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NEIGHBORHOOD PROPERTIES FOR CERTAIN $p-$VALENT ANALYTIC FUNCTIONS ASSOCIATED WITH $q-p-$VALENT BERNARDI INTEGRAL OPERATOR OF COMPLEX ORDER

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Abstract. In this paper, we introduce and investigate two new subclasses of p-valent analytic functions of complex order defined by using q-p-valent Bernardi integral operator. Also we obtain coefficient estimates and consequent inclusion relationships involving the (q, m, δ)-neighborhoods of these subclasses.

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1. Introduction

Let $\mathcal{A}(p, m)$ denote the class of analytic functions of the form:

$$f(\omega) = \omega^p + \sum_{k=p+m}^{\infty} a_k \, \omega^k \quad (p, m \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

which are p-valent in the open unit disc $\mathbb{U} = \{\omega \in \mathbb{C} : |\omega| < 1\}$. We note that $\mathcal{A}(p, 1) = \mathcal{A}(p)$, $\mathcal{A}(1, m) = \mathcal{A}(m)$ and $\mathcal{A}(1, 1) = \mathcal{A}$. Also, let $\mathcal{T}(p, m)$ denote the subclass of $\mathcal{A}(p, m)$ consisting of analytic and p-valent functions which can expressed in the form:

$$f(\omega) = \omega^p - \sum_{k=p+m}^{\infty} a_k \, \omega^k \quad (a_k \; > 0; p, m \in \mathbb{N}), \tag{2}$$

with $\mathcal{T}(p, 1) = \mathcal{T}(p)$, $\mathcal{T}(1, m) = \mathcal{T}(m)$ and $\mathcal{T}(1, 1) = \mathcal{T}$.

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For $f \in \mathcal{A}(p,m)$ given by (1) and $0 < q < 1$, the $q-$derivative of $f(\omega)$ is given by (see [1],[9],[10],[12],[15], [16],[18], [19],[20],[29],[35],[36] and [38])

$$D_{p,q}f(\omega) = \begin{cases} \frac{f(\omega)-f(q\omega)}{(1-q)\omega} & for\ \omega \neq 0, \\ f'(0) & for\ \omega = 0, \end{cases} \tag{3}$$

provided that $f'(0)$ exists. From (1) and (3), we deduce that

$$D_{p,q}f(\omega) = [p]_q\, \omega^{p-1} + \sum_{k=p+m}^{\infty} [k]_q\, a_k\, \omega^{k-1}, \tag{4}$$

where

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + ... + q^{k-1}, [0]_q = 0, 0 < q < 1. \tag{5}$$

We note that

$$\lim_{q\to 1-} D_{p,q}f(\omega) = \lim_{q\to 1-} \frac{f(\omega) - f(q\omega)}{(1-q)\,\omega} = f'(\omega)$$

for a function f which is differentiable in a given subset of $\mathbb{C}$. Further, for $p = 1$, we have $D_{1,q}f(\omega) = D_q f(\omega)$ (see [33] and [34]). The $q-$Jackson definite integral of the function $f(\omega)$ is defined by

$$\int_0^{\omega} f(t)\, d_q t = \omega(1-q) \sum_{k=0}^{\infty} q^k f\left(\omega q^k\right), \quad \omega \in \mathbb{C}, \tag{6}$$

provided that the series converges (see [18] and [19]). For a function f given by (1), we observe that

$$\int_0^{\omega} f(t)\, d_q t = \frac{\omega^{p+1}}{[p+1]_q} + \sum_{k=p+m}^{\infty} \frac{a_k\ \omega^{k+1}}{[k+1]_q}$$

and

$$\lim_{q\to 1-} \int_0^{\omega} f(t)\, d_q t = \frac{\omega^{p+1}}{p+1} + \sum_{k=p+m}^{\infty} \frac{a_k\ \omega^{k+1}}{k+1} = \int_0^{\omega} f(t)\, dt,$$

where $\int_0^{\omega} f(t)\, dt$ is the ordinary integral.

We use the $q-$Jackson definite integral of the function $f(\omega) \in \mathcal{A}(p,m)$ to define the $q-p-$valent Bernardi integral operator $F_{\nu,p,q}$ in the following definition.

