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NORMAL AND COSETS OF (γ, ∂)-FUZZY HX-SUBGROUPS

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ABSTRACT. In this paper, the concept of (γ, ∂)-fuzzy HX-subgroups is introduced. We present some properties of (γ, ∂)-normal fuzzy HX-subgroups and we discuss some results related to (γ, ∂)-fuzzy cosets.

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1. Introduction

As a generalization of subsets notion in ordinary set theory, Zadeh [1] introduced fuzzy subsets concepts. Many algebraic structures have been fuzzified. And it was the first in 1971 by Rosenfeld [2] discussed a fuzzy subgroup concept. Most works recent on fuzzy subgroups used definition of Rosenfeld. Massa'deh and Hatamleh [3, 4] investigated the notions of fuzzy subgroups with operators and L-Q-fuzzy quotient, for more details of fuzzy subgroups and related results see [5, 6, 7]. In 1987, Li Hongxing [8] gave $HX-$group concepts and Luo chengzhong et al [9] introduced the fuzzy $HX-$group concepts. Here after, more important interesting results about a fuzzy $HX-$subgroups and its properties have been obtained (see, e.g., [10, 11, 12, 13, 14] Some extensions of fuzzy subgroups emerged. Bhakat and Das [15] discussed $(\epsilon, \epsilon \vee q)-$ fuzzy subgroup concept, while. In 2003, Yao [16] introduced the concept of $(\lambda, \mu)-$fuzzy normal subgroups as a generalization of $(\epsilon, \epsilon \vee q)-$fuzzy subgroup. Moreover, Chinnadurai and Arul mozhi [17] introduced and studied a $(\eta, \delta)-$bipolar fuzzy ideal and bi-ideal.

In this paper, we conduct a study a bout $(\gamma, \partial) - HX-$ fuzzy subgroups. The paper contains some properties of $(\gamma, \partial) - HX-$fuzzy subgroups, $(\gamma, \partial) - HX-$

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normal fuzzy subgroups and $(\gamma, \partial)-$fuzzy cosets and discussed some of their related results.

2. PRELIMINARIES

Definition 2.1. [1] If G is a non empty set. A fuzzy subset λ of G is a function $\lambda : G \to [0, 1]$.

Definition 2.2. [8] An $HX-$ group on G is a non empty set $\nu \subset 2^G - \{\phi\}$ such that ν is a group with respect to the algebraic operation defined by $XY = \{xy; x \in X$ and $y \in Y\}$, and E is the unit element .

Definition 2.3. [9] If $\nu \subset 2^G - \{\phi\}$ is an $HX-$group on G, and λ is a fuzzy subset defined on G. A fuzzy set δ_λ defined on ν is called fuzzy $HX-$subgroup on ν if for any $X, Y \in \nu$

(1) $\delta_\lambda(XY) \geq \min\{\delta_\lambda(X), \delta_\lambda(Y)\}$
(2) $\delta_\lambda(X^{-1}) = \delta_\lambda(X)$

where $\delta_\lambda(X) = \max\{\lambda(x)$, for all $x \in X \subset G\}$.

Definition 2.4. [2] If λ is a fuzzy subset of ν and $\alpha \in [0, 1]$, suppose that $\lambda_\alpha = \{X \in \nu, \lambda(X) \geq \alpha\}$. Then λ_α is called a level subset of λ.

Lemma 2.5. [11] *A fuzzy subset λ of ν is a fuzzy $HX-$subgroup if and only if $\lambda_\alpha \neq \phi$ is a crisp $HX-$subgroup of ν for every $\alpha \in [0, 1]$.*

Proof. Straightforward. □

Proposition 2.6. *If λ is a fuzzy subset of ν. Then λ is normal fuzzy $HX-$subgroup of ν if and only if $\lambda_\alpha \neq \phi$ is a normal $HX-$subgroup of ν for all $\alpha \in [0, 1]$.*

Proof. Straightforward. □

3. (γ, ∂) FUZZY AND NORMAL FUZZY $HX-$SUBGROUP

Definition 3.1. If λ is a fuzzy subset of ν. λ is called a $(\gamma, \partial)-$ fuzzy $HX-$subgroup of ν if:

(1) $\lambda(XY) \vee \gamma \geq \lambda(X) \wedge \lambda(Y) \wedge \partial$
(2) $\lambda(X^{-1}) \vee \gamma \geq \lambda(X) \wedge \partial$ for all $X, Y \in \nu$.

