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DISTRIBUTION OF VALUES OF DIFFERENCE OPERATORS CONCERNING WEAKLY WEIGHTED SHARING

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Abstract. Using the conception of weakly weighted sharing we discussed the value distribution of the differential product functions constructed with a polynomial and difference operator of entire function. Here we established two uniqueness result on product of difference operators when two such functions share a small function.

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1. Introduction and Definitions

Here we deal with mero-morphic and entire functions which are defined on complex plane. We adopt the standard definitions and notations of Nevanlinna's theory of mero-morphic functions (see [3, 11]). The Nevanlinna characteristic function of a non-constant mero-morphic function is denoted by $T(r,\xi)$, and $S(r,\xi)$ is any quantity satisfying $S(r,\xi) = o\{T(r,\xi)\}$ where $r(\to\infty) \in \mathbb{R}^+ \setminus E$ (measure of E is finite).

Definition 1.1. [3, 11] Deficiency of $\alpha \in \mathbb{C}$ with respect to a mero-morphic function ξ is denoted by $\delta(\alpha,\xi)$ and defined as $\delta(\alpha,\xi) = \underline{\lim}_{r\to\infty} \frac{m(r,\frac{1}{\xi-\alpha})}{T(r,\xi)} = 1 - \overline{\lim}_{r\to\infty} \frac{N(r,\alpha;\xi)}{T(r,\xi)}$.

By the Nevanlinna's SFT, it can be easily show that,

$$\sum_{\alpha\in\mathbb{C}\cup\{\infty\}} \delta(\alpha,\xi) \leq 2. \tag{1.1}$$

Definition 1.2. [3, 11] Order of a mero-morphic function ξ is denoted by $\sigma(\xi)$ and define by $\sigma(\xi) = \lim_{r\to\infty} sup \frac{\log T(r,\xi)}{\log r}$.

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Let ξ and ζ are non-constant mero-morphic functions and $\alpha \in \mathbb{C}$. We say ξ and ζ share α CM(Counting Multiplicities) if zeros of $\xi = \alpha$ and $\zeta = \alpha$ are same in value and multiplicities. And we say ξ and ζ share α IM(Ignoring Multiplicities) if zeros of $\xi = \alpha$ and $\zeta = \alpha$ are same only in value.

Definition 1.3. [4] Let $\tau \in \mathbb{N} \cup \{0\} \cup \{\infty\}$ and the counting function for the zeros of $\xi = \alpha$ with multiplicity $\leq \tau$ is denoted by $N(r, \alpha; \xi| \leq \tau)$ and the counting function for the zeros of $\xi = \alpha$ with multiplicity $\geq \tau$ is denoted by $N(r, \alpha; \xi| \geq \tau)$ ($\overline{N}(r, \alpha; \xi| \leq \tau)$ and $\overline{N}(r, \alpha; \xi| \geq \tau)$ are corresponding reduced counting function). The counting function for the zeros of $\xi = \alpha$, where multiplicity λ is counted λ times if $\lambda \leq \tau$ and τ times if $\lambda > \tau$ is denoted by $N(r, \alpha; \xi|\tau)$ and $N(r, \alpha; \xi|\tau) = \overline{N}(r, \alpha; \xi) + \overline{N}(r, \alpha; \xi| \geq 2) + ... + \overline{N}(r, \alpha; \xi| \geq \tau)$.

The set of all α-points of $\xi(z)$ with multiplicity upto τ is denoted by $E_{\tau)}(\alpha, \xi)$ ($\overline{E}_{\tau)}(\alpha, \xi)$ is corresponding reduce counting function) and if $E_{\tau)}(\alpha, \xi) = E_{\tau)}(\alpha, \zeta)$, then we say that $\xi(z)$ and $\zeta(z)$ share the value α with weight upto τ.

Definition 1.4. [3, 11] Let $\xi(z)$ and $\alpha(z)$ be two mero-morphic functions. $\alpha(z)$ be a small function of $\xi(z)$ if $T(r, \alpha(z)) = S(r, \xi)$.

Definition 1.5. [1] Difference operator of a function is denoted by $\Delta_\omega \xi$ and defined by $\Delta_\omega \xi(z) = \xi(z+\omega) - \xi(z)$, where $\omega \in \mathbb{C} \setminus \{0\}$ and p-order difference operator is given by $\Delta_\omega^p \xi$ and defined by $\Delta_\omega^p \xi(z) = \Delta_\omega^{p-1}(\Delta_\omega \xi(z))$, where $p(\geq 2) \in \mathbb{N}$.

We can also deduce that, $\Delta_\omega^p \xi = \sum_{i=0}^{p} \binom{p}{i} \xi(z + (p-i)\omega)$.

Definition 1.6. [6] Let ξ and ζ be two non-constant mero-morphic functions and $\alpha \in \mathbb{C}$. The counting function of all common zeros with the same multiplicities of $\xi = \alpha$ and $\zeta = \alpha$ is denoted by $N(r, \alpha)(E)$ and the counting function of all common zeros in ignorance of multiplicities is denoted by $N(r, \alpha)(0)$ ($\overline{N}(r, \alpha)(E)$ and $\overline{N}(r, \alpha)(0)$ are corresponding reduce counting functions). We defined that ξ and ζ share α CM weakly, if,
$\overline{N}(r, \alpha; \xi) + \overline{N}(r, \alpha; \zeta) - 2\overline{N}(r, \alpha)(E) = S(r, \xi) + S(r, \zeta)$,
and said ξ and ζ share α IM weakly, if,
$\overline{N}(r, \alpha; \xi) + \overline{N}(r, \alpha; \zeta) - 2\overline{N}(r, \alpha)(0) = S(r, \xi) + S(r, \zeta)$.

In 2006, S. Lin and W. Lin [6] introduced the conception of weakly weighted sharing:

Definition 1.7. [6] Let ξ and ζ be two non-constant mero-morphic functions and $\alpha \in S(\xi) \cap S(\zeta)$, $\tau \in \mathbb{Z}^+ \cup \{\infty\}$ and if
$\overline{N}(r, \alpha; \xi)(\leq \tau) + \overline{N}(r, \alpha; \zeta)(\leq \tau) - 2\overline{N}(r, \alpha)(E(\leq \tau)) = S(r, \xi) + S(r, \zeta)$,
$\overline{N}(r, \alpha; \xi)(\geq \tau+1) + \overline{N}(r, \alpha; \zeta)(\geq \tau+1) - 2\overline{N}(r, \alpha)(0(\geq \tau+1)) = S(r, \xi) + S(r, \zeta)$,
or, if $\tau = 0$ then,
$\overline{N}(r, \alpha; \xi) + \overline{N}(r, \alpha; \zeta) - 2\overline{N}(r, \alpha)(0) = S(r, \xi) + S(r, \zeta)$,
and we call that ξ and ζ weakly share α with weight τ and the notion will be denoted by $\omega(\alpha, \tau)$.

Let ξ and ζ share 1 IM weakly. Then the counting function of 1 points of ξ with multiplicities greater than of 1 points of ζ is denoted by $\overline{N}(r,1;\xi|L)$. $\overline{N}(r,1;\zeta|L)$ is defined similarly.

Many research papers are all ready published on shift function[7] and difference operator[1]. We are interested on the product of difference operators which is given by $\prod_{i=1}^{\eta}(\Delta_{\omega}^{p}\xi)^{\mu_i}$ where η, $\mu_i(i=1,2,...,\eta)\in\mathbb{Z}^{+}\cup\{0\}$ and throughout the paper we use $\rho=\sum_{i=1}^{n}\mu_i$.

In 2016, P. Sahoo and B. Saha [7] studied the distribution of value of difference-differential polynomial with shift function and developed the following uniqueness results concerning with CM sharing of a small function:

Theorem 1.8. [7] *Let $\xi(z)$ and $\zeta(z)$ be two transcendental entire function of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to ξ and ζ. Let $\omega\in\mathbb{C}\setminus\{0\}$, and n, $\lambda(\geq 1)$, $\tau(\geq 0)\in\mathbb{N}\cup\{0\}$, where $n\geq 2\tau+\lambda+6$. If $[\xi^{n}(\xi^{\lambda}-1)\xi(z+\omega)]^{(\tau)}$ and $[\zeta^{n}(\zeta^{\lambda}-1)\zeta(z+\omega)]^{(\tau)}$ share α CM, then, $\xi=t\zeta$ where $t^{\lambda}=1$.*

Theorem 1.9. [7] *Let $\xi(z)$ and $\zeta(z)$ be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to ξ and ζ. Let $\omega\in\mathbb{C}\setminus\{0\}$, and n, $\lambda(\geq 1)$, $\tau(\geq 0)\in\mathbb{N}\cup\{0\}$, where $n\geq 2\tau+\lambda+6$ when $\lambda\leq\tau+1$ and $n\geq 4\tau-\lambda+10$ when $\lambda>\tau+1$. If $[\xi^{n}(\xi-1)^{\lambda}\xi(z+\omega)]^{(\tau)}$ and $[\zeta^{n}(\zeta-1)^{\lambda}\zeta(z+\omega)]^{(\tau)}$ share α CM, then, $\xi\equiv\zeta$; or, ξ and ζ satisfying the algebraic equation $R(\xi,\zeta)=0$, where $R(\xi,\zeta)$ is given by $R(\phi_1,\phi_2)=\phi_1^{n}(\phi_1-1)^{\lambda}\phi_1(z+\omega)-\phi_2^{n}(\phi_2-1)^{\lambda}\phi_2(z+\omega)$.*

In 2018, H.P. Waghamore [8], introduce the product of shift functions with some polynomials and proved the following result on τ-th order difference differential polynomial of transcendental entire functions concerning with sharing values:

Theorem 1.10. [8] *Let $\xi(z)$ and $\zeta(z)$ be two transcendental entire function of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to ξ and ζ. Let $\omega_i\in\mathbb{C}\setminus\{0\}$, $\mu_i\in\mathbb{Z}^{+}\cup\{0\}$ where $i=1,2,...,\eta$ and n, $\lambda(\geq 1)$, $\tau(\geq 0)$, $\eta\in\mathbb{N}\cup\{0\}$, where $n\geq 2\tau+\lambda+\rho+5$. If $[\xi^{n}(\xi^{\lambda}-1)\prod_{i=1}^{\eta}\xi(z+\omega_i)^{\mu_i}]^{(\tau)}$ and $[\zeta^{n}(\zeta^{\lambda}-1)\prod_{i=1}^{\eta}\zeta(z+\omega_i)^{\mu_i}]^{(\tau)}$ share α CM, then, $\xi=t\zeta$ where $t^{\lambda}=1$.*

Theorem 1.11. [8] *Let $\xi(z)$ and $\zeta(z)$ be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to ξ and ζ. Let $\omega_i\in\mathbb{C}\setminus\{0\}$, $\mu_i\in\mathbb{Z}^{+}\cup\{0\}$ where $i=1,2,...,\eta$ and n, $\lambda(\geq 1)$, $\tau(\geq 0)$, $\eta\in\mathbb{N}\cup\{0\}$, where $n\geq 2\tau+\lambda+\rho+5$ when $\lambda\leq\tau+1$ and $n\geq 4\tau-\lambda+\rho+9$ when $\lambda>\tau+1$. If $[\xi^{n}(\xi-1)^{\lambda}\prod_{i=1}^{\eta}\xi(z+\omega_i)^{\mu_i}]^{(\tau)}$ and $[\zeta^{n}(\zeta-1)^{\lambda}\prod_{i=1}^{\eta}\zeta(z+\omega_i)^{\mu_i}]^{(\tau)}$ share α CM, then, $\xi\equiv\zeta$; or, ξ and ζ satisfying the algebraic equation $R(\xi,\zeta)=0$, where $R(\xi,\zeta)$ is given by $R(\phi_1,\phi_2)=\phi_1^{n}(\phi_1-1)^{\lambda}\prod_{i=1}^{\eta}\phi_1(z+\omega_i)^{\mu_i}-\phi_2^{n}(\phi_2-1)^{\lambda}\prod_{i=1}^{\eta}\phi_2(z+\omega_i)^{\mu_i}$.*

NOTE: There is no clear discussion for taking the conditions $\lambda \leq \tau+1$ and $\lambda > \tau+1$ in the theorem 1.9 and the theorem 1.11. We clearly explain all the cases, $\lambda \leq \tau+1$, $\lambda = \tau+2$ and $\lambda > \tau+2$ separately in our second theorem.

We take transcendental entire functions with zeros of multiplicity atleast ι and study the distribution of values of differential polynomials are formed with product of difference operators of transcendental entire functions according to weakly weighted sharing of a small function. We present our main results in the following section:

2. Lemmas

Now, we present some lemmas which will be needed in a sequel.

Lemma 2.1. [10] *Let $\xi(z)$ be a non-constant mero-morphic function and a_0, a_1,..., $a_n(\not\equiv 0)$ be mero-morphic functions such that $T(r,a_i) = S(r,\xi)$ where $i = 0,1,...,n$. Then, $T(r, \sum_{i=0}^{n} a_i\xi^i) = nT(r,\xi) + S(r,\xi)$.*

Lemma 2.2. [12] *Let τ and p be positive integers and $\xi(z)$ be a non-constant mero-morphic function then,*
$N(r,0;\xi^{(\tau)} \mid p) \leq T(r,\xi^{(\tau)}) - T(r,\xi) + N(r,0;\xi \mid \tau+p) + S(r,\xi)$;
$N(r,0;\xi^{(\tau)} \mid p) \leq \tau\overline{N}(r,\infty;\xi) + N(r,0;\xi \mid \tau+p) + S(r,\xi)$.

Lemma 2.3. [6] *Let ξ and ζ be two non-constant mero-morphic functions share $\omega(1,\Gamma)$ where $\Gamma \in \mathbb{Z}^+ \cup \{0\} \cup \{\infty\}$ and let, $\Omega = (\frac{\xi^{(2)}}{\xi^{(1)}} - \frac{2\xi^{(1)}}{\xi-1}) - (\frac{\zeta^{(2)}}{\zeta^{(1)}} - \frac{2\zeta^{(1)}}{\zeta-1})$. If $\Omega \not\equiv 0$, then,*
(i) $T(r,\xi) \leq N(r,\infty;\xi|2) + N(r,\infty;\zeta|2) + N(r,0;\xi|2) + N(r,0;\zeta|2) + S(r,\xi) + S(r,\zeta)$ *when* $2 \leq \Gamma \leq \infty$;
(ii) $T(r,\xi) \leq N(r,\infty;\xi|2)+N(r,\infty;\zeta|2)+N(r,0;\xi|2)+N(r,0;\zeta|2)+\overline{N}(r,1;\xi|L)+S(r,\xi) + S(r,\zeta)$ *when* $\Gamma = 1$;
(iii) $T(r,\xi) \leq N(r,\infty;\xi|2)+N(r,\infty;\zeta|2)+N(r,0;\xi|2)+N(r,0;\zeta|2)+2\overline{N}(r,1;\xi|L)+\overline{N}(r,1;\zeta|L) + S(r,\xi) + S(r,\zeta)$ *when* $\Gamma = 0$;
and same inequalities are hold for $T(r,\zeta)$.

Lemma 2.4. [9] *Let ξ and ζ are non-constant mero-morphic functions share $\omega(1,1)$. Then, $\overline{N}(r,1;\xi|L) \leq \frac{1}{2}\overline{N}(r,0;\xi) + \frac{1}{2}\overline{N}(r,\infty;\xi) + S(r,\xi)$.*

Lemma 2.5. [9] *Let ξ and ζ are non-constant mero-morphic functions share $\omega(1,0)$. Then, $\overline{N}(r,1;\xi|L) \leq \overline{N}(r,0;\xi) + \overline{N}(r,\infty;\xi) + S(r,\xi)$.*

Lemma 2.6. [2] *Let $\xi(z)$ be a mero-morphic function of finite order(σ) and $\omega \in \mathbb{C}\backslash\{0\}$ be fixed. Then for each $\epsilon(>0)$, we have, $m(r, \frac{\xi(z+\omega)}{\xi(z)}) + m(r, \frac{\xi(z)}{\xi(z+\omega)}) = O(r^{\sigma+\epsilon-1}) = S(r,\xi)$.*

Lemma 2.7. *Let $\xi(z)$ be a transcendental entire function of finite order(σ) and n, λ, p, η, $\mu_i \in \mathbb{Z}^+$, where $i = 1,2,...,\eta$, $\omega \in \mathbb{C} \setminus \{0\}$, and $\Delta_\omega^p\xi \not\equiv 0$. Then, $T(r,\xi^n(\xi^\lambda - 1)\prod_{i=1}^{\eta}(\Delta_\omega^p\xi)^{\mu_i}) = T(r,\xi^n(\xi^\lambda-1)\xi^\rho) = (n+\lambda+\rho)T(r,\xi) + S(r,\xi)$.*

Proof. Since ξ is transcendental entire function, then $\xi^n(\xi^\lambda - 1)\prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}$ is also an entire function. Using Lemma 2.4, we can deduce that,

$$\begin{aligned} & T(r, \xi^n(\xi^\lambda - 1)\prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}) = m(r, \xi^n(\xi^\lambda - 1)\prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}) \\ = \quad & m(r, \xi^n) + m(r, \xi^\lambda - 1) + m(r, \prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}) \\ = \quad & nm(r, \xi) + \lambda m(r, \xi) + \rho m(r, \Delta_\omega^p \xi) \\ = \quad & nm(r, \xi) + \lambda m(r, \xi) + \rho m(r, \xi) + \rho m(r, \frac{\Delta_\omega^p f}{f}) \\ \leq \quad & (n + \lambda + \rho)T(r, \xi) + S(r, \xi). \end{aligned} \tag{2.1}$$

Otherwise we deduce using Lemma 2.4 and FFT that,

$$\begin{aligned} & (n + \lambda + \rho)T(r, \xi) = T(r, \xi^n(\xi^\lambda - 1)\xi^\rho) = m(r, \xi^n(\xi^\lambda - 1)\xi^\rho) \\ \leq \quad & m(r, \xi^n(\xi^\lambda - 1)\prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}) + m(r, \frac{\xi^\rho}{\prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}}) \\ \leq \quad & T(r, \xi^n(\xi^\lambda - 1)\prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}) + T(r, \frac{\xi^\rho}{\prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}}) + S(r, \xi) \\ \leq \quad & T(r, \xi^n(\xi^\lambda - 1)\prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}) + T(r, \frac{\prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}}{\xi^\rho}) + S(r, \xi) \\ \leq \quad & T(r, \xi^n(\xi^\lambda - 1)\prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}) + S(r, \xi). \end{aligned} \tag{2.2}$$

Hence, combining (2.1) and (2.2) we have, $T(r, \xi^n(\xi^\lambda - 1)\prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}) = T(r, \xi^n(\xi^\lambda - 1)\xi^\rho) = (n + \lambda + \rho)T(r, \xi) + S(r, \xi)$. □