Definition 1.1. Let ν be a real number such that $\nu > -p\,(p \in \mathbb{N})$. The $q-p-$valent Bernardi integral operator $F_{\nu,p,q}$ is defined by

$$F_{\nu,p,q}(\omega) = \frac{[\nu+p]_q}{\omega^{\nu}} \int_0^{\omega} t^{\nu-1} f(t)\, d_q t \quad (\nu > -p; f(\omega) \in \mathcal{A}(p,m)). \tag{7}$$

For a function f given by (1), we have

$$F_{\nu,p,q}(\omega) = \omega^p + \sum_{k=p+m}^{\infty} \frac{[\nu+p]_q}{[\nu+k]_q} a_k\ \omega^k \quad (\nu > -p; p \in \mathbb{N}). \tag{8}$$

We note that:

(1) $\lim_{q\to 1-} F_{\nu,p,q}(\omega) = F_{\nu,p}(\omega)\,(\nu > -p)$, where $F_{\nu,p}(\omega)$ is the $p-$valent Bernardi integral operator (see [31], [32] and [8]);

(2) $F_{\nu,1,q}(\omega) = F_{\nu,q}(\omega)$ (see [24]);

(3) $\lim_{q\to 1-} F_{\nu,1,q}(\omega) = F_{\nu}(\omega)\,(\nu > -1)$ (see [13] and [21]).

By using the operator $F_{\nu,p,q}(\omega)$ we define the class $\mathcal{S}_m(\nu,p,q,\lambda,\gamma,\beta)$ as follows.

Definition 1.2. *Let $f \in \mathcal{T}(p,m)$. Then we say that $f \in \mathcal{S}_m(\nu,p,q,\lambda,\gamma,\beta)$ if it satisfies the following inequality:*

$$\left|\frac{1}{\gamma}\left[\frac{(1-\lambda)\,\omega D_{p,q}(F_{\nu,p,q}(\omega)) + \lambda\omega D_{p,q}(\omega D_{p,q}(F_{\nu,p,q}(\omega)))}{(1-\lambda)\,F_{\nu,p,q}(\omega) + \lambda\omega D_{p,q}(F_{\nu,p,q}(\omega))} - [p]_q\right]\right| < \beta \qquad (9)$$

$$(\nu > -p;\gamma \in \mathbb{C}^* = \mathbb{C}\setminus\{0\};p,m \in \mathbb{N};0<q<1;\ 0 \le \lambda \le 1;\ 0<\beta\le 1).$$

We note that:

(1) $\lim_{q\to 1-} \mathcal{S}_m(\nu,p,q,\lambda,\gamma,\beta) = \mathcal{S}_m(\nu,p,\lambda,\gamma,\beta)$

$$\mathcal{S}_m(\nu,p,\lambda,\gamma,\beta) = \left\{f \in \mathcal{T}(p,m) : \left|\frac{1}{\gamma}\left[\frac{\omega F'_{\nu,p}(\omega) + \lambda\omega^2 F''_{\nu,p}(\omega)}{(1-\lambda)\,F_{\nu,p}(\omega) + \lambda\omega F'_{\nu,p}(\omega)} - p\right]\right| < \beta\right\}$$

$$(\nu > -p;\gamma \in \mathbb{C}^*;p,m \in \mathbb{N};\ 0 \le \lambda \le 1;\ 0<\beta\le 1);$$

(2) $\mathcal{S}_m(\nu,1,q,\lambda,\gamma,\beta) = \mathcal{S}_m(\nu,q,\lambda,\gamma,\beta)$

$$\mathcal{S}_m(\nu,q,\lambda,\gamma,\beta) = \left\{f \in \mathcal{T}(m) : \left|\frac{1}{\gamma}\left[\frac{(1-\lambda)\,\omega D_q(F_{\nu,q}(\omega)) + \lambda\omega D_q(\omega D_q(F_{\nu,q}(\omega)))}{(1-\lambda)\,F_{\nu,q}(\omega) + \lambda\omega D_q(F_{\nu,q}(\omega))} - 1\right]\right| < \beta\right\}$$

$$(\nu > -1;\gamma \in \mathbb{C}^*;m \in \mathbb{N};0<q<1;\ 0 \le \lambda \le 1;\ 0<\beta\le 1);$$

