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MESHLESS AND HOMOTOPY PERTURBATION METHODS FOR ONE DIMENSIONAL INVERSE HEAT CONDUCTION PROBLEM WITH NEUMANN AND ROBIN BOUNDARY CONDITIONS

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Abstract. In this article, we investigate the solution of the inverse problem for one dimensional heat equation with Neumann and Robin boundary conditions, that is, we determine the temperature and source term with given initial and boundary conditions. Three radial basis functions(RBFs) have been used for numerical solution, and Homotopy perturbation method for analytic solution. Numerical solutions which are obtained by considering each of the three RBFs are compared to the exact solution. For appropriate value of shape parameter c, numerical solutions best approximates exact solutions. Furthermore, we have shown the impact of noisy data on the numerical solution of u and f.

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1. Introduction

Let $I \subset \mathbb{R}$ be a bounded domain and let x_0 be a fixed point in I. We will determine the function $u(x,t)$ (called temperature) and the source term $f(t)$ (called surface heat flux) satisfying the heat(diffusion) equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(t), 0 < t \leq T \text{ and } x \in I = (a,b) \tag{1}$$

subject to

$$\textbf{IC}: \qquad u(x,0) = g(x), \tag{2}$$

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$$\mathbf{BC}: \qquad \alpha u(x,t) + \beta \frac{\partial u(x,t)}{\partial x} = \begin{cases} h_1(t), & \text{if } x = a \\ h_2(t), & \text{if } x = b \end{cases} \tag{3}$$

α and $\beta(\neq 0)$ are constants.

$$\mathbf{AC}: \qquad u(x_0, t) = E(t). \tag{4}$$

Problem (1) together with (2), (3), and (4) is called inverse problem. The name arises due to the determination of surface heat flux($f(t)$) and temperature ($u(x,t)$) from temperature measurements at one or more interior locations. Though our problem is source determination inverse problem, inverse problems may be subdivided into determination of boundary value, initial value, material properties, source, and shape [1]. Inverse heat conduction problems arise in many physical applications where heat transfer occurs [2].

The numerical method we apply to solve (1), (2), (3), (4) is meshless method. Meshless methods for the solution of PDEs can be grouped into methods based on RBF interpolation, and the least squares technique. Hardy in [3] introduced the radial basis functions interpolation to approximate two-dimensional geographical surfaces based on scattered data. Then Kansa in[4] investigated a meshless method based on multiquadrics RBFs for the numerical solution of PDEs. Later, Golberg et al. extended the idea[5]. The existence, uniqueness, and convergence of the RBFs approximation was discussed by Franke and Schaback[6], Madych and Nelson[7], and Micchelli[8]. Meshless methods based on RBFs have been applied to solve Rosenau equation [9], fractional diffusion equation[10], Poisson and Helmholtz equation[11], two dimensional heat equation[12], compressible Euler equation with application in finite-rate Chemistry[13], one dimensional advection diffusion equation[14] and nonlinear integral equations[15]. The advantage of meshless method over mesh methods for example, finite difference methods, finite element methods and finite volume method, is: does not require domain discretization. These methods were applied for inverse heat conduction problems[16, 17, 18]. The mentioned mesh methods have been extensively used to find the solution of PDEs. In comparison to mesh methods, meshless method is widely used to solve problems in recent years.

The inverse problem for one dimensional diffusion equation with Dirichlet boundary conditions has been studied in [19, 20, 21, 22]. In [19] and [20] the authors determined the source term using moving least square method and Gaussian radial basis function respectively. In [21] the authors determined the unknown temperature at $x = 0$ and section of initial condition at $t = 0$ for Neumann boundary condition. The authors in [23] discussed inverse problems for multidimensional heat equations where the value of the unknown at a single point on the boundary is given. Pyatkov and Safonov studied some classes of inverse problems of recovering a source[24]. In general, meshless methods are applicable to compute the solution of problems arising in engineering and physics[25, 26, 27, 28, 29], and in economics[30].