Lemma 2.8. *Let $\xi(z)$ be a transcendental entire function of finite order (σ) and n, λ, p, η, $\mu_i \in \mathbb{Z}^+$, where $i = 1, 2, ..., \eta$, $\omega \in \mathbb{C} \setminus \{0\}$, and $\Delta_\omega^p \xi \not\equiv 0$. Then, $T(r, \xi^n(\xi - 1)^\lambda \prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}) = T(r, \xi^n(\xi - 1)^\lambda \xi^\rho) = (n + \lambda + \rho)T(r, \xi) + S(r, \xi)$.*

Proof. The lemma will be proved from the line of the Lemma 2.7. □

Lemma 2.9. *Let $\xi(z)$ and $\zeta(z)$ be two transcendental entire functions of finite order and multiplicity of zeros of ξ and ζ is atleast ι. Let $\omega \in \mathbb{C} \setminus \{0\}$, $\iota, \mu_i \in \mathbb{Z}^+$ where $i = 1, 2, ..., \eta$ and n, $\lambda(\geq 1)$, $\tau(\geq 0)$ p, $\eta \in \mathbb{N} \cup \{0\}$ and $\Delta_\omega^p \xi \not\equiv 0$, $\Delta_\omega^p \zeta \not\equiv 0$. Let $\mathcal{F} = [\xi^n(\xi^\lambda - 1)\prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}]^{(\tau)}$ and $\mathcal{G} = [\zeta^n(\zeta^\lambda - 1)\prod_{i=1}^{\eta}(\Delta_\omega^p \zeta)^{\mu_i}]^{(\tau)}$. If there exists non zero constants α_1 and α_2 such that $\overline{N}(r, \alpha_1; \mathcal{F}) = \overline{N}(r, 0; \mathcal{G})$ and $\overline{N}(r, \alpha_2; \mathcal{G}) = \overline{N}(r, 0; \mathcal{F})$, then $n \leq \frac{2}{\iota}(\tau + 1) + \lambda + \rho$.*

Proof. We assume that $\mathcal{F}_1 = \xi^n(\xi^\lambda - 1)\prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}$ and
$\mathcal{G}_1 = \zeta^n(\zeta^\lambda - 1)\prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}$, and by Nevanlinna's SFT, we have,

$$T(r, \mathcal{F}) \quad \leq \quad \overline{N}(r, 0; \mathcal{F}) + \overline{N}(r, \alpha_1; \mathcal{F}) + S(r, \mathcal{F})$$

$$\leq \overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G})+S(r,\mathcal{F}). \tag{2.3}$$

Using Lemma 2.2, Lemma 2.6 and Lemma 2.7 on equation (2.3),

$$\begin{aligned}
&(n+\lambda+\rho)T(r,\xi)\leq T(r,\mathcal{F})-\overline{N}(r,0;\mathcal{F})+N(r,0;\mathcal{F}_1|\tau+1)\\
\leq\ & \overline{N}(r,0;\mathcal{G})+N(r,0;\mathcal{F}_1|\tau+1)+S(r,\xi)\\
\leq\ & \overline{N}(r,0;\mathcal{G}_1|\tau+1)+\overline{N}(r,0;\mathcal{F}_1|\tau+1)+S(r,\xi)\\
\leq\ & \overline{N}(r,0;\zeta^n|\tau+1)+\overline{N}(r,0;\zeta^\lambda-1|\tau+1)\\
+\ & \overline{N}(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}|\tau+1)+S(r,\zeta)+\overline{N}(r,0;\xi^n|\tau+1)\\
+\ & \overline{N}(r,0;\xi^\lambda-1|\tau+1)+\overline{N}(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p\xi)^{\mu_i}|\tau+1)+S(r,\xi)\\
\leq\ & \frac{1}{\iota}(\tau+1)T(r,\zeta)+\lambda T(r,\zeta)+\rho T(r,\zeta)+\frac{1}{\iota}(\tau+1)T(r,\xi)\\
+\ & \lambda T(r,\xi)+\rho T(r,\xi)+S(r,\xi)+S(r,\zeta)\\
\leq\ & (\frac{1}{\iota}(\tau+1)+\lambda+\rho)(T(r,\xi)+T(r,\zeta))+S(r,\xi)+S(r,\zeta).
\end{aligned} \tag{2.4}$$

Similarly we can deduce,

$$\begin{aligned}
(n+\lambda+\rho)T(r,\zeta) \quad \leq\ & (\frac{1}{\iota}(\tau+1)+\lambda+\rho)(T(r,\xi)+T(r,\zeta))\\
+\ & S(r,\xi)+S(r,\zeta).
\end{aligned} \tag{2.5}$$

Adding (2.4) and (2.5), we have, $(n+\lambda+\rho)(T(r,\xi)+T(r,\zeta))\leq 2[\frac{1}{\iota}(\tau+1)+\lambda+\rho](T(r,\xi)+T(r,\zeta))+S(r,\xi)+S(r,\zeta)$, which implies that $n\leq\frac{2}{\iota}(\tau+1)+\lambda+\rho$. Hence the lemma. □

Lemma 2.10. *Let $\xi(z)$ and $\zeta(z)$ be two transcendental entire function of finite order and multiplicity of zeros of ξ and ζ is atleast ι. Let $\omega\in\mathbb{C}\setminus\{0\}$, $\iota,\mu_i\in\mathbb{Z}^+$ where $i=1,2,...,\eta$ and n, $\lambda(\geq 1)$, $\tau(\geq 0)$ p, $\eta\in\mathbb{N}\cup\{0\}$ and $\Delta_\omega^p\xi\not\equiv 0$, $\Delta_\omega^p\zeta\not\equiv 0$. Let $\mathcal{F}=[\xi^n(\xi-1)^\lambda\prod_{i=1}^{\eta}(\Delta_\omega^p\xi)^{\mu_i}]^{(\tau)}$ and $\mathcal{G}=[\zeta^n(\zeta-1)^\lambda\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}]^{(\tau)}$. If there exists non zero constants α_1 and α_2 such that $\overline{N}(r,\alpha_1;\mathcal{F})=\overline{N}(r,0;\mathcal{G})$ and $\overline{N}(r,\alpha_2;\mathcal{G})=\overline{N}(r,0;\mathcal{F})$, then $n\leq\frac{2}{\iota}(\tau+1)+\lambda+\rho$ when $\lambda\leq\tau+1$, or, $n\leq\frac{2}{\iota}(\tau+1)+2\tau+\rho-\lambda+2$ when $\lambda>\tau+1$.*

Proof. We assume that $\mathcal{F}_1=\xi^n(\xi-1)^\lambda\prod_{i=1}^{\eta}(\Delta_\omega^p\xi)^{\mu_i}$ and $\mathcal{G}_1=\zeta^n(\zeta-1)^\lambda\prod_{i=1}^{\eta}(\Delta_\omega^p\xi)^{\mu_i}$, and by Nevanlinna's SFT, we have,

$$\begin{aligned}
T(r,\mathcal{F}) \quad \leq\ & \overline{N}(r,0;\mathcal{F})+\overline{N}(r,\alpha_1;\mathcal{F})+S(r,\mathcal{F})\\
\leq\ & \overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G})+S(r,\mathcal{F}).
\end{aligned} \tag{2.6}$$

Using Lemma 2.2, Lemma 2.6 and Lemma 2.7 on inequality (2.6),

$$\begin{aligned}
&(n+\lambda+\rho)T(r,\xi)\leq T(r,\mathcal{F})-\overline{N}(r,0;\mathcal{F})+N(r,0;\mathcal{F}_1|\tau+1)\\
&\leq\overline{N}(r,0;\mathcal{G})+N(r,0;\mathcal{F}_1|\tau+1)+S(r,\xi)
\end{aligned}$$

$$\leq \overline{N}(r,0;\mathcal{G}_1|\tau+1)+\overline{N}(r,0;\mathcal{F}_1|\tau+1)+S(r,\xi), \tag{2.7}$$

Case 1. $\lambda \leq \tau+1$
Then from (2.7), we have,

$$\begin{aligned}
&(n+\lambda+\rho)T(r,\xi) \leq \overline{N}(r,0;\zeta^n|\tau+1)+\overline{N}(r,0;(\zeta-1)^\lambda|\tau+1)\\
+\quad & \overline{N}(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}|\tau+1)+S(r,\zeta)+\overline{N}(r,0;\xi^n|\tau+1)\\
+\quad & \overline{N}(r,0;(\xi-1)^\lambda|\tau+1)+\overline{N}(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p\xi)^{\mu_i}|\tau+1)+S(r,\xi)\\
\leq\quad & \frac{1}{\iota}(\tau+1)T(r,\zeta)+\lambda T(r,\zeta)+\rho T(r,\zeta)+\frac{1}{\iota}(\tau+1)T(r,\xi)\\
+\quad & \lambda T(r,\xi)+\rho T(r,\xi)+S(r,\xi)+S(r,\zeta)\\
\leq\quad & (\frac{1}{\iota}(\tau+1)+\lambda+\rho)(T(r,\xi)+T(r,\zeta))+S(r,\xi)+S(r,\zeta).
\end{aligned} \tag{2.8}$$

Similarly we can deduce,

$$\begin{aligned}
(n+\lambda+\rho)T(r,\zeta) \quad\leq\quad & (\frac{1}{\iota}(\tau+1)+\lambda+\rho)(T(r,\xi)+T(r,\zeta))\\
+\quad & S(r,\xi)+S(r,\zeta).
\end{aligned} \tag{2.9}$$