(3) $\lim_{q\to 1-} \mathcal{S}_m(\nu,1,q,\lambda,\gamma,\beta) = \mathcal{S}_m(\nu,\lambda,\gamma,\beta)$

$$\mathcal{S}_m(\nu,\lambda,\gamma,\beta) = \left\{f \in \mathcal{T}(m) : \left|\frac{1}{\gamma}\left[\frac{\omega F'_{\nu}(\omega) + \lambda\omega^2 F''_{\nu}(\omega)}{(1-\lambda)\,F_{\nu}(\omega) + \lambda\omega F'_{\nu}(\omega)} - 1\right]\right| < \beta\right\}$$

$$(\nu > -1;\gamma \in \mathbb{C}^*;m \in \mathbb{N};0 \le \lambda \le 1;\ 0<\beta\le 1).$$

Definition 1.3. *Let $f \in \mathcal{T}(p,m)$. Then we say that $f \in \mathcal{K}_m(\nu,p,q,\lambda,\gamma,\beta)$ if it satisfies the following inequality:*

$$\left|\frac{1}{\gamma}\left[(1-\lambda)\,\frac{F_{\nu,p,q}(\omega)}{\omega^p} + \lambda\frac{D_{p,q}(F_{\nu,p,q}(\omega))}{[p]_q\,\omega^{p-1}} - 1\right]\right| < \beta \qquad (10)$$

$$(\nu > -p;\gamma \in \mathbb{C}^*;p,m \in \mathbb{N};0<q<1;\ 0 \le \lambda \le 1;\ 0<\beta\le 1).$$

We note that:

(1) $\lim_{q\to 1-}\mathcal{K}_m(\nu,p,q,\lambda,\gamma,\beta)=\mathcal{K}_m(\nu,p,\lambda,\gamma,\beta)$

$$\mathcal{K}_m(\nu,p,\lambda,\gamma,\beta) = \left\{ f\in\mathcal{T}(p,m) : \left|\frac{1}{\gamma}\left[(1-\lambda)\frac{F_{\nu,p}(\omega)}{\omega^p}+\lambda\frac{\omega F'_{\nu,p}(\omega)}{p\omega^{p-1}}-1\right]\right|<\beta\right\}$$

$$(\nu>-p;\gamma\in\mathbb{C}^*;p,m\in\mathbb{N};\ 0\le\lambda\le1;\ 0<\beta\le1);$$

(2) $\mathcal{K}_m(\nu,1,q,\lambda,\gamma,\beta)=\mathcal{K}_m(\nu,q,\lambda,\gamma,\beta)$

$$\mathcal{K}_m(\nu,q,\lambda,\gamma,\beta) = \left\{ f\in\mathcal{T}(m) : \left|\frac{1}{\gamma}\left[(1-\lambda)\frac{F_{\nu,q}(\omega)}{\omega}+\lambda D_q(F_{\nu,q}(\omega))-1\right]\right|<\beta\right\}$$

$$(\nu>-1;\gamma\in\mathbb{C}^*;m\in\mathbb{N};0<q<1;\ 0\le\lambda\le1;\ 0<\beta\le1);$$

(3) $\lim_{q\to 1-}\mathcal{K}_m(\nu,1,q,\lambda,\gamma,\beta)=\mathcal{K}_m(\nu,\lambda,\gamma,\beta)$

$$\mathcal{K}_m(\nu,\lambda,\gamma,\beta)=\left\{ f\in\mathcal{T}(m) : \left|\frac{1}{\gamma}\left[(1-\lambda)\frac{F_{\nu}(\omega)}{\omega}+\lambda\omega F'_{\nu}(\omega)-1\right]\right|<\beta\right\}$$

$$(\nu>-1;\gamma\in\mathbb{C}^*;m\in\mathbb{N};0\le\lambda\le1;\ 0<\beta\le1).$$

Now, following the earlier investigations by Goodman [17], Ruscheweyh [30] and others including Altintaş and Owa [2, 4], Altintaş et al. [3, 5, 6], Mugrusundaramoorthy and Srivastava [23], Riana and Srivastava [28], Prajapat et al. [27] and Srivastava and Orhan [37] (see also, [11], [14], [22], [25] and [26]), we define the (m,δ) −neighborhood of a function $f\in\mathcal{T}(p,m)$ given by (2) as follows:

$$\mathcal{N}^p_{m,\delta}(f) = \left\{ g\in\mathcal{T}(p,m) : g(\omega)=\omega^p-\sum_{k=p+m}^{\infty}b_k\,\omega^k \text{ and } \sum_{k=p+m}^{\infty}k\left|a_k-b_k\right|\le\delta\right\}. \tag{11}$$