Since, a $(0, 1)-$ fuzzy $HX-$subgroup is just fuzzy $HX-$subgroup, while a (γ, ∂) fuzzy $HX-$subgroup is a generalization of fuzzy $HX-$subgroup.

Lemma 3.2. *Let λ be a $(\gamma, \partial)-$fuzzy $HX-$subgroup of ν, then $\lambda(E) \vee \gamma \geq \lambda(X) \wedge \partial$ for all $X \in \nu$ and E is the identity in ν.*

Proof. Straightforward. □

Corollary 3.3. *If λ is a $(\gamma, \partial)-$fuzzy $HX-$subgroup of ν. Then*

(1) *If $\lambda(X) \geq \partial$, then $\lambda(E) \geq \partial$for some $X \in \nu$.*
(2) *If $\lambda(X) \leq \partial$ for all $X \in \nu$ and $I = \{X; \gamma < \lambda(X) < \partial\} \neq \phi$, then $\lambda(E) = \max\{\lambda(X); X \in \nu\}$.*

(3) *If* $\lambda(E) \leq \gamma$, *then* $\lambda(X) \leq \gamma$ *for all* $X \in \nu$.

Proof. (1) For some $\lambda(X) \leq \lambda(E)$, if $\lambda(X) \geq \partial$ then $\lambda(E) \vee \gamma \geq \lambda(X) \wedge \partial$ by Lemma3.2; thus $\lambda(E) \geq \partial$.

(2) When $X \in I$ by Lemma 3.2, $\lambda(E) \vee \gamma \geq \lambda(X) \wedge \partial = \lambda(X)$ that is $\lambda(E) \geq \lambda(X)$. Then $\lambda(E) > \gamma$ while, if $X \notin I, \lambda(X) \geq \gamma$ due to $\lambda(X) < \partial.\lambda(E) = \lambda(E) \vee \gamma \geq \lambda(X) \wedge \partial = \lambda(X)$ by Lemma 3.2. Thus, $\lambda(X) \leq \lambda(E)$ for all $X \in \nu$ that is, $\lambda(E) = \max\{\lambda(X); X \in \nu\}$.

(3) If $\lambda(E) \geq \gamma$, thus $\gamma = \lambda(E) \vee \gamma \geq \lambda(X) \wedge \partial$ for all $X \in \nu$, then $\lambda(X) \leq \gamma$ due to $\gamma < \partial$.

□

Proposition 3.4. *If* λ *is a* $(\gamma, \partial)-$ *fuzzy* $HX-$ *subgroup of* ν *and* $\gamma < \lambda(E) < \partial$. *Then:*

(1) $\lambda(X) \leq \lambda(E)$ *hold for all* $X \in \nu$.
(2) $\lambda(X) = \lambda(E)$ *for all* $X \in \lambda_{\lambda(E)}$.

Proof. (1) If $\lambda(X) \geq \lambda(E)$ for some $X \in \nu$, then $\lambda(E) \geq \partial$ by Corollary 3.3 (1), which is contradiction. Hence $\lambda(E) < \partial$ for all $X \in \nu$. Since $E \in I, \lambda(X) \leq \lambda(E)$ satisfy for all $X \in \nu$ by Corollary 3.3(2).

(2) $\lambda(X) \geq \lambda(E)$ for every $X \in \lambda_{\lambda(E)}$ by (1) $\lambda(X) \leq \lambda(E)$ then $\lambda(X) = \lambda(E)$.