Combining (2.8) and (2.9) we have,

$$\begin{aligned}
(n+\lambda+\rho)(T(r,\xi)+T(r,\zeta)) \quad\leq\quad & 2[\frac{1}{\iota}(\tau+1)+\lambda+\rho](T(r,\xi)+T(r,\zeta))\\
+\quad & S(r,\xi)+S(r,\zeta),
\end{aligned}$$

which implies that $n \leq \frac{2}{\iota}(\tau+1)+\lambda+\rho$.
Case 2. $\lambda > \tau+1$,
Then from (2.7), we have,

$$\begin{aligned}
&(n+\lambda+\rho)T(r,\xi) \leq \overline{N}(r,0;\zeta^n|\tau+1)+\overline{N}(r,0;(\zeta-1)^\lambda|\tau+1)\\
+\quad & \overline{N}(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}|\tau+1)+S(r,\zeta)+\overline{N}(r,0;\xi^n|\tau+1)\\
+\quad & \overline{N}(r,0;(\xi-1)^\lambda|\tau+1)+\overline{N}(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p\xi)^{\mu_i}|\tau+1)+S(r,\xi)\\
\leq\quad & \frac{1}{\iota}(\tau+1)T(r,\zeta)+(\tau+1)T(r,\zeta)+\rho T(r,\zeta)+\frac{1}{\iota}(\tau+1)T(r,\xi)\\
+\quad & (\tau+1)T(r,\xi)+\rho T(r,\xi)+S(r,\xi)+S(r,\zeta)\\
\leq\quad & (\frac{1}{\iota}(\tau+1)+\tau+\rho+1)(T(r,\xi)+T(r,\zeta))\\
+\quad & S(r,\xi)+S(r,\zeta).
\end{aligned} \tag{2.10}$$

Similarly we can deduce,

$$\begin{aligned}(n+\lambda+\rho)T(r,\zeta) &\leq (\frac{1}{\iota}(\tau+1)+\tau+1+\rho)(T(r,\xi)+T(r,\zeta)) \\ &+ S(r,\xi)+S(r,\zeta). \qquad (2.11)\end{aligned}$$

Adding (2.10) and (2.11) we have, $(n+\lambda+\rho)(T(r,\xi)+T(r,\zeta)) \leq 2[\frac{1}{\iota}(\tau+1)+\tau+\rho+1](T(r,\xi)+T(r,\zeta))+S(r,\xi)+S(r,\zeta)$, which implies that $n \leq \frac{2}{\iota}(\tau+1)+2\tau+\rho-\lambda+2$. Hence the lemma. □

3. Main results

Theorem 3.1. *Let $\xi(z)$ and $\zeta(z)$ be two transcendental entire functions of finite order and multiplicity of zeros of $\xi(z)$ and $\zeta(z)$ is atleast ι. Let $\alpha(z)(\not\equiv 0)$ be a small function with respect to $\xi(z)$ and $\zeta(z)$ and let $\omega \in \mathbb{C}\setminus\{0\}$, $\iota, \mu_i \in \mathbb{Z}^+$ where $i=1,2,...,\eta$ and n, $\lambda(\geq 1)$, $\tau(\geq 0)$ p, $\eta \in \mathbb{N}\cup\{0\}$ and $\Delta_\omega^p \xi \not\equiv 0$, $\Delta_\omega^p \zeta \not\equiv 0$. Let $[\xi^n(\xi^\lambda-1)\prod_{i=1}^{\eta}(\Delta_\omega^p\xi)^{\mu_i}]^{(\tau)}$ and $[\zeta^n(\zeta^\lambda-1)\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}]^{(\tau)}$ share $\omega(\alpha,\Gamma)$, then, for one of following conditions,*
(i) $\Gamma \geq 2$, $n > \frac{2}{\iota}(\tau+2)+\lambda+\rho$;
(ii) $\Gamma = 1$, $n > \frac{1}{2}(\frac{1}{\iota}(5\tau+9)+3\lambda+3\rho)$;
(ii) $\Gamma = 0$, $n > \frac{1}{\iota}(5\tau+7)+4\lambda+4\rho$;
either $\xi(z)\equiv\zeta(z)$; or,
$\xi(z)=\kappa\zeta(z)$ where κ is a variable and $\zeta^\lambda = \frac{\kappa^{n+\lambda}\prod_{i=1}^{\eta}(\Delta_\omega^p\kappa\zeta)^{\mu_i}-\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}}{\kappa^n\prod_{i=1}^{\eta}(\Delta_\omega^p\kappa\zeta)^{\mu_i}-\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}}$.

Proof. Let us assume that $\mathcal{F}_1 = \xi^n(\xi^\lambda-1)\prod_{i=1}^{\eta}(\Delta_\omega^p\xi)^{\mu_i}$,
$\mathcal{G}_1 = \zeta^n(\zeta^\lambda-1)\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}$, and $\mathcal{F} = \frac{\mathcal{F}_1^{(\tau)}}{\alpha(z)}$, $\mathcal{G} = \frac{\mathcal{G}_1^{(\tau)}}{\alpha(z)}$. Then $\mathcal{F}$ and $\mathcal{G}$ are transcendental mero-morphic function that share $\omega(1,\Gamma)$, except the zeros and poles of $\alpha(z)$. From Lemma 2.2 and Lemma 2.7, we have,
$N(r,0;\mathcal{F}|2) \leq N(r,0;\mathcal{F}_1^{(\tau)}|2)+S(r,\xi)$
$\leq T(r,\mathcal{F}_1^{(\tau)})-(n+\lambda+\rho)T(r,\xi)+N(r,0;\mathcal{F}_1|\tau+2)+S(r,\xi)$
$\leq T(r,\mathcal{F})-(n+\lambda+\rho)T(r,\xi)+N(r,0;\mathcal{F}_1|\tau+2)+S(r,\xi)$,
Hence,

$$(n+\lambda+\rho)T(r,\xi) \leq T(r,\mathcal{F})+N(r,0;\mathcal{F}_1|\tau+2)-N(r,0;\mathcal{F}|2)+S(r,\xi). \quad (3.1)$$

Again from Lemma 2.2, we have,

$$\begin{aligned}N(r,0;\mathcal{F}|2) &\leq N(r,0;\mathcal{F}_1^{(\tau)}|2)+S(r,\xi) \\ &\leq N(r,0;\mathcal{F}_1|\tau+2)+S(r,\xi). \qquad (3.2)\end{aligned}$$

Let us consider, $\Omega = (\frac{\mathcal{F}^{(2)}}{\mathcal{F}^{(1)}}-\frac{2\mathcal{F}^{(1)}}{\mathcal{F}-1})-(\frac{\mathcal{G}^{(2)}}{\mathcal{G}^{(1)}}-\frac{2\mathcal{G}^{(1)}}{\mathcal{G}-1})$.
Case 1. $\Omega \not\equiv 0$,
Since $\mathcal{F}$ and $\mathcal{G}$ share $\omega(1,\Gamma)$, we discuss all the cases of Lemma 2.3,
Case 1.1. $\Gamma \geq 2$,
From (i) of Lemma 2.3, and inequalities (3.1) and (3.2), we have,

$$(n+\lambda+\rho)T(r,\xi) \leq N(r,0;\mathcal{F}|2)+N(r,\infty;\mathcal{F}|2)+N(r,0;\mathcal{G}|2)$$

$$
\begin{aligned}
&+ \quad N(r,\infty;\mathcal{G}|2) + S(r,\xi) + S(r,\zeta)\\
&\leq \quad N(r,0;\mathcal{F}_1|\tau+2) + N(r,0;\mathcal{G}_1|\tau+2) + S(r,\xi) + S(r,\zeta)\\
&\leq \quad N(r,0;\xi^n|\tau+2) + N(r,0;\xi^\lambda-1|\tau+2)\\
&+ \quad N(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p\xi)^{\mu_i}|\tau+2) + S(r,\xi) + N(r,0;\zeta^n|\tau+2)\\
&+ \quad N(r,0;\zeta^\lambda-1|\tau+2) + N(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}|\tau+2) + S(r,\zeta)\\
&\leq \quad (\frac{1}{\iota}(\tau+2)+\lambda+\rho)(T(r,\xi)+T(r,\zeta)) + S(r,\xi) + S(r,\zeta). \qquad (3.3)
\end{aligned}
$$

Similarly we can show that,

$$(n+\lambda+\rho)T(r,\zeta) \leq (\frac{1}{\iota}(\tau+2)+\lambda+\rho)(T(r,\xi)+T(r,\zeta))+S(r,\xi)+S(r,\zeta). \quad (3.4)$$

Combining (3.3) and (3.4) we have, $[n-(\frac{2}{\iota}(\tau+2)+\lambda+\rho)](T(r,\xi)+T(r,\zeta)) \leq S(r,\xi)+S(r,\zeta)$, which is contradiction as $n > \frac{2}{\iota}(\tau+2)+\lambda+\rho$.
Case 1.2. $\Gamma = 1$,
From (ii) of Lemma 2.3 and inequalities (3.1),(3.2), and Lemma 2.4,

$$
\begin{aligned}
&\quad\quad (n+\lambda+\rho)T(r,\xi) \leq N(r,0;\mathcal{F}|2) + N(r,\infty;\mathcal{F}|2) + N(r,0;\mathcal{G}|2)\\
&+ \quad N(r,\infty;\mathcal{G}|2) + \overline{N}(r,1;\mathcal{F}|L) + S(r,\xi) + S(r,\zeta)\\
&\leq \quad N(r,0;\mathcal{F}_1|\tau+2) + N(r,0;\mathcal{G}_1|\tau+2) + \frac{1}{2}N(r,0;\mathcal{F}_1|\tau+1)\\
&+ \quad S(r,\xi) + S(r,\zeta)\\
&\leq \quad N(r,0;\xi^n|\tau+2) + N(r,0;\xi^\lambda-1|\tau+2)\\
&+ \quad N(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p\xi)^{\mu_i}|\tau+2) + \frac{1}{2}N(r,0;\xi^n|\tau+1)\\
&+ \quad \frac{1}{2}N(r,0;\xi^\lambda-1|\tau+1) + \frac{1}{2}N(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p\xi)^{\mu_i}|\tau+1)\\
&+ \quad N(r,0;\zeta^n|\tau+2) + N(r,0;\zeta^\lambda-1|\tau+2)\\
&+ \quad N(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}|\tau+2) + S(r,\xi) + S(r,\zeta)\\
&\leq \quad \frac{1}{2}(\frac{1}{\iota}(3\tau+5)+3\lambda+3\rho)T(r,\xi) + (\frac{1}{\iota}(\tau+2)+\lambda+\rho)T(r,\zeta)\\
&+ \quad S(r,\xi) + S(r,\zeta). \qquad (3.5)
\end{aligned}
$$