In particular, if

$$h(\omega)=\omega^p \quad (p\in\mathbb{N}), \tag{12}$$

we immediately have

$$\mathcal{N}^p_{m,\delta}(h)=\left\{ g\in\mathcal{T}(p,m) : g(\omega)=\omega^p-\sum_{k=p+m}^{\infty}b_k\,\omega^k \text{ and } \sum_{k=p+m}^{\infty}k\left|b_k\right|\le\delta\right\}. \tag{13}$$

Now, we define the (q,m,δ) −neighborhood of a function $f\in\mathcal{T}(p,m)$ given by (2) as follows (see [7])

$$\mathcal{N}^{p,q}_{m,\delta}(f) = \left\{ g\in\mathcal{T}(p,m) : g(\omega)=\omega^p-\sum_{k=p+m}^{\infty}b_k\,\omega^k \text{ and } \sum_{k=p+m}^{\infty}[k]_q\left|a_k-b_k\right|\le\delta\right\}. \tag{14}$$

In particular, if $h(\omega)$ given by (12), we immediately have

$$\mathcal{N}_{m,\delta}^{p,q}(h) = \left\{ g \in \mathcal{T}(p,m) : g(\omega) = \omega^p - \sum_{k=p+m}^{\infty} b_k\ \omega^k \text{ and } \sum_{k=p+m}^{\infty} [k]_q\,|b_k\ | \leq \delta \right\}. \tag{15}$$

We note that $\lim_{q\to 1-} \mathcal{N}_{m,\delta}^{p,q}(f) = \mathcal{N}_{m,\delta}^{p}(f)$ and $\lim_{q\to 1-} \mathcal{N}_{m,\delta}^{p,q}(h) = \mathcal{N}_{m,\delta}^{p}(h)$ (see [5]).

2. Coefficient bounds

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\gamma \in \mathbb{C}^*$, $p, m \in \mathbb{N}$, $0 < q < 1$, $0 \leq \lambda \leq 1$, $0 < \beta \leq 1$ and $\nu > -p$.

In our present investigation of the inclusion relations involving $\mathcal{N}_{m,\delta}^{p,q}(h)$, we shall require Lemmas 2.1 and 2 below.

Lemma 2.1. *Let $f \in \mathcal{T}(p,m)$ be given by (2). Then $f \in \mathcal{S}_m(\nu,p,q,\lambda,\gamma,\beta)$ if and only if*

$$\begin{aligned}&\sum_{k=p+m}^{\infty} \left([k]_q + \beta|\gamma| - [p]_q\right)\left[1 + \lambda\left([k]_p - 1\right)\right]\frac{[\nu+p]_q}{[\nu+k]_q} a_k \\ &\leq \beta|\gamma|\left[1 + \lambda\left([k]_p - 1\right)\right].\end{aligned} \tag{16}$$

Proof. Let $f(\omega) \in \mathcal{S}_m(\nu,p,q,\lambda,\gamma,\beta)$. Then we have

$$\begin{aligned}&\Re\left\{\frac{(1-\lambda)\,\omega D_{p,q}\left(F_{\nu,p,q}(\omega)\right) + \lambda\omega D_{p,q}\left(\omega D_{p,q}\left(F_{\nu,p,q}(\omega)\right)\right)}{(1-\lambda)\,F_{\nu,p,q}(\omega) + \lambda\omega D_{p,q}\left(F_{\nu,p,q}(\omega)\right)} - [p]_q\right\} \\ &> -\beta|\gamma| \ \ (\omega \in \mathbb{U}),\end{aligned} \tag{17}$$

or, equivalently,

$$\Re\left\{\frac{-\sum\limits_{k=p+m}^{\infty}\left([k]_q - [p]_q\right)\left[1 + \lambda\left([k]_p - 1\right)\right]\frac{[\nu+p]_q}{[\nu+k]_q} a_k\ \omega^{k-p}}{\left[1 + \lambda\left([p]_p - 1\right)\right] - \sum\limits_{k=p+m}^{\infty}\left[1 + \lambda\left([k]_p - 1\right)\right]\frac{[\nu+p]_q}{[\nu+k]_q} a_k\ \omega^{k-p}}\right\} > -\beta|\gamma|. \tag{18}$$