So far we have discussed meshless methods for solving partial differential equations(PDEs). There are analytic methods that have been applied to solve

PDEs. We may mention Adomain decomposition method and Homotopy perturbation method. In this article we use the well known Homotopy perturbation method since it is simple to use. It is the one that provides series solution to linear and nonlinear PDEs [31, 32, 33, 34]. This method is in recursive sequence forms which can be used to get the closed form of the solutions [35, 36]. It has been applied for solving PDEs arising in the transmission of nerve impulses[37] and modeling of flow in porous media[38]. It computed the solution of non-linear fractional PDE[39] and non-linear system of second order boundary value problems[40].

The paper is organized as follows. In section two we discuss about RBFs, meshless method based on RBFs and effect of noisy data. In section three we see Homotopy perturbation method. In section four we include application of meshless method. Finally conclusion is drawn in section five.

2. Meshless Method based On RBFs

Define a function $\phi(r) : [0, \infty) \to \mathbb{R}$. Let $S = \{x_1, \cdots, x_N\}$ be the set of N distinct collocation points, where $S_{int} = \{x_2, \cdots, N_{N-1}\}$ are interior points and $S_{bdry} = S \setminus S_{int}$ are boundary points. We may describe interior point, boundary point and point x_0 graphically in figure(1).

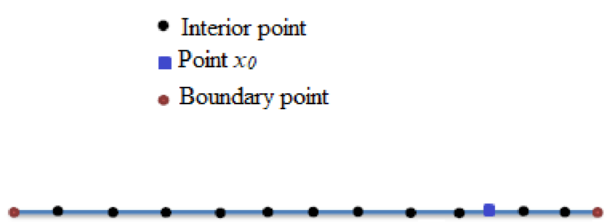

FIGURE 1. Collocation points and point x_0

We use the notations $\phi_k(x) = \phi(|x - x_k|)$, for $k = 1, 2, \cdots, N$.

In this paper the approximate function $u^a(x, t)$ of $u(x, t)$ can be represented as

$$u^a(x, t) = \sum_{k=1}^{N} \lambda_k(t)\phi_k(x), \forall x \in I \text{ and } t \in [0, T], \tag{5}$$

where $\phi_k(x)$ is radial basis function(RBF) and $\lambda_k(t)$ is unknown RBF coefficient.

A radial basis function ϕ on $[0, \infty)$ is called positive definite if for all choices of finite distinct points $x_1, x_2, x_3, \cdots, x_N$, the matrix M is positive definite[41] where

$$M = M_{i,k} = \phi_k(x_i).$$

A function $\phi_k(x)$ is conditionally positive definite of order m[42] , if for all sets $\{x_1, x_2, \cdots, x_N\}$ of distinct points, and all vectors $\nu(\neq 0)$ satisfying

$$\sum_{k=1}^{N} \nu_k P(x_k) = 0$$

for any polynomial P of degree at most $m-1$, we have

$$\sum_{i=1}^{N} \sum_{j}^{N} \nu_i \nu_j \phi(x_i - x_j) > 0.$$

If a matrix M is positive definite, then $det(M) \neq 0$. Hence, if the radial basis function ϕ in (5) is positive definite, the following system of linear equation is solvable.

$$\begin{bmatrix} \phi_1(x_1) & \cdots & \phi_N(x_1) \\ \vdots & \vdots & \vdots \\ \phi_1(x_N) & \cdots & \phi_N(x_N) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix} = \begin{bmatrix} u_1^a \\ \vdots \\ u_N^a \end{bmatrix} \tag{6}$$

We can also show that equation (6) is solvable if ϕ is conditionally positive definite of order one[43]. The above system of linear equations is obtained from (5) at the collocation points. In this paper the known RBFs are considered, which are listed in table(1). These RBFs are infinitely differentiable and depend

Table 1. Radial basis functions

No	RBFs	Definition[20, 14, 44]	
1	Gaussian (GA)	$\phi(r) = e^{-cr^2}$	Positive definite[45]
2	Hardy multiquadrics (HMQ)	$\phi(r) = \sqrt{r^2 + c^2}$	Conditionally positive definite of order 1[46]
3	Inverse multiquadrics (IMQ)	$\phi(r) = (\sqrt{r^2 + c^2})^{-1}$	Positive definite[47]

on the shape parameter $c > 0$[48, 49]. Shape parameter controls the fitting of a smoothing surface to the data, and affects the condition number of the coefficient matrix in equation (6). Even though the optimal choice of shape parameter is still an open problem[50], researchers proposed different methods to compute the optimal value for c [46, 51]. As it is indicated in [52], shape parameter depends on number of grid points, distribution of grid points, interpolation function and condition number of a matrix. We have been chosen shape parameters in table (2) for the problems we consider in this article.