Similarly we can show that,

$$(n+\lambda+\rho)T(r,\zeta) \leq \frac{1}{2}(\frac{1}{\iota}(3\tau+5)+3\lambda+3\rho)T(r,\zeta)$$

$$+(\frac{1}{\iota}(\tau+2)+\lambda+\rho)T(r,\xi)+S(r,\xi)+S(r,\zeta). \tag{3.6}$$

Combining (3.5) and (3.6) we have, $[n-\frac{1}{2}(\frac{1}{\iota}(5\tau+9)+3\lambda+3\rho)](T(r,\xi)+T(r,\zeta)) \leq S(r,\xi)+S(r,\zeta)$, which is contradiction as $n>\frac{1}{2}(\frac{1}{\iota}(5\tau+9)+3\lambda+3\rho)$.
Case 1.3. $\Gamma=0$,
From (*iii*) of Lemma 2.3 and inequalities (3.1),(3.2), and Lemma 2.5,

$$\begin{aligned}
& (n+\lambda+\rho)T(r,\xi) \leq N(r,0;\mathcal{F}|2)+N(r,\infty;\mathcal{F}|2)+N(r,0;\mathcal{G}|2) \\
+ \quad & N(r,\infty;\mathcal{G}|2)+2\overline{N}(r,1;\mathcal{F}|L)+\overline{N}(r,1;\mathcal{G}|L)+S(r,\xi)+S(r,\zeta) \\
\leq \quad & N(r,0;\mathcal{F}_1|\tau+2)+N(r,0;\mathcal{G}_1|\tau+2)+2N(r,0;\mathcal{F}_1|\tau+1) \\
+ \quad & N(r,0;\mathcal{G}_1|\tau+1)+S(r,\xi)+S(r,\zeta) \\
\leq \quad & N(r,0;\xi^n|\tau+2)+N(r,0;\xi^\lambda-1|\tau+2) \\
+ \quad & N(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p\xi)^{\mu_i}|\tau+2)+2N(r,0;\xi^n|\tau+1) \\
+ \quad & 2N(r,0;\xi^\lambda-1|\tau+1)+2N(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p\xi)^{\mu_i}|\tau+1)+S(r,\xi) \\
+ \quad & N(r,0;\zeta^n|\tau+2)+N(r,0;\zeta^\lambda-1|\tau+2) \\
+ \quad & N(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}|\tau+2)+N(r,0;\zeta^n|\tau+1) \\
+ \quad & N(r,0;\zeta^\lambda-1|\tau+1)+N(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}|\tau+1)+S(r,\zeta) \\
\leq \quad & (\frac{1}{\iota}(3\tau+4)+3\lambda+3\rho)T(r,\xi)+(\frac{1}{\iota}(2\tau+3)+2\lambda+2\rho)T(r,\zeta) \\
+ \quad & S(r,\xi)+S(r,\zeta).
\end{aligned} \tag{3.7}$$

Similarly we can show that,

$$\begin{aligned}
& (n+\lambda+\rho)T(r,\zeta) \leq (\frac{1}{\iota}(3\tau+4)+3\lambda+3\rho)T(r,\zeta)+S(r,\zeta) \\
& +(\frac{1}{\iota}(2\tau+3)+2\lambda+2\rho)T(r,\xi)+S(r,\xi).
\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) we have, $[n-(\frac{1}{\iota}(5\tau+7)+4\lambda+4\rho)](T(r,\xi)+T(r,\zeta)) \leq S(r,\xi)+S(r,\zeta)$, which is contradiction as $n>\frac{1}{\iota}(5\tau+7)+4\lambda+4\rho$.

Case 2. $\Omega\equiv 0$,
Now integrating twice we find, $\frac{1}{\mathcal{G}-1}=\frac{U}{\mathcal{F}-1}+V$, where $U(\neq 0)$ and V are two complex constants. Which implies that,

$$\mathcal{G}=\frac{(V+1)\mathcal{F}+(U-V-1)}{V\mathcal{F}+(U-V)}, \tag{3.9}$$

and

$$\mathcal{F} = \frac{(V-U)\mathcal{G} + (U-V-1)}{V\mathcal{G} - (V-1)}. \tag{3.10}$$

Now we discuss following subcases:
Subcase 2.1. Let $V \neq 0, -1$. We obtain from (3.9), $\overline{N}(r, \frac{V+1}{V}; \mathcal{G}) = \overline{N}(r, \infty; \mathcal{F})$. Using 2^{nd} part of Lemma 2.2 on SFT we have,

$$\begin{aligned}
T(r,\mathcal{G}) &\leq \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,\frac{V+1}{V};\mathcal{G}) + S(r,\mathcal{G}) \\
&\leq \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,\infty;\mathcal{F}) + S(r,\mathcal{G}) \\
&\leq N(r,0;\zeta^n|\tau+1) + N(r,0;\zeta^\lambda - 1|\tau+1) \\
&+ N(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p \zeta)^{\mu_i}|\tau+1) + S(r,\xi) + S(r,\zeta),
\end{aligned}$$

hence,

$$(n+\lambda+\rho)T(r,\zeta) \leq (\frac{1}{\iota}(\tau+1) + \lambda + \rho)T(r,\zeta) + S(r,\xi) + S(r,\zeta). \tag{3.11}$$

We assume that $U-V-1 \neq 0$, then follows from (3.13) that $N(r, \frac{-U+V-1}{V+1}; \mathcal{F}) = N(r, 0; \mathcal{G})$. From Nevanlinna's SFT we have,

$$\begin{aligned}
T(r,\mathcal{F}) &\leq \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,\frac{-U+V-1}{V+1};\mathcal{F}) + S(r,\mathcal{F}) \\
&\leq \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}) + S(r,\mathcal{F}) \\
&\leq N(r,0;\xi^n|\tau+1) + N(r,0;\xi^\lambda - 1|\tau+1) \\
&+ N(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}|\tau+1) + S(r,\xi) + N(r,0;\zeta^n|\tau+1) \\
&+ N(r,0;\zeta^\lambda - 1|\tau+1) + N(r,0;\prod_{i=1}^{\eta}(\Delta_\omega^p \zeta)^{\mu_i}|\tau+1) + S(r,\zeta),
\end{aligned}$$

hence,

$$(n+\lambda+\rho)T(r,\xi) \leq (\frac{1}{\iota}(\tau+1)+\lambda+\rho)(T(r,\xi)+T(r,\zeta)) + S(r,\xi) + S(r,\zeta). \tag{3.12}$$

Combining (3.11) and (3.12), we deduce that, $(n+\lambda+\rho)(T(r,\xi)+T(r,\zeta)) \leq (\frac{1}{\iota}(\tau+1)+\lambda+\rho)T(r,\xi) + 2(\frac{1}{\iota}(\tau+1)+\lambda+\rho)T(r,\zeta) + S(r,\xi) + S(r,\zeta)$, which implies a contradiction. Therefore we assume $U-V-1=0$, then it follows from (3.9) that, $\overline{N}(r, \frac{-1}{V}; \mathcal{F}) = \overline{N}(r, \infty; \mathcal{G})$. Using Nevanlinna's SFT we have,

$$\begin{aligned}
T(r,\mathcal{F}) &\leq \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,\frac{-1}{V};\mathcal{F}) + S(r,\mathcal{F}) \\
&\leq \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,\infty;\mathcal{G}) + S(r,\mathcal{F}) + S(r,\mathcal{G}) \\
&\leq N(r,0;\xi^n|\tau+1) + N(r,0;\xi^\lambda - 1|\tau+1)
\end{aligned}$$

$$+ \quad N(r,0;\prod_{i=1}^{\eta}(\Delta_{\omega}^{p}\xi)^{\mu_i}|\tau+1) + S(r,\xi) + S(r,\zeta),$$

hence,

$$\begin{aligned}(n+\lambda+\rho)T(r,\xi) &\leq (\frac{1}{\iota}(\tau+1)+\lambda+\rho)T(r,\xi)\\ &+ S(r,\xi)+S(r,\zeta). \end{aligned} \tag{3.13}$$