□

Lemma 3.5. *If* λ *is a* $(\gamma, \partial)-$*fuzzy subset of* ν, *then* λ *is a* $(\gamma, \partial)-$*fuzzy* $HX-$*subgroup of* ν *iff* $\lambda_\alpha \neq \phi$ *is a* $HX-$*subgroup of* ν *for all* $\alpha \in (\gamma, \partial]$.

Proof. Straightforward. □

Theorem 3.6. *If* λ *is a* $(\gamma, \partial)-$ *fuzzy* $HX-$*subgroup of* ν *and* $X \in \nu$ *then:*

(1) *If* $\lambda(X) \geq \partial$, *then* $\lambda(X^{-1}) \geq \partial$.
(2) *If* $\gamma < \lambda(X) < \partial$, *then* $\partial(X) = \lambda(X^{-1})$.
(3) *If* $\lambda(X) \leq \gamma$, *then* $\lambda(X^{-1}) \leq \gamma$.

Proof. By Lemma 3.5 $\lambda_\alpha \neq \phi$ is a $HX-$subgroup of ν for all $\alpha \in (\gamma, \partial]$.

(1) If $\lambda(X) \geq \partial$, then $X \in \lambda_\partial$. Since λ_∂ is a$HX-$subgroup of $\nu, X^{-1} \in \lambda_\partial$, that is $\lambda(X^{-1}) \geq \partial$.

(2) If $\gamma < \lambda(X) < \partial$, then $\lambda(X) = \lambda(X) \vee \gamma \geq \lambda(X^{-1}) \wedge \partial = \lambda(X^{-1})$. If $\lambda(X^{-1}) \wedge \partial = \partial$, then $\lambda(X^{-1}) \geq \partial$; that is, $\lambda(X) \geq \partial$ which is contradictory to that $\lambda(X) < \partial$. Hence $\lambda(X) \geq \lambda(X^{-1})$. Furthermore, since $X \in \lambda_{\lambda(X)}$ and $\lambda_{\lambda(X)}$ is a $HX-$subgroup of ν we have $X^{-1} \in \lambda_{\lambda(X)}$. Therefore, $\lambda(X^{-1}) \geq \lambda(X)$ and we get $\lambda(X) = \lambda(X^{-1})$.

(3) Assume $\lambda(X^{-1}) > \gamma$. Let $\beta_0 = \min\{\lambda(X^{-1}), \partial\}$. Then $\gamma < \beta_0 \leq \partial$, and $\lambda(X^{-1}) \geq \beta_0$. Thus $X^{-1} \in \lambda_{\beta_0}$, by Lemma 3.5, λ_{β_0} is a $HX-$subgroup of ν and thus $X \in \lambda_{\beta_0}$ therefore $\gamma < \beta_0 \leq \lambda(X)$ which is a contradiction to that $\lambda(X) \leq \gamma$. Hence $\lambda(X^{-1}) \leq \gamma$.

□

Corollary 3.7. *If λ is a $(\gamma,\partial)-$fuzzy $HX-$subgroup of ν and $X, Y \in \nu$.*

1. *If $\lambda(X), \lambda(Y) \geq \partial$, then $\lambda(XY) \geq \partial$.*
2. *If $\gamma < \lambda(X) < \partial, \lambda(X) < \lambda(Y)$, then $\lambda(XY) = \lambda(X) = \lambda(YX)$.*
3. *If $\lambda(X) \leq \gamma, \lambda(Y) > \gamma$, then $\lambda(XY) \leq \gamma$ and $\lambda(YX) \leq \gamma$.*

Proof. For all $\beta \in (\gamma, \partial]\lambda_\beta \neq \phi$ is a $HX-$subgroup of ν by Lemma 3.5.