TABLE 2. Shape parameters

No	RBFs	Shape parameters	
		Neumann BCs	Robin BCs
1	Gaussian (GA)	0.061	0.01
2	Hardy multiquadrics (HMQ)	6	9.2
3	Inverse multiquadrics (IMQ)	7.4	13.8

We now make our problems suitable to compute numerical solution via meshless method based on RBFs. So, from equation (1) and (4) we have

$$E^{'}(t) = \frac{\partial u(x_0,t)}{\partial t} = \frac{\partial^2 u(x_0,t)}{\partial x^2} + f(t) \tag{7}$$

From equation(7) we get

$$f(t) = E^{'}(t) - \left[\frac{\partial^2 u(x_0,t)}{\partial x^2}\right] \tag{8}$$

Consequently, equation(1) becomes

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + E^{'}(t) - \left[\frac{\partial^2 u(x_0,t)}{\partial x^2}\right] \tag{9}$$

Substituting equation(5) into (9),(2), (3), and (8) we obtain respectively

$$\sum_{k=1}^{N}\left[\lambda_k^{'}(t)\phi_k(x)\right] = \sum_{k=1}^{N}\left[\lambda_k(t)\phi_k^{''}(x)\right] + E^{'}(t) - \sum_{k=1}^{N}\left[\lambda_k(t)\phi_k^{''}(x_0)\right], \forall x \in I \tag{10}$$

$$\sum_{k=1}^{N}[\lambda_k(0)\phi_k(x)] = g(x), \forall x \in I \tag{11}$$

$$\sum_{k=1}^{N}\lambda_k(t)[\alpha\phi_k(x) + \beta\phi_k^{'}(x)] = \begin{cases} h_1(t), & \text{if } x = a \\ h_2(t), & \text{if } x = b, \end{cases} \tag{12}$$

and

$$f(t)=E^{'}(t)-\sum_{k=1}^{N}\lambda_k(t)\left[\phi_k^{''}(x_0)\right] \tag{13}$$

Taking $t_{n+1}=t_1+n\Delta t$ for $n=1,2,\cdots,M-1$, where $t_1=0$ and $t_M=T$, and applying forward difference operator to time in equation (10) and (13) we get respectively

$$\begin{aligned}\sum_{k=1}^{N}\left[\lambda_k(t_{n+1})-\lambda_k(t_n)\right]\phi_k(x) &= \sum_{k=1}^{N}\triangle t\lambda_k(t_n)\left[\phi_k^{''}(x)\right]+E(t_{n+1})-E(t_n)-\\ &\sum_{k=1}^{N}\triangle t\lambda_k(t_n)\left[\phi_k^{''}(x_0)\right] \end{aligned}\tag{14}$$

and

$$f^a(t_n)=\frac{E(t_{n+1})-E(t_n)}{\triangle t}-\sum_{k=1}^{N}\lambda_k(t_n)\left[\phi_k^{''}(x_0)\right]\text{ for } n=1,2,\cdots,M-1. \tag{15}$$

And using backward difference operator we have

$$f^a(t_M)=\frac{E(t_M)-E(t_{M-1})}{\triangle t}-\sum_{k=1}^{N}\lambda_k(t_M)\left[\phi_k^{''}(x_0)\right]. \tag{16}$$

The RBF coefficients $\lambda_k(t_n)$ can be obtained iteratively from (11),(12), and (14) for $k=1,2,\cdots,N$ and $n=1,2,\cdots,M-1$.