Setting $\omega = r\,(0 \leq r < 1)$ in (18), we observe that the expression in the denominator of the left hand side of (18) is positive for $r = 0$ and also for all $0 \leq r < 1$. Thus, by letting $r \longrightarrow 1-$ through real values, (18) leads us to the desired assertion of Lemma 2.1.

Conversely, by applying the hypothesis (16) and letting $|\omega| = 1$, we find from (9) that

$$\left|\frac{(1-\lambda)\,\omega D_{p,q}\left(F_{\nu,p,q}(\omega)\right) + \lambda\omega D_{p,q}\left(\omega D_{p,q}\left(F_{\nu,p,q}(\omega)\right)\right)}{(1-\lambda)\,F_{\nu,p,q}(\omega) + \lambda\omega D_{p,q}\left(F_{\nu,p,q}(\omega)\right)} - [p]_q\right|$$

$$= \left| \frac{\sum\limits_{k=p+m}^{\infty} \left([k]_q - [p]_q \right) \left[1 + \lambda \left([k]_p - 1 \right) \right] \frac{[\nu+p]_q}{[\nu+k]_q} a_k \ \omega^{k-p}}{\left[1 + \lambda \left([p]_p - 1 \right) \right] - \sum\limits_{k=p+m}^{\infty} \left[1 + \lambda \left([k]_p - 1 \right) \right] \frac{[\nu+p]_q}{[\nu+k]_q} a_k \ \omega^{k-p}} \right|$$

$$\leq \frac{\sum\limits_{k=p+m}^{\infty} \left([k]_q - [p]_q \right) \left[1 + \lambda \left([k]_p - 1 \right) \right] \frac{[\nu+p]_q}{[\nu+k]_q} a_k \ |\omega|^{k-p}}{\left[1 + \lambda \left([p]_p - 1 \right) \right] - \sum\limits_{k=p+m}^{\infty} \left[1 + \lambda \left([k]_p - 1 \right) \right] \frac{[\nu+p]_q}{[\nu+k]_q} a_k \ |\omega|^{k-p}}$$

$$\leq \frac{\sum\limits_{k=p+m}^{\infty} \left([k]_q - [p]_q \right) \left[1 + \lambda \left([k]_p - 1 \right) \right] \frac{[\nu+p]_q}{[\nu+k]_q} a_k}{\left[1 + \lambda \left([p]_p - 1 \right) \right] - \sum\limits_{k=p+m}^{\infty} \left[1 + \lambda \left([k]_p - 1 \right) \right] \frac{[\nu+p]_q}{[\nu+k]_q} a_k} = \beta \left| \gamma \right|.$$

Hence, by the maximum modulus theorem, we have $f(\omega) \in \mathcal{S}_m(\nu, p, q, \lambda, \gamma, \beta)$, which evidently completes the proof of Lemma 2.1. □

Similarly, we can prove the following lemma.

Lemma 2.2. *Let $f \in \mathcal{T}(p, m)$ be given by (2). Then $f \in \mathcal{K}_m(\nu, p, q, \lambda, \gamma, \beta)$ if and only if*

$$\sum_{k=p+m}^{\infty} \left([p]_q + \lambda \left([k]_p - [p]_q \right) \right) \frac{[\nu+p]_q}{[\nu+k]_q} a_k \ \leq \beta [p]_q \left| \gamma \right|. \tag{19}$$

3. Neighborhoods for the classes $\mathcal{S}_m(\nu, p, q, \lambda, \gamma, \beta)$ and $\mathcal{K}_m(\nu, p, q, \lambda, \gamma, \beta)$

In this section, we determine inclusion relations for each of the classes $\mathcal{S}_m(\nu, p, q, \lambda, \gamma, \beta)$ and $\mathcal{K}_m(\nu, p, q, \lambda, \gamma, \beta)$ involving $(q, m, \delta)-$neighborhood defined by (14) and (15).