1. Since $\lambda(X), \lambda(Y) \geq \partial$, thus $X, Y \in \lambda_\partial$ and since λ_∂ is a$HX-$subgroup of ν, we have $XY \in \lambda_\partial$, that is $\lambda(XY) \geq \partial$.
2. If $\lambda(X) = \beta_1, \lambda(Y) = \beta_2$ and $\lambda(XY) = \beta_3$. Then $\gamma < \beta_1 \leq \partial$ and $\beta_2 > \beta_1$. Now, we have $\gamma < \lambda(X) < \lambda(Y)$ then $X, Y \in \lambda_\beta$ and thus $XY \in \lambda_\beta$ by Lemma3.5 (1) therefore $\lambda(XY) \geq \beta_1$ hence $\beta_3 \geq \beta_1$. If $\beta_3 > \beta_1$, let $\beta_0 = \min\{\beta_2, \beta_3, \partial\}$. Then $\gamma < \beta_0 \leq \partial$ and $XY, Y \in \lambda_{\beta_0}$. By Lemma 3.5, λ_{β_0} is a $HX-$subgroup of ν, thus $X = XYY^{-1} \in \lambda_{\beta_0}$. Hence $\beta_1 < \beta_3 \leq \beta_0 \leq \lambda(X)$ which is a contradiction. Therefore, $\beta_3 = \beta_1$, that is $\lambda(XY) = \lambda(X)$ by the same, $\lambda(X) = \lambda(YX)$.
3. Assume $\lambda(XY) > \gamma$. Let $\beta_0 = \min\{\lambda(XY), \lambda(Y), \partial\}$ then $\gamma < \beta_0 \leq \partial$ and $XY, Y \in \lambda_{\beta_0}$. By Lemma 3.5, λ_{β_0} is a $HX-$subgroup of ν, thus $X = XYY^{-1} \in \lambda_{\beta_0}$. It follows that $\gamma < \beta_0 \leq \lambda(X)$, which is a contradiction. Thus $\lambda(XY) \leq \gamma$. By the same $\lambda(YX) \leq \gamma$.

□

Definition 3.8. If λ is a $(\gamma,\partial)-$ fuzzy $HX-$subgroup of ν, λ is a $(\gamma,\partial)-$ normal fuzzy $HX-$subgroup of ν if $\lambda(XYX^{-1}) \vee \gamma \geq \lambda(Y) \wedge \partial$ for all $X, Y \in \nu$.

Proposition 3.9. *If λ is a $(\gamma,\partial)-$ fuzzy $HX-$subgroup of ν, then*

1. *λ is a $(\gamma,\partial)-$ normal fuzzy $HX-$subgroup of ν if and only if $\lambda(XY) \vee \gamma \geq \lambda(YX) \wedge \partial$ for all $X, Y\nu$.*
2. *λ is a $(\gamma,\partial)-$normal fuzzy $HX-$subgroup of ν, iff $\lambda_\beta \neq \phi$is a normal $HX-$subgroup of ν for all $\beta \in (\gamma, \partial]$.*

Proof. Straightforward. □

Theorem 3.10. *If λ is a $(\gamma,\partial)-$ normal fuzzy $HX-$subgroup of ν, then:*

1. *If $\lambda(X) \geq \partial$, then $\lambda(YXY^{-1}) \geq \partial$ for all$X, Y \in \nu$.*
2. *If $\gamma < \lambda(X) < \partial, \lambda(X) < \lambda(Y)$, then $\lambda(YXY^{-1}) = \lambda(X)$ for all $X, Y \in \nu$.*
3. *If $Y \in \nu$ and $\gamma < \lambda(XY) < \partial$, then $\lambda(XY) = \lambda(YX)$.*
4. *If $Y \in \nu$ and $\lambda(XY) \geq \partial$, then $\lambda(YX) \geq \partial$.*
5. *If $Y \in \nu$ and $\lambda(XY) \leq \gamma$, then $\lambda(YX) \leq \gamma$.*

Proof. (1) If $\lambda(X) \geq \partial$, then $X \in \lambda_\partial$. By Proposition 3.9(2), λ_∂ is a normal $HX-$subgroup of ν and hence $YXY^{-1} \in \lambda_\partial$. Thus $\lambda(YXY^{-1})$.