Thus, equation(14) can be written as

$$\begin{aligned}u^a(x,t_{n+1}) &= u^a(x,t_n)+\sum_{k=1}^{N}\triangle t\lambda_k(t_n)\left[\phi_k^{''}(x)\right]+E(t_{n+1})-E(t_n))-\\ &\sum_{k=1}^{N}\triangle t\lambda_k(t_n)\left[\phi_k^{''}(x_0)\right]\end{aligned}\tag{17}$$

The following notations have been used. $r_k=|x-x_k|$ and $r_{0,k}=|(x_0-x_k|$, where x is the collocation points.

Gaussian(GA)

$$\begin{aligned}u^a(x,t_{n+1}) &= u^a(x,t_n)+\textstyle\sum_{k=1}^{N}\triangle t\lambda_k(t_n)\left[-2ce^{-cr_k^2}[1-2cr_k^2]\right]+E(t_{n+1})\\ &\quad -E(t_n)-\textstyle\sum_{k=1}^{N}\triangle t\lambda_k(t_n)\left[-2ce^{-cr_{0,k}^2}[1-2cr_{0,k}^2]\right]\\ f^a(t_n) &= \frac{E(t_{n+1})-E(t_n)}{\triangle t}-\textstyle\sum_{k=1}^{N}\lambda_k(t_n)\left[-2ce^{-cr_{0,k}^2}[1-2cr_{0,k}^2]\right]\text{ for}\\ &\quad n=1,2,\cdots,M-1.\\ f^a(t_M) &= \frac{E(t_M)-E(t_{M-1})}{\triangle t}-\textstyle\sum_{k=1}^{N}\lambda_k(t_M)\left[-2ce^{-cr_{0,k}^2}[1-2cr_{0,k}^2]\right]\end{aligned}$$

Hardy Multiquadrics(HMQ)

$$\begin{aligned} u^a(x,t_{n+1}) &= u^a(x,t_n) + \textstyle\sum_{k=1}^{N} \triangle t\lambda_k(t_n)\left[\dfrac{c^2}{(r_k^2+c^2)^{\frac{3}{2}}}\right] + E(t_{n+1}) - E(t_n) \\ &\quad - \textstyle\sum_{k=1}^{N} \triangle t\lambda_k(t_n)\left[\dfrac{c^2}{(r_{0,k}^2+c^2)^{\frac{3}{2}}}\right] \\ f^a(t_n) &= \dfrac{E(t_{n+1})-E(t_n)}{\triangle t} - \textstyle\sum_{k=1}^{N} \lambda_k(t_n)\left[\dfrac{c^2}{(r_{0,k}^2+c^2)^{\frac{3}{2}}}\right] \text{ for} \\ &\quad n = 1,2,\cdots,M-1. \\ f^a(t_M) &= \dfrac{E(t_M)-E(t_{M-1})}{\triangle t} - \textstyle\sum_{k=1}^{N} \lambda_k(t_M)\left[\dfrac{c^2}{(r_{0,k}^2+c^2)^{\frac{3}{2}}}\right] \end{aligned}$$

Inverse Multiquadrics(IMQ)

$$\begin{aligned} u^a(x,t_{n+1}) &= u^a(x,t_n) + \textstyle\sum_{k=1}^{N} \triangle t\lambda_k(t_n)\left[\dfrac{2r_k^2-c^2}{(r_k^2+c^2)^{\frac{5}{2}}}\right] + E(t_{n+1}) - E(t_n) \\ &\quad - \textstyle\sum_{k=1}^{N} \triangle t\lambda_k(t_n)\left[\dfrac{2r_{0,k}^2-c^2}{(r_{0,k}^2+c^2)^{\frac{5}{2}}}\right] \\ f^a(t_n) &= \dfrac{E(t_{n+1})-E(t_n)}{\triangle t} - \textstyle\sum_{k=1}^{N} \lambda_k(t_n)\left[\dfrac{2r_{0,k}^2-c^2}{(r_{0,k}^2+c^2)^{\frac{5}{2}}}\right] \text{ for} \\ &\quad n = 1,2,\cdots,M-1. \\ f^a(t_M) &= \dfrac{E(t_M)-E(t_{M-1})}{\triangle t} - \textstyle\sum_{k=1}^{N} \lambda_k(t_M)\left[\dfrac{2r_{0,k}^2-c^2}{(r_{0,k}^2+c^2)^{\frac{5}{2}}}\right] \end{aligned}$$