Theorem 3.1. *Let $f \in \mathcal{T}(p, m)$ be in the class $\mathcal{S}_m(\nu, p, q, \lambda, \gamma, \beta)$, then*

$$\mathcal{S}_m(\nu, p, q, \lambda, \gamma, \beta) \subset \mathcal{N}_{m,\delta}^{p,q}(h), \tag{20}$$

where $h(\omega)$ is given by (12) and the parameter δ is given by

$$\delta = \frac{[p+m]_q \beta \left| \gamma \right| \left[1 + \lambda \left([p]_q - 1 \right) \right] [\nu+p+m]_q}{\left([p+m]_q + \beta \left| \gamma \right| - [p]_q \right) \left[1 + \lambda \left([p+m]_q - 1 \right) \right] [\nu+p]_q} \qquad \left([p]_q > \left| \gamma \right| \right). \tag{21}$$

Proof. Let $f(\omega) \in \mathcal{S}_m(\nu, p, q, \lambda, \gamma, \beta)$. Then, by using assertion (16) of Lemma 2.1, we have

$$\left([p+m]_q + \beta \left| \gamma \right| - [p]_q \right) \left[1 + \lambda \left([p+m]_q - 1 \right) \right] \frac{[\nu+p]_q}{[\nu+p+m]_q} \sum_{k=p+m}^{\infty} a_k$$

$$\leq \sum_{k=p+m}^{\infty} \left([k]_q + \beta|\gamma| - [p]_q\right) \left[1 + \lambda\left([k]_q - 1\right)\right] \frac{[\nu+p]_q}{[\nu+k]_q} a_k$$

$$\leq \beta|\gamma| \left[1 + \lambda\left([p]_q - 1\right)\right], \tag{22}$$

which readily yields

$$\sum_{k=p+m}^{\infty} a_k \leq \frac{\beta|\gamma| \left[1 + \lambda\left([p]_q - 1\right)\right] [\nu+p+m]_q}{\left([p+m]_q + \beta|\gamma| - [p]_q\right) \left[1 + \lambda\left([p+m]_q - 1\right)\right] [\nu+p]_q}. \tag{23}$$

Making use of (16), in conjunction with (23), we obtain

$$\begin{aligned}
&\left[1 + \lambda\left([p+m]_q - 1\right)\right] \frac{[\nu+p]_q}{[\nu+p+m]_q} \sum_{k=p+m}^{\infty} [k]_q a_k \\
&\leq \beta|\gamma| \left[1 + \lambda\left([p]_q - 1\right)\right] \\
&+ \left([p]_q - \beta|\gamma|\right) \left[1 + \lambda\left([p+m]_q - 1\right)\right] \frac{[\nu+p]_q}{[\nu+p+m]_q} \sum_{k=p+m}^{\infty} a_k \\
&\leq \beta|\gamma| \left[1 + \lambda\left([p]_q - 1\right)\right] + \frac{\left([p]_q - \beta|\gamma|\right) \beta|\gamma| \left[1 + \lambda\left([p]_q - 1\right)\right]}{\left([p+m]_q + \beta|\gamma| - [p]_q\right)} \\
&= \frac{[p+m]_q \beta|\gamma| \left[1 + \lambda\left([p]_q - 1\right)\right]}{\left([p+m]_q + \beta|\gamma| - [p]_q\right)}.
\end{aligned}$$

Hence

$$\sum_{k=p+m}^{\infty} [k]_q a_k \leq \frac{[p+m]_q \beta|\gamma| \left[1 + \lambda\left([p]_q - 1\right)\right] [\nu+p+m]_q}{\left([p+m]_q + \beta|\gamma| - [p]_q\right) \left[1 + \lambda\left([p+m]_q - 1\right)\right] [\nu+p]_q} = \delta, \tag{24}$$

which, by means of the definition (15), establishes the inclusion (20) asserted by Theorem 3.1. □

In a similar manner, by applying (19) of Lemma 2.2 instead of (16) of Lemma 2.1 to functions in the class $\mathcal{K}_m(\nu, p, q, \lambda, \gamma, \beta)$, we can prove the following inclusion relationship.