(2) If $\lambda(X) = \beta$. Then $\gamma < \beta < \partial$. By Proposition 3.9(2) λ_β is a normal $HX-$subgroup of ν and hence $YXY^{-1} \in \lambda_\beta$. Thus $\lambda(YXY^{-1}) \geq \beta = \lambda(X)$. Suppose that $\lambda(YXY^{-1}) > \beta$, put $\beta_0 = \min\{\lambda(YXY^{-1}), \partial\}$. Then $\gamma < \beta_0 < \partial$. By Proposition3.9(2), λ_{β_0} is a normal $HX-$subgroup

of ν and thus $YXY^{-1} \in \lambda_{\beta_0}$. Then $X = Y^{-1}(YXY^{-1})Y \in \lambda_{\beta_0}$, we get $\lambda(X) \geq \beta_0 > \beta$ which is a contradiction to $\lambda(X) = \beta$. Then $\lambda(YXY^{-1}) = \lambda(X)$.

(3) If $\gamma < \lambda(XY) < \partial$, then $\lambda(YX) = \lambda(X^{-1}(XY)X) = \lambda(XY)$ by (2), we get $\lambda(XY) = \lambda(YX)$.

(4) If $\lambda(XY) \geq \partial$, then $XY \in \lambda_\partial$ but $X \in \lambda_\partial$ is a normal $HX-$subgroup of ν by Proposition 3.9(2), $YX = X^{-1}(YX)X \in \lambda_\partial$; thus $\lambda(YX) \geq \partial$.

(5) Assume $\lambda(YX) > \gamma$ on the contrary. If $\lambda(YX) \geq \partial$, then by (1) $\lambda(XY) \geq \partial$ and its contradiction to $\lambda(XY) \leq \gamma$, therefore $\lambda(YX) \leq \gamma$. □

4. COSETS OF a $(\gamma, \partial)-$ FUZZY $HX-$SUBGROUP

Definition 4.1. If λ is a $(\gamma, \partial)-$fuzzy $HX-$subgroup of ν and $X \in \nu$. A fuzzy subsets $X\lambda$ and λX of ν define respectively by:
$(X\lambda)(Y) = (\lambda(X^{-1}Y) \vee \gamma) \wedge \partial$
$(\lambda X)(Y) = (\lambda(YX^{-1}) \vee \gamma) \wedge \partial$.
For all $Y \in \nu$. $X\lambda$ and λX will be called a left and right cosets of λ respectively.

Remark 4.1. $E\lambda = \lambda E, \gamma \leq (\lambda X)(Y) \leq \partial$ and $\gamma(X\partial)(Y) \leq \partial$ which are valid for all $Y \in \nu$.

Corollary 4.2. *If λ, δ are a $(\gamma, \partial)-$ fuzzy $HX-$subgroup of ν. Then:*

(1) $X(Y\lambda) = (XY)\lambda$.
(2) $(\lambda X)Y = \lambda(XY)$.
(3) $X\lambda = Y\delta$ *iff* $E\lambda = (X^{-1}Y)\delta$ *and* $E\lambda = (Y^{-1}X)\delta$.
(4) $\lambda X = \delta Y$ *iff* $\lambda E = \delta(X^{-1}Y)$ *and* $\lambda E = \delta(Y^{-1}X)$.
(5) *If* $\lambda = \delta$, *then* $X\lambda = Y\lambda$ *iff* $E\lambda = (X^{-1}Y)\lambda$ *and* $E\lambda = (Y^{-1}X)\delta$.
(6) $\lambda X = \lambda Y$ *iff* $\lambda E = \lambda(X^{-1}Y)$ *and* $\lambda E = \lambda(Y^{-1}X)$. *For all* $X, Y \in \nu$.

Proof. Straightforward. □

Remark 4.2. If λ is a $(\gamma, \partial)-$fuzzy $HX-$subgroup of ν. Then

(1) $\gamma < \lambda(X) < \partial$, then $(E\lambda)(X) = (\lambda X)$.
(2) If $\lambda(X) \geq \partial$, then $(E\lambda)(X) = \partial$.
(3) If $\lambda(X) \leq \gamma$, then $(E\lambda)(X) = \gamma$.