Combining (3.11) and (3.13) we deduce that, $(n+\lambda+\rho)(T(r,\xi)+T(r,\zeta)) \leq (\frac{1}{\iota}(\tau+1)+\lambda+\rho)(T(r,\xi)+T(r,\zeta))+S(r,\xi)+S(r,\zeta)$, which implies a contradiction.
Subcase 2.2. $V=-1$.We obtain from (3.9) and (3.10) that, $\mathcal{G}=\frac{U}{U+1-\mathcal{F}}$ and $\mathcal{F}=\frac{(U+1)\mathcal{G}-U}{\mathcal{F}}$. If $U+1\neq 0$, then, $\overline{N}(r,U+1;\mathcal{F})=\overline{N}(r,\infty;\mathcal{G})$ and $\overline{N}(r,\frac{U}{U+1};\mathcal{G})=\overline{N}(r,0;\mathcal{F})$. Now following the same argument as Subcase 2.1., we arrive at a contradiction. Therefore $U+1=0$ and it implies that $\mathcal{F}\mathcal{G}=1$. Hence, $\mathcal{F}_1^{(\tau)}\mathcal{G}_1^{(\tau)}=\alpha^2(z)$,

$$\begin{aligned} &\Rightarrow \quad [\xi^n(\xi^\lambda-1)\prod_{i=1}^{\eta}(\Delta_{\omega}^{p}\xi)^{\mu_i}]^{(\tau)}[\zeta^n(\zeta^\lambda-1)\prod_{i=1}^{\eta}(\Delta_{\omega}^{p}\zeta)^{\mu_i}]^{(\tau)}=\alpha^2(z)\\ &\Rightarrow \quad [\xi^n(\xi-1)(\xi^{\lambda-1}+\xi^{\lambda-2}+...+1)\prod_{i=1}^{\eta}(\Delta_{\omega}^{p}\xi)^{\mu_i}]^{(\tau)}\times\\ &\qquad [\zeta^n(\zeta-1)(\zeta^{\lambda-1}+\zeta^{\lambda-2}+...+1)\prod_{i=1}^{\eta}(\Delta_{\omega}^{p}\zeta)^{\mu_i}]^{(\tau)}=\alpha^2(z). \end{aligned} \tag{3.14}$$

From (3.14) we deduce that, $N(r,0;\xi)=S(r,\xi)=N(r,1;\xi)$, then, $\delta(0,\xi)=1=\delta(1,\xi)=\delta(\infty,\xi)$. Hence, $\sum_{\alpha=0,1,\infty}\delta(\alpha,\xi)=3$, which contradict the inequality (1.1).
Subcase 2.3. $V=0$. We obtain from (3.9) and (3.10) that, $\mathcal{G}=\frac{\mathcal{F}+U-1}{U}$ and $\mathcal{F}=U\mathcal{G}+1-U$. If $U-1\neq 0$, then, $\overline{N}(r,1-U;\mathcal{F})=\overline{N}(r,0;\mathcal{G})$ and $\overline{N}(r,\frac{U-1}{U};\mathcal{G})=\overline{N}(r,0;\mathcal{F})$. Now following the same argument as Subcase 2.1. we arrive at a contradiction. Therefore $U-1=0$ and it implies that $\mathcal{F}=\mathcal{G}$. Hence,
$[\xi^n(\xi^\lambda-1)\prod_{i=1}^{\eta}(\Delta_{\omega}^{p}\xi)^{\mu_i}]^{(\tau)}=[\zeta^n(\zeta^\lambda-1)\prod_{i=1}^{\eta}(\Delta_{\omega}^{p}\zeta)^{\mu_i}]^{(\tau)}$.
Integrating, we find,
$[\xi^n(\xi^\lambda-1)\prod_{i=1}^{\eta}(\Delta_{\omega}^{p}\xi)^{\mu_i}]^{(\tau-1)}=[\zeta^n(\zeta^\lambda-1)\prod_{i=1}^{\eta}(\Delta_{\omega}^{p}\zeta)^{\mu_i}]^{(\tau-1)}+c_{\tau-1}$, where $c_{\tau-1}$ is complex constant. If $c_{\tau-1}\neq 0$, then using Lemma 2.9, we find that $n\leq\frac{2}{\iota}(\tau+1)+\lambda+\rho$ which is contradiction as $n>\frac{2}{\iota}(\tau+2)+\lambda+\rho$. Then $c_{\tau-1}=0$. Proceeding the process upto τ-times we find that,

$$\xi^n(\xi^\lambda-1)\prod_{i=1}^{\eta}(\Delta_{\omega}^{p}\xi)^{\mu_i}=\zeta^n(\zeta^\lambda-1)\prod_{i=1}^{\eta}(\Delta_{\omega}^{p}\zeta)^{\mu_i}. \tag{3.15}$$

We assume that $\xi=\kappa\zeta$ and if κ is complex constant and $\kappa\neq 1$, then $\Delta_{\omega}^{p}\xi=\kappa\Delta_{\omega}^{p}\zeta$. Hence from (3.15), we deduce that, $(\kappa\zeta)^n((\kappa\zeta)^\lambda-1)\kappa^\rho\prod_{i=1}^{\eta}(\Delta_{\omega}^{p}\zeta)^{\mu_i}=\zeta^n(\zeta^\lambda-1)\prod_{i=1}^{\eta}(\Delta_{\omega}^{p}\zeta)^{\mu_i}\Rightarrow\zeta^\lambda=\frac{\kappa^{n+\rho}-1}{\kappa^{n+\lambda+\rho}-1}$.

Since, ζ is entire function, then zeros of $\kappa^{n+\rho}-1=0$ and $\kappa^{n+\lambda+\rho}-1=0$ are coincided. But it is impossible as $n>\frac{2}{\iota}(\tau+2)+\lambda+\rho$. Then $\kappa=1$. Hence, $\xi(z)\equiv\zeta(z)$.
Again if κ be a variable, then, $\Delta_\omega^p\xi=\Delta_\omega^p\kappa\zeta$. Hence from (3.15), we deduce that, $(\kappa\zeta)^n((\kappa\zeta)^\lambda-1)\prod_{i=1}^{\eta}(\Delta_\omega^p\kappa\zeta)^{\mu_i}=\zeta^n(\zeta^\lambda-1)\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}\Rightarrow\zeta^\lambda=\frac{\kappa^{n+\lambda}\prod_{i=1}^{\eta}(\Delta_\omega^p\kappa\zeta)^{\mu_i}-\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}}{\kappa^n\prod_{i=1}^{\eta}(\Delta_\omega^p\kappa\zeta)^{\mu_i}-\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}}$. This complete the proof of the theorem. □

Theorem 3.2. *Let $\xi(z)$ and $\zeta(z)$ be two transcendental entire functions of finite order and multiplicity of zeros of $\xi(z)$ and $\zeta(z)$ is atleast ι. Let $\alpha(z)(\not\equiv 0)$ be a small function with respect to $\xi(z)$ and $\zeta(z)$ and let $\omega\in\mathbb{C}\setminus\{0\}$, $\iota,\mu_i\in\mathbb{Z}^+$ where $i=1,2,...,\eta$ and n, $\lambda(\geq 1)$, $\tau(\geq 0)$ p, $\eta\in\mathbb{N}\cup\{0\}$ and $\Delta_\omega^p\xi\not\equiv 0$, $\Delta_\omega^p\zeta\not\equiv 0$. Let $[\xi^n(\xi-1)^\lambda\prod_{i=1}^{\eta}(\Delta_\omega^p\xi)^{\mu_i}]^{(\tau)}$ and $[\zeta^n(\zeta-1)^\lambda\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}]^{(\tau)}$ share $\omega(\alpha,\Gamma)$, then, for one of following condition,*
(i) $\Gamma\geq 2$,
(a) $\lambda\leq\tau+2$ *and* $n>\frac{2}{\iota}(\tau+2)+\lambda+\rho$;
(b) $\lambda>\tau+2$ *and* $n>\frac{2}{\iota}(\tau+2)+2\tau-\lambda+\rho+4$;
(ii) $\Gamma=1$,
(a) $\lambda\leq\tau+1$ *and* $n>\frac{1}{2}(\frac{1}{\iota}(5\tau+9)+3\lambda+3\rho)$;
(b) $\lambda=\tau+2$ *and* $n>\frac{1}{2}(\frac{1}{\iota}(5\tau+9)+2\lambda+\tau+3\rho+1)$;
(c) $\lambda>\tau+2$ *and* $n>\frac{1}{2}(\frac{1}{\iota}(5\tau+9)+5\tau-2\lambda+3\rho+9)$;
(iii) $\Gamma=0$,
(a) $\lambda\leq\tau+1$ *and* $n>\frac{1}{\iota}(5\tau+7)+4\lambda+4\rho$;
(b) $\lambda=\tau+2$ *and* $n>\frac{1}{\iota}(5\tau+7)+\lambda+3\tau+4\rho+3$;
(c) $\lambda>\tau+2$ *and* $n>\frac{1}{\iota}(5\tau+7)+5\tau-\lambda+4\rho+7$;
either, $\xi(z)=\kappa\zeta(z)$, where κ is a constant and $\kappa^\chi=1$, where $\chi=GCD\{\lambda+n+\rho,\lambda+n+\rho-1,...,\lambda+n+\rho-i,...,n+\rho\}$; or,
$\xi(z)=\kappa\zeta(z)$, where κ is a variable, then ξ and ζ satisfy the algebraic equation $R(\xi,\zeta)=0$, where, $R(\phi_1,\phi_2)=\phi_1^n(\phi_1-1)^\lambda\prod_{i=1}^{\eta}(\Delta_\omega^p\phi_1)^{\mu_i}-\phi_2^n(\phi_2-1)^\lambda\prod_{i=1}^{\eta}(\Delta_\omega^p\phi_2)^{\mu_i}$.