Theorem 3.2. *Let $f \in \mathcal{T}(p, m)$ be in the class $\mathcal{K}_m(\nu, p, q, \lambda, \gamma, \beta)$, then*

$$\mathcal{K}_m(\nu, p, q, \lambda, \gamma, \beta) \subset \mathcal{N}_{m,\delta}^{p,q}(h), \tag{25}$$

where $h(\omega)$ is given by (12) and the parameter δ is given by

$$\delta = \frac{[p+m]_q [p]_q \beta|\gamma| [\nu+p+m]_q}{\left[[p]_q + \lambda\left([p+m]_q - [p]_q\right)\right] [\nu+p]_q}. \tag{26}$$

4. Neighborhoods for the classes $\mathcal{S}_m^{(\alpha)}(\nu,p,q,\lambda,\gamma,\beta)$ and $\mathcal{K}_m^{(\alpha)}(\nu,p,q,\lambda,\gamma,\beta)$

In this section, we determine the neighborhood for each of the classes $\mathcal{S}_m^{(\alpha)}(\nu,p,q,\lambda,\gamma,\beta)$ and $\mathcal{K}_m^{(\alpha)}(\nu,p,q,\lambda,\gamma,\beta)$, which we define as follows. A function $f(\omega)\in\mathcal{T}(p,m)$ is said to be in the class $\mathcal{S}_m^{(\alpha)}(\nu,p,q,\lambda,\gamma,\beta)$ if there exists a function $\rho(\omega)\in\mathcal{S}_m(\nu,p,q,\lambda,\gamma,\beta)$ such that

$$\left|\frac{f(\omega)}{\rho(\omega)}-1\right|<[p]_q-\alpha \quad \left(\omega\in\mathbb{U};\ 0\le\alpha<[p]_q\right). \tag{27}$$

Analogously, a function $f(\omega)\in\mathcal{T}(p,m)$ is said to be in the class $\mathcal{K}_m^{(\alpha)}(\nu,p,q,\lambda,\gamma,\beta)$, if there exists a function $\rho(\omega)\in\mathcal{K}_m(\nu,p,q,\lambda,\gamma,\beta)$ such that the inequality (27) holds true.

Theorem 4.1. *Let $f(\omega)\in\mathcal{T}(p,m)$ be in the class $\mathcal{S}_m(\nu,p,q,\lambda,\gamma,\beta)$ and*

$$\alpha=[p]_q-\frac{\delta\left([p+m]_q+\beta|\gamma|-[p]_q\right)\left[1+\lambda\left([p+m]_q-1\right)\right][\nu+p]_q}{[p+m]_q\left\{\left([p+m]_q+\beta|\gamma|-[p]_q\right)\left[1+\lambda\left([p+m]_q-1\right)\right][\nu+p]_q-\beta|\gamma|\left[1+\lambda\left([p]_q-1\right)\right][\nu+p+m]_q\right\}}, \tag{28}$$

then

$$\mathcal{N}_{m,\delta}^{p,q}(h)\subset\mathcal{S}_m^{(\alpha)}(\nu,p,q,\lambda,\gamma,\beta), \tag{29}$$

where

$$\delta\le[p]_q[p+m]_q\left\{1-\frac{\beta|\gamma|\left[1+\lambda\left([p]_q-1\right)\right][\nu+p+m]_q}{\left([p+m]_q+\beta|\gamma|-[p]_q\right)\left[1+\lambda\left([p+m]_q-1\right)\right][\nu+p]_q}\right\}. \tag{30}$$