Theorem 4.3. *Let λ be a $(\gamma, \partial)-$fuzzy $HX-$subgroup of ν and $\lambda(E) \geq \partial$, then $X \in \lambda_\partial$ if and only if $X\lambda = E\lambda$.*

Proof. $\Rightarrow$ Suppose that $X \in \lambda_\partial$, since λ_∂ is a $HX-$ subgroup of ν by proposition 3.9(2), then $X^{-1} \in \lambda_\partial$, thus $\lambda(X^{-1}) \geq \partial$ by Theorem 3.6(1). If $Y \in \nu$, we have three cases.
Case 1. If $\lambda(Y) \geq \partial$, then $Y \in \lambda_\partial$ and $X^{-1}Y \in \lambda_\partial$. That is, $\lambda(X^{-1}Y) \geq \partial$ by Corollary 3.7(1) then:

$$\begin{aligned}(X\lambda)(Y) &= (\lambda(X^{-1}Y) \vee \gamma) \wedge \partial \\ &= \partial \\ &= (\lambda(Y) \vee \gamma) \wedge \partial \\ &= (E\lambda)(Y).\end{aligned}$$

Case 2. If $\gamma < \lambda(X) < \partial$, then $\lambda(X^{-1}Y) = \lambda(Y)$ by Corollary 3.7(2). Thus

$$\begin{aligned}(X\lambda)(Y) &= (\lambda(X^{-1}Y) \vee \gamma) \wedge \partial \\ &= (\lambda(Y) \vee \gamma) \wedge \partial \\ &= (E\lambda)(Y).\end{aligned}$$

Case 3. If $\lambda(X) \leq \gamma$, then $\lambda(X^{-1}Y) \leq \gamma$ by Corollary 3.7(2). Thus

$$\begin{aligned}(X\lambda)(Y) &= (\lambda(X^{-1}Y) \vee \gamma) \wedge \partial \\ &= \partial \\ &= (\lambda(Y) \vee \gamma) \wedge \partial \\ &= (E\lambda)(Y).\end{aligned}$$

In summary $Y\lambda = E\lambda$. $\Leftrightarrow$ Suppose that $Y\lambda = E\lambda$, then we have:

$$\begin{aligned}(Y\lambda)(Y) &= (E\lambda)(Y) \\ (\lambda(Y^{-1}Y) \vee \gamma) \wedge \partial &= (\lambda(Y) \vee \gamma) \wedge \partial \\ (\lambda(E) \vee \gamma) \wedge \partial &= (\lambda(Y) \vee \gamma) \wedge \partial \\ \partial &= (\lambda(Y) \vee \gamma) \wedge \partial \qquad (\text{ Hence } \lambda(E) \geq \partial)\end{aligned}$$

Then $\lambda(Y) \geq \partial$ and hence $Y \in \lambda_\partial$.

Similarly, if λ be a $(\gamma, \partial)-$fuzzy $HX-$subgroup of ν and $\lambda(E) \geq \partial$, then $X \in \lambda_\partial$ iff $\lambda X = \lambda E$. □

Proposition 4.4. *Let λ be a $(\gamma, \partial)-$fuzzy $HX-$subgroup of ν and $Y \in \lambda_\partial$ then $Y\lambda = E\lambda = \lambda Y$.*

Proof. $Y \in \lambda_\partial$, then $\lambda(E) \geq \partial$ by Corollary 3.3 and thus $Y\lambda = E\lambda$ and $\lambda E = \lambda Y$. It follows $Y\lambda = E\lambda = \lambda Y$. □

