structure is integrable. The main results concern invariant submanifolds of $N(k)$-contact metric manifolds which are locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ under the conditions $\tilde{Z}\cdot W_2=0$, $W_2=0$ and W_2-semi symmetric condition. With respect to these conditions, invariant submanifold reduces to Einstein manifold. In this case, submanifold has the constant scalar curvature. Further the square of Ricci curvature tensor in the submanifold has been expressed in terms of the Ricci tensor. This work opens further research on anti invariant submanifolds of manifolds with different contact structures using the conditions and methods used here.

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