Proof. Assume that $f(\omega)\in\mathcal{N}_{m,\delta}^{p,q}(h)$. We find that from (14) that

$$\sum_{k=p+m}^{\infty}[k]_q|a_k-b_k|\le\delta, \tag{31}$$

which readily implies that

$$\sum_{k=p+m}^{\infty}|a_k-b_k|\le\frac{\delta}{[p+m]_q}. \tag{32}$$

Next, since $\rho(\omega)\in\mathcal{S}_m(\nu,p,q,\lambda,\gamma,\beta)$, by using (23), we have

$$\sum_{k=p+m}^{\infty}b_k\le\frac{\beta|\gamma|\left[1+\lambda\left([p]_q-1\right)\right][\nu+p+m]_q}{\left([p+m]_q+\beta|\gamma|-[p]_q\right)\left[1+\lambda\left([p+m]_q-1\right)\right][\nu+p]_q}, \tag{33}$$

so that

$$\begin{aligned}\left|\frac{f(\omega)}{\rho(\omega)}-1\right| &\leq \frac{\sum\limits_{k=p+m}^{\infty}|a_k-b_k|}{1-\sum\limits_{k=p+m}^{\infty}b_k}\\ &\leq \frac{\delta\left([p+m]_q+\beta|\gamma|-[p]_q\right)\left[1+\lambda\left([p+m]_q-1\right)\right][\nu+p]_q}{\left([p+m]_q+\beta|\gamma|-[p]_q\right)\left[1+\lambda\left([p+m]_q-1\right)\right][\nu+p]_q-\beta|\gamma|\left[1+\lambda\left([p]_q-1\right)\right][\nu+p+m]_q}\\ &= [p]_q-\alpha,\end{aligned}$$

provided that α is given by (28). Thus, by the above definition, $f(\omega)\in\mathcal{S}_m^{(\alpha)}(\nu,p,q,\lambda,\gamma,\beta)$. This completes the proof of Theorem 4.1. □

The proof of Theorem 4.2 below is similar to the proof of Theorem 4.1, we omit the details involved.

Theorem 4.2. *Let $f(\omega)\in\mathcal{T}(p,m)$ be in the class $\mathcal{K}_m(\nu,p,q,\lambda,\gamma,\beta)$ and*

$$\alpha=[p]_q-\frac{\delta\left[[p]_q+\lambda\left([p+m]_q-[p]_q\right)\right][\nu+p]_q}{[p+m]_q\left\{\left[[p]_q+\lambda\left([p+m]_q-[p]_q\right)\right][\nu+p]_q-\beta|\gamma|[p]_q[\nu+p+m]_q\right\}}, \tag{34}$$

then

$$\mathcal{N}_{m,\delta}^{p,q}(h)\subset\mathcal{K}_m^{(\alpha)}(\nu,p,q,\lambda,\gamma,\beta), \tag{35}$$

where

$$\delta\leq[p]_q[p+m]_q\left\{1-\frac{[p]_q\beta|\gamma|[\nu+p+m]_q}{\left[[p]_q+\lambda\left([p+m]_q-[p]_q\right)\right][\nu+p]_q}\right\}. \tag{36}$$

Remark 4.1. Letting $q\to1-$ in Theorems 1, 2, 3 and 4, respectively, we obtain new results for the classes $\mathcal{S}_m(\nu,p,\lambda,\gamma,\beta)$, $\mathcal{K}_m(\nu,p,\lambda,\gamma,\beta)$, $\mathcal{S}_m^{(\alpha)}(\nu,p,\lambda,\gamma,\beta)$ and $\mathcal{K}_m^{(\alpha)}(\nu,p,\lambda,\gamma,\beta)$, respectively.

Remark 4.2. Taking $p=1$ in Theorems 1, 2, 3 and 4, respectively, we obtain new results for the classes $\mathcal{S}_m(\nu,q,\lambda,\gamma,\beta)$, $\mathcal{K}_m(\nu,q,\lambda,\gamma,\beta)$, $\mathcal{S}_m^{(\alpha)}(\nu,q,\lambda,\gamma,\beta)$ and $\mathcal{K}_m^{(\alpha)}(\nu,q,\lambda,\gamma,\beta)$, respectively.

Remark 4.3. Letting $q\to1-$ and taking $p=1$ in Theorems 1, 2, 3 and 4, respectively, we obtain new results for the classes $\mathcal{S}_m(\nu,\lambda,\gamma,\beta)$, $\mathcal{K}_m(\nu,\lambda,\gamma,\beta)$, $\mathcal{S}_m^{(\alpha)}(\nu,\lambda,\gamma,\beta)$ and $\mathcal{K}_m^{(\alpha)}(\nu,\lambda,\gamma,\beta)$, respectively.

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