Theorem 4.5. *If λ be a $(\gamma, \partial)-$fuzzy $HX-$subgroup of ν. Then:*

(1) *$X\lambda_\partial = Y\lambda_\partial$ if and only if $X\lambda = Y\lambda$ provided $\lambda_\partial \neq \phi$ where $X, Y \in \nu$.*
(2) *If $\gamma < \lambda(E) < \partial$, then $\lambda(X) = \lambda(E)$ iff $X\lambda = E\lambda$.*

Proof.

$$\begin{aligned}1. X\lambda_\partial &= Y\lambda_\partial \\ &\Leftrightarrow Y^{-1}X \in \lambda_\partial \\ &\Leftrightarrow (Y^{-1}X)\lambda = E\lambda \qquad (\text{ Theorem 3.6}) \\ &\Leftrightarrow X\lambda = Y\lambda \qquad (\text{ Corollary 4.2(5)})\end{aligned}$$

2. $\Rightarrow$ Suppose that $\lambda(X) = \lambda(E)$, that is, $X \in \lambda_{\lambda(E)}$. By Lemma 3.5, $\lambda_{\lambda(E)}$ is a $HX-$subgroup of ν. We have $X^{-1} \in \lambda_{\lambda(E)}$, thus, $\lambda(X^{-1}) \geq \lambda(E)$ and so $\lambda(X^{-1}) = \lambda(E)$ take $Y \in \nu$. We have $\lambda(Y) \leq \lambda(E)$ by Corollary 3.3. There are three cases:

Case1. $\lambda(Y) \leq \gamma$. Thus $\lambda(X^{-1}Y) \leq \gamma$ by Corollary 3.7(3) and so:

$$\begin{aligned}(X\lambda)(Y) &= (\lambda(X^{-1}Y)\vee\gamma)\wedge\partial\\ &= \partial\\ &= (\lambda(Y)\vee\gamma)\wedge\partial\\ &= (E\lambda)(Y).\end{aligned}$$

Case 2. If $\gamma < \lambda(Y) < \lambda(E) = \lambda(X^{-1})$, then $\lambda(X^{-1}Y) = \lambda(Y)$ by Corollary 3.7(2) and so

$$\begin{aligned}(X\lambda)(Y) &= (\lambda(X^{-1}Y)\vee\gamma)\wedge\partial\\ &= (\lambda(Y)\vee\gamma)\wedge\partial\\ &= (E\lambda)(Y).\end{aligned}$$

Case 3. If $\lambda(X) = \lambda(E)$ we have $X^{-1}, Y \in \lambda_{\lambda(E)}$ ($\lambda_{\lambda(E)}$ is $HX-$subgroup of ν). Then $X^{-1}Y \in \lambda_{\lambda(E)}$ and thus $\lambda(X^{-1}Y) \geq \lambda(E)$. Hence $\lambda(X^{-1}Y) = \lambda(E)$ and we get $\lambda(X^{-1}Y) = \lambda(Y) = \lambda(E)$. Then

$$\begin{aligned}(X\lambda)(Y) &= (\lambda(X^{-1}Y)\vee\gamma)\wedge\partial\\ &= (\lambda(Y)\vee\gamma)\wedge\partial\\ &= (E\lambda)(Y).\end{aligned}$$

Therefore $X\lambda = E\lambda$.

$\Leftrightarrow$ Suppose that $X\lambda = E\lambda$. Then

$$\begin{aligned}(X\lambda)(X) &= (E\lambda)(X)\\ (\lambda(X^{-1}X)\vee\gamma)\wedge\partial &= (\lambda(X)\vee\gamma)\wedge\partial\\ \lambda(E) &= (\lambda(X)\vee\gamma)\wedge\partial \qquad (\gamma<\lambda(E)<\partial)\\ \lambda(E) &= \lambda(X).\end{aligned}$$

□

Corollary 4.6. *If λ be a $(\gamma,\partial)-$fuzzy $HX-$subgroup of ν then $X\lambda_{\lambda(E)} = Y\lambda_{\lambda(E)}$ iff $X\lambda = Y\lambda$ provided $\gamma < \lambda(E) < \partial$ for any $X, Y \in \nu$.*

Proof.