Proof. Let us assume that $\mathcal{F}_1=\xi^n(\xi-1)^\lambda\prod_{i=1}^{\eta}(\Delta_\omega^p\xi)^{\mu_i}$ and $\mathcal{G}_1=\zeta^n(\zeta-1)^\lambda\prod_{i=1}^{\eta}(\Delta_\omega^p\zeta)^{\mu_i}$, and $\mathcal{F}=\frac{\mathcal{F}_1^{(\tau)}}{\alpha(z)}$ and $\mathcal{G}=\frac{\mathcal{G}_1^{(\tau)}}{\alpha(z)}$. Then $\mathcal{F}$ and $\mathcal{G}$ are transcendental mero-morphic function that share $\omega(1,\Gamma)$, except the zeros and poles of $\alpha(z)$. From Lemma 2.2 and Lemma 2.7,
$N(r,0;\mathcal{F}|2)\leq N(r,0;\mathcal{F}_1^{(\tau)}|2)+S(r,\xi)$
$\leq T(r,\mathcal{F}_1^{(\tau)})-(n+\lambda+\rho)T(r,\xi)+N(r,0;\mathcal{F}_1|\tau+2)+S(r,\xi)$
$\leq T(r,\mathcal{F})-(n+\lambda+\rho)T(r,\xi)+N(r,0;\mathcal{F}_1|\tau+2)+S(r,\xi)$,
hence,

$$(n+\lambda+\rho)T(r,\xi)\leq T(r,\mathcal{F})+N(r,0;\mathcal{F}_1|\tau+2)-N(r,0;\mathcal{F}|2)+S(r,\xi). \quad (3.16)$$

Again from Lemma 2.2, we have,

$$N(r,0;\mathcal{F}|2)\quad\leq\quad N(r,0;\mathcal{F}_1^{(\tau)}|2)+S(r,\xi)$$

$$\leq \quad N(r,0;\mathcal{F}_1|\tau+2)+S(r,\xi). \tag{3.17}$$

Let us consider, $\Omega=(\frac{\mathcal{F}^{(2)}}{\mathcal{F}^{(1)}}-\frac{2\mathcal{F}^{(1)}}{\mathcal{F}-1})-(\frac{\mathcal{G}^{(2)}}{\mathcal{G}^{(1)}}-\frac{2\mathcal{G}^{(1)}}{\mathcal{G}-1})$.
Case 1. $\Omega\not\equiv 0$,
Since $\mathcal{F}$ and $\mathcal{G}$ share $\omega(1,\Gamma)$, we discuss all the cases of Lemma 2.3,
Subcase 1.1. $\Gamma\geq 2$. Then from (3.16) and (3.17), we deduce that,

$$\begin{aligned}&(n+\lambda+\rho)T(r,\xi)\leq N(r,0;\mathcal{F}|2)+N(r,\infty;\mathcal{F}|2)+N(r,0;\mathcal{G}|2)\\ +\quad & N(r,\infty;\mathcal{G}|2)+S(r,\xi)+S(r,\zeta)\\ \leq\quad & N(r,0;\mathcal{F}_1|\tau+2)+N(r,0;\mathcal{G}_1|\tau+2)+S(r,\xi)+S(r,\zeta),\end{aligned} \tag{3.18}$$

Subsubcase 1.1.1. $\lambda\leq\tau+2$. Then from (3.18), we deduce that,

$$\begin{aligned}(n+\lambda+\rho)T(r,\xi) \quad &\leq \quad (\frac{1}{\iota}(\tau+2)+\lambda+\rho)(T(r,\xi)+T(r,\zeta))\\ &+ \quad S(r,\xi)+S(r,\zeta).\end{aligned} \tag{3.19}$$

Similarly we can show that,

$$(n+\lambda+\rho)T(r,\zeta)\leq(\frac{1}{\iota}(\tau+2)+\lambda+\rho)(T(r,\xi)+T(r,\zeta))+S(r,\xi)+S(r,\zeta). \tag{3.20}$$

Combining (3.19) and (3.20) we have, $[n-(\frac{2}{\iota}(\tau+2)+\lambda+\rho)](T(r,\xi)+T(r,\zeta))\leq S(r,\xi)+S(r,\zeta)$, which is contradiction as $n>\frac{2}{\iota}(\tau+2)+\lambda+\rho$.
Subcase 1.1.2. $\lambda>\tau+2$. Then from (3.13), we deduce that,

$$\begin{aligned}(n+\lambda+\rho)T(r,\xi) \quad &\leq \quad (\frac{1}{\iota}(\tau+2)+\tau+2+\rho)(T(r,\xi)+T(r,\zeta))\\ &+ \quad S(r,\xi)+S(r,\zeta).\end{aligned} \tag{3.21}$$

Similarly we can show that,

$$(n+\lambda+\rho)T(r,\zeta)\leq(\frac{1}{\iota}(\tau+2)+\tau+2+\rho)(T(r,\xi)+T(r,\zeta))+S(r,\xi)+S(r,\zeta). \tag{3.22}$$

Combining (3.21) and (3.22) we have,

$$[n-(\frac{2}{\iota}(\tau+2)+2\tau-\lambda+\rho+4)](T(r,\xi)+T(r,\zeta))+S(r,\xi)+S(r,\zeta),$$

which is contradiction as $n>\frac{2}{\iota}(\tau+2)+2\tau-\lambda+\rho+4$.
Subcase 1.2. $\Gamma=1$. From (ii) of Lemma 2.3 and inequalities (3.16), (3.17) and Lemma 2.4,

$$\begin{aligned}&(n+\lambda+\rho)T(r,\xi)\leq N(r,0;\mathcal{F}|2)+N(r,\infty;\mathcal{F}|2)+N(r,0;\mathcal{G}|2)\\ +\quad & N(r,\infty;\mathcal{G}|2)+\overline{N}(r,1;\mathcal{F}|L)+S(r,\xi)+S(r,\zeta)\\ \leq\quad & N(r,0;\mathcal{F}_1|\tau+2)+N(r,0;\mathcal{G}_1|\tau+2)+\frac{1}{2}N(r,0;\mathcal{F}_1|\tau+1)\\ +\quad & S(r,\xi)+S(r,\zeta),\end{aligned} \tag{3.23}$$

Subsubcase 1.2.1. $\lambda\leq\tau+1$. Then from (3.23), we deduce that,

$$(n+\lambda+\rho)T(r,\xi) \quad \leq \quad \frac{1}{2}(\frac{1}{\iota}(3\tau+5)+3\lambda+3\rho)T(r,\xi)+S(r,\xi)$$

$$+ \quad (\frac{1}{\iota}(\tau+2)+\lambda+\rho)T(r,\zeta)+S(r,\zeta). \qquad (3.24)$$

Similarly we can show that,

$$\begin{aligned}(n+\lambda+\rho)T(r,\zeta) &\leq \frac{1}{2}(\frac{1}{\iota}(3\tau+5)+3\lambda+3\rho)T(r,\zeta)+S(r,\zeta)\\ &+ (\frac{1}{\iota}(\tau+2)+\lambda+\rho)T(r,\xi)+S(r,\xi). \qquad (3.25)\end{aligned}$$

Combining (3.28) and (3.29) we have, $[n-\frac{1}{2}(\frac{1}{\iota}(5\tau+9)+3\lambda+3\rho)](T(r,\xi)+T(r,\zeta))\leq S(r,\xi)+S(r,\zeta)$, which is contradiction as $n>\frac{1}{2}(\frac{1}{\iota}(5\tau+9)+3\lambda+3\rho)$.
Subsubcase 1.2.2. $\lambda=\tau+2$. Then from (3.23), we deduce that,

$$\begin{aligned}&(n+\lambda+\rho)T(r,\xi)\leq\frac{1}{2}(\frac{1}{\iota}(3\tau+5)+2\lambda+\tau+3\rho+1)T(r,\xi)\\ &+(\frac{1}{\iota}(\tau+2)+\lambda+\rho)T(r,\zeta)+S(r,\xi)+S(r,\zeta). \qquad (3.26)\end{aligned}$$

Similarly we can show that,

$$\begin{aligned}&(n+\lambda+\rho)T(r,\zeta)\leq\frac{1}{2}(\frac{1}{\iota}(3\tau+5)+2\lambda+\tau+3\rho+1)T(r,\zeta)\\ &+(\frac{1}{\iota}(\tau+2)+\lambda+\rho)T(r,\xi)+S(r,\zeta)+S(r,\xi). \qquad (3.27)\end{aligned}$$

Combining (3.26) and (3.27) we have, $[n-\frac{1}{2}(\frac{1}{\iota}(5\tau+9)+2\lambda+\tau+3\rho+1)](T(r,\xi)+T(r,\zeta))\leq S(r,\xi)+S(r,\zeta)$, which is contradiction as $n>\frac{1}{2}(\frac{1}{\iota}(5\tau+9)+2\lambda+\tau+3\rho+1)$.
Subcase 1.2.3. $\lambda>\tau+2$. Then from (3.23), we deduce that,

$$\begin{aligned}(n+\lambda+\rho)T(r,\xi) &\leq \frac{1}{2}(\frac{1}{\iota}(3\tau+5)+3\tau+5+3\rho)T(r,\xi)+S(r,\xi)\\ &+ (\frac{1}{\iota}(\tau+2)+\tau+2+\rho)T(r,\zeta)+S(r,\zeta). \qquad (3.28)\end{aligned}$$

Similarly we can show that,

$$\begin{aligned}(n+\lambda+\rho)T(r,\zeta) &\leq \frac{1}{2}(\frac{1}{\iota}(3\tau+5)+3\tau+5+3\rho)T(r,\zeta)+S(r,\zeta)\\ &+ (\frac{1}{\iota}(\tau+2)+\tau+2+\rho)T(r,\xi)+S(r,\xi) \qquad (3.29)\end{aligned}$$