$$\begin{aligned}X\lambda_{\lambda(E)} &= Y\lambda_{\lambda(E)}\\ &\Leftrightarrow Y^{-1}X \in \lambda_{\lambda(E)}\\ &\Leftrightarrow (Y^{-1}X)\lambda = E\lambda\\ &\Leftrightarrow X\lambda = E\lambda.\end{aligned}$$

□

Theorem 4.7. *If λ be a $(\gamma,\partial)-$fuzzy $HX-$subgroup of $\nu, X\lambda = Y\lambda$ and $\gamma < \lambda(X) < \partial$ then $\lambda(X) = \lambda(Y)$for any $X, Y \in \nu$.*

Proof. $X\lambda = Y\lambda$ then $(Y^{-1}X)\lambda = E\lambda$. We have two cases:

Case 1. If $\lambda(Y) \geq \partial$ then by Theorem 3.6, $X^{-1}Y \in \lambda_\partial$, thus $(X^{-1}Y) \geq \partial$. By Corollary 3.7 and $\gamma < \lambda(X) < \partial, \lambda(X) = \lambda(X(X^{-1}Y)) = \lambda(Y)$.

Case 2. If $\lambda(E) < \partial$ and $\gamma < \lambda(X) < \partial$, then $\lambda(X) < \lambda(E) < \partial$ and then $\lambda(X^{-1}Y) = \lambda(E)$ by Thereom 3.10(2).

If $\lambda(X) < \lambda(E) = \lambda(X^{-1}Y)$ therefore $\lambda(X) = \lambda(X(X^{-1}Y)) = \lambda(Y)$.
If $\lambda(X) = \lambda(E) = \lambda(X^{-1}Y)$ then $X, X^{-1}Y \in \lambda_{\lambda(E)}$, hence λ is a $(\gamma,\partial)-$fuzzy $HX-$subgroup of $\nu, \lambda_{\lambda(E)}$ is a $HX-$subgroup of ν. Therefore $Y = X(X^{-1}Y) \in \lambda_{\lambda(E)}$ and then $\lambda(Y) \geq \lambda(E)$, then $\lambda(X) = \lambda(E) = \lambda(Y)$. □

Corollary 4.8. *If λ be a $(\gamma,\partial)-$fuzzy $HX-$subgroup of ν, such that $\beta \in (\gamma,\partial], \lambda_\beta \neq \phi$. If $X\lambda = Y\lambda$ then $X\lambda_\beta = Y\lambda_\beta$.*

Proof. λ_β is a $HX-$subgroup by Lemma 3.5 and $\lambda_\beta \neq \phi$ and $\gamma < \beta \leq \lambda(E)$, by $X\lambda = Y\lambda$, we know that $(X\lambda)(Y) = (Y\lambda)(Y)$ that is
$(\lambda(X^{-1}Y)\vee\gamma)\wedge\partial = (\lambda(E)\vee\gamma)\wedge\partial = \lambda(E)\vee\gamma$. If $\lambda(E) \geq \partial$ then If $\lambda(X^{-1}Y) \geq \partial$, that is $X^{-1}Y \in \lambda_\partial \subseteq \lambda_\beta$ and hence $X\lambda_\beta = Y\lambda_\beta$.

If $\lambda(E) < \partial$, then $\lambda(X^{-1}Y) = \lambda(E)$ that is $X^{-1}Y \in \lambda_{\lambda(E)}$ by $\lambda_{\lambda(E)} \subseteq \lambda_\beta$, then $X^{-1}Y \in \lambda_\beta$. Therefore $X\lambda_\beta = Y\lambda_\beta$.

□

5. CONCLUDING REMARKS

A $(\gamma,\partial)-$fuzzy subgroups concepts discussed by Yuen et al. In this paper, we studied a $(\gamma,\partial)-$fuzzy $HX-$subgroup, normal and cosets with suitable properties. We extended these ideas to the intuitionistic bipolar fuzzy $HX-$subgroup and discussed some properties of them.

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