Combining (3.28) and (3.29) we have, $[n-\frac{1}{2}(\frac{1}{\iota}(5\tau+9)+5\tau-2\lambda+3\rho+9)](T(r,\xi)+T(r,\zeta))\leq S(r,\xi)+S(r,\zeta)$, which is contradiction as $n>\frac{1}{2}(\frac{1}{\iota}(5\tau+9)+5\tau-2\lambda+3\rho+9)$.
Subcase 1.3. $\Gamma=0$. From (iii) of Lemma 2.3 and inequalities (3.16), (3.17) and Lemma 2.5, we have,

$$\begin{aligned}&(n+\lambda+\rho)T(r,\xi)\leq N(r,0;\mathcal{F}|2)+N(r,\infty;\mathcal{F}|2)+N(r,0;\mathcal{G}|2)\\ + \quad & N(r,\infty;\mathcal{G}|2)+2\overline{N}(r,1;\mathcal{F}|L)+\overline{N}(r,1;\mathcal{G}|L)+S(r,\xi)+S(r,\zeta)\\ \leq \quad & N(r,0;\mathcal{F}_1|\tau+2)+N(r,0;\mathcal{G}_1|\tau+2)+2N(r,0;\mathcal{F}_1|\tau+1)\end{aligned}$$

$$+ \quad N(r,0;\mathcal{G}_1|\tau+1)+S(r,\xi)+S(r,\zeta). \tag{3.30}$$

Subsubcase 1.3.1. $\lambda \leq \tau+1$. Then from (3.30), we deduce that,

$$(n+\lambda+\rho)T(r,\xi) \leq (\frac{1}{\iota}(3\tau+4)+3\lambda+3\rho)T(r,\xi)+S(r,\xi)$$
$$+ \quad (\frac{1}{\iota}(2\tau+3)+2\lambda+2\rho)T(r,\zeta)+S(r,\zeta). \tag{3.31}$$

Similarly we can show that,

$$(n+\lambda+\rho)T(r,\zeta) \leq (\frac{1}{\iota}(3\tau+4)+3\lambda+3\rho)T(r,\zeta)+S(r,\zeta)$$
$$+ \quad (\frac{1}{\iota}(2\tau+3)+2\lambda+2\rho)T(r,\xi)+S(r,\xi). \tag{3.32}$$

Combining (3.31) and (3.32) we have, $[n-(\frac{1}{\iota}(5\tau+7)+4\lambda+4\rho)](T(r,\xi)+T(r,\zeta)) \leq S(r,\xi)+S(r,\zeta)$, which is contradiction as $n > (\frac{1}{\iota}(5\tau+7)+4\lambda+4\rho)$.
Subsubcase 1.3.2. $\lambda = \tau+2$. Then from (3.30), we deduce that,

$$(n+\lambda+\rho)T(r,\xi) \leq (\frac{1}{\iota}(3\tau+4)+\lambda+2\tau+3\rho+2)T(r,\xi)$$
$$+ \quad (\frac{1}{\iota}(2\tau+3)+\lambda+\tau+2\rho+1)T(r,\zeta)+S(r,\xi)+S(r,\zeta). \tag{3.33}$$

Similarly we can show that,

$$(n+\lambda+\rho)T(r,\zeta) \leq (\frac{1}{\iota}(3\tau+4)+\lambda+2\tau+3\rho+2)T(r,\zeta)$$
$$+ \quad (\frac{1}{\iota}(2\tau+3)+\lambda+\tau+2\rho+1)T(r,\xi)+S(r,\zeta)+S(r,\xi). \tag{3.34}$$

Combining (3.33) and (3.34) we have, $[n-(\frac{1}{\iota}(5\tau+7)+\lambda+3\tau+4\rho+3)](T(r,\xi)+T(r,\zeta)) \leq S(r,\xi)+S(r,\zeta)$, which is contradiction as $n > \frac{1}{\iota}(5\tau+7)+\lambda+3\tau+4\rho+3$.
Subcase 1.3.2. $\lambda > \tau+2$. Then from (3.30), we deduce that,

$$(n+\lambda+\rho)T(r,\xi) \leq (\frac{1}{\iota}(3\tau+4)+3\tau+4+3\rho)T(r,\xi)+S(r,\xi)$$
$$+ \quad (\frac{1}{\iota}(2\tau+3)+2\tau+3+2\rho)T(r,\zeta)+S(r,\zeta). \tag{3.35}$$

Similarly we can show that,

$$(n+\lambda+\rho)T(r,\zeta) \leq (\frac{1}{\iota}(3\tau+4)+3\tau+4+3\rho)T(r,\zeta)+S(r,\zeta)$$
$$+ \quad (\frac{1}{\iota}(2\tau+3)+2\tau+3+2\rho)T(r,\xi)+S(r,\xi). \tag{3.36}$$

Combining (3.35) and (3.36) we have, $[n-(\frac{1}{\iota}(5\tau+7)+5\tau-\lambda+4\rho+7)](T(r,\xi)+T(r,\zeta)) \leq S(r,\xi)+S(r,\zeta)$, which is contradiction as $n > \frac{1}{\iota}(5\tau+7)+5\tau-\lambda+4\rho+7$.
Case 2. $\Omega \equiv 0$,
Using same technic and proceeding similarly as Case 2. of Theorem 3.1, we ultimately obtain, $\mathcal{F}=\mathcal{G}$. Then,
$[\xi^n(\xi-1)^\lambda \prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}]^{(\tau)} = [\zeta^n(\zeta-1)^\lambda \prod_{i=1}^{\eta}(\Delta_\omega^p \zeta)^{\mu_i}]^{(\tau)}$.

Integrating, we find,
$[\xi^n(\xi-1)^\lambda \prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}]^{(\tau-1)} = [\zeta^n(\zeta-1)^\lambda \prod_{i=1}^{\eta}(\Delta_\omega^p \zeta)^{\mu_i}]^{(\tau-1)} + c_{\tau-1}$,
where $c_{\tau-1}$ is complex constant. If $c_{\tau-1} \neq 0$, then using Lemma 2.10, we find that $n \leq \frac{2}{\iota}(\tau+1)+\lambda+\rho$, or, $n \leq \frac{2}{\iota}(\tau+1)+2\tau+\rho-\lambda+2$, which is contradiction as $n > \frac{2}{\iota}(\tau+2)+\lambda+\rho$, or, $n > \frac{2}{\iota}(\tau+2)+2\tau+\rho-\lambda+4$ accordingly. Then $c_{\tau-1} = 0$. Proceeding the process upto τ-times we find that,

$$\xi^n(\xi-1)^\lambda \prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i} = \zeta^n(\zeta-1)^\lambda \prod_{i=1}^{\eta}(\Delta_\omega^p \zeta)^{\mu_i}. \tag{3.37}$$

We assume that, $\xi = \kappa\zeta$ and if κ is complex constant, then $\Delta_\omega^p \xi = \kappa \Delta_\omega^p \zeta$. Hence from (3.37), we deduce that,
$(\kappa\zeta)^n(\kappa\zeta-1)^\lambda \kappa^\rho \prod_{i=1}^{\eta}(\Delta_\omega^p \zeta)^{\mu_i} = \zeta^n(\zeta-1)^\lambda \prod_{i=1}^{\eta}(\Delta_\omega^p \zeta)^{\mu_i}$
$\Rightarrow \zeta^n \prod_{i=1}^{\eta}(\Delta_\omega^p \zeta)^{\mu_i}[\zeta^\lambda(\kappa^{n+\lambda+\rho}-1) - \binom{\lambda}{1}\zeta^{\lambda-1}(\kappa^{n+\lambda+\rho-1}-1)+...+$
$(-1)^i\binom{\lambda}{i}\zeta^{\lambda-i}(\kappa^{n+\lambda+\rho-i}-1)+...+(-1)^\lambda(\kappa^{n+\rho}-1)] = 0$. Since, $\zeta^n \prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i} \not\equiv 0$. Then,
$\zeta^\lambda(\kappa^{n+\lambda+\rho}-1) - \binom{\lambda}{1}\zeta^{\lambda-1}(\kappa^{n+\lambda+\rho-1}-1)+...+$
$(-1)^i\binom{\lambda}{i}\zeta^{\lambda-i}(\kappa^{n+\lambda+\rho-i}-1)+...+(-1)^\lambda(\kappa^{n+\rho}-1) = 0$, which implies that $\kappa^\chi = 1$, where $\chi = GCD\{\lambda+n+\rho, \lambda+n+\rho-1, ..., \lambda+n+\rho-i, ..., n+\rho\}$. If κ a variable, then $\Delta_\omega^p \xi = \Delta_\omega^p \kappa\zeta$. Then from (3.37) we say that ξ and ζ satisfy the algebraic equation $R(\xi,\zeta) = 0$, where, $R(\phi_1,\phi_2) = \phi_1^n(\phi_1-1)^\lambda \prod_{i=1}^{\eta}(\Delta_\omega^p \phi_1)^{\mu_i} - \phi_2^n(\phi_2-1)^\lambda \prod_{i=1}^{\eta}(\Delta_\omega^p \phi_2)^{\mu_i}$. Hence the theorem. $\square$

Corollary 3.3. *If we replace product of difference operators* $\prod_{i=1}^{\eta}(\Delta_\omega^p \xi)^{\mu_i}$ *by* $\prod_{i=1}^{\eta}(\Delta_{\omega_i}^p \xi)^{\mu_i}$ *where* $\omega_i \in \mathbb{C}\setminus\{0\}$*, then we can find same results for each of two theorems.*

Remark 3.1. If $\lambda = 1$, then theorem 3.1 and theorem 3.2 will be equivalent with each other.

Remark 3.2. In both of theorem 3.1 and theorem 3.2, value of n will be continuously decreasing for increasing value of ι.

4. Open Problems

We can pose following problems from our results,
1. Can n be further reduced in theorem 3.1 and theorem 3.2?
2. Is it possible to replace the transcendental entire functions in theorem 3.1 and theorem 3.2 by transcendental mero-morphic functions?

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