Effect of Noisy Data. Here, we discuss about the numerical solutions $u^a(x,t)$ and $f^a(x,t)$ if there is error on additional condition(**AC**). To illustrate this we introduce the error function $E(t)\chi(t)$, where $\chi(t)$ is a noisy parameter. In this way equation (4) and (7) becomes respectively

$$u(x_0,t) = E(t)[1+\chi(t)] \text{ and } f(t) = \frac{d}{dt}[E(t)(1+\chi(t))] - \sum_{k=1}^{N} \lambda_k(t)\phi_k^{''}(x_0).$$

It follows that

$$\begin{aligned} u^a(x,t_{n+1}) &= u^a(x,t_n) + \sum_{k=1}^{N} \triangle t\lambda_k(t_n)\left[\phi_k^{''}(x)\right] + E(t_{n+1})[1+\chi(t_{n+1})] \\ &\quad -E(t_n)[1+\chi(t_n)] - \sum_{k=1}^{N} \triangle t\lambda_k(t_n)\left[\phi_k^{''}(x_0)\right] \end{aligned}$$

$$f^a(t_n) = \frac{1}{\Delta t}[E(t_{n+1})(1+\chi(t_{n+1})) - E(t_n)(1+\chi(t_n))] -$$

$$\sum_{k=1}^{N}\lambda_k(t_n)\phi_k(x_0) \text{ for } n=1,2,\cdots,M-1.$$

and

$$f^a(t_M) = \frac{1}{\Delta t}[E(t_M)(1+\chi(t_M)) - E(t_{M-1})(1+\chi(t_{M-1}))] - \sum_{k=1}^{N}\lambda_k(t_M)\phi_k(x_0).$$

Gaussian(GA)

$$u^a(x,t_{n+1}) = u^a(x,t_n) + \sum_{k=1}^{N}\triangle t\lambda_k(t_n)\left[-2ce^{-cr_k^2}[1-2cr_k^2]\right] + E(t_{n+1})(1+\chi(t_{n+1})) - E(t_n)(1+\chi(t_n)) - \sum_{k=1}^{N}\triangle t\lambda_k(t_n)\left[-2ce^{-cr_{0,k}^2}[1-2cr_{0,k}^2]\right].$$

$$f^a(t_n) = \frac{1}{\Delta t}[E(t_{n+1})(1+\chi(t_{n+1})) - E(t_n)(1+\chi(t_n))] - \sum_{k=1}^{N}\lambda_k(t_n)\left[-2ce^{-cr_{0,k}^2}[1-2cr_{0,k}^2]\right] \text{ for } n=1,2,\cdots,M-1.$$

$$f^a(t_M) = \frac{1}{\Delta t}[E(t_M)(1+\chi(t_M)) - E(t_{M-1})(1+\chi(t_{M-1}))] - \sum_{k=1}^{N}\lambda_k(t_M)\left[-2ce^{-cr_{0,k}^2}[1-2cr_{0,k}^2]\right]$$

Hardy Multiquadrics(HMQ)

$$u^a(x,t_{n+1}) = u^a(x,t_n) + \sum_{k=1}^{N}\triangle t\lambda_k(t_n)\left[\frac{c^2}{(r_k^2+c^2)^{\frac{3}{2}}}\right] + E(t_{n+1})(1+\chi(t_{n+1})) - E(t_n)(1+\chi(t_n)) - \sum_{k=1}^{N}\triangle t\lambda_k(t_n)\left[\frac{c^2}{(r_{0,k}^2+c^2)^{\frac{3}{2}}}\right].$$

$$f^a(t_n) = \frac{1}{\Delta t}[E(t_{n+1})(1+\chi(t_{n+1})) - E(t_n)(1+\chi(t_n))] - \sum_{k=1}^{N}\lambda_k(t_n)\left[\frac{c^2}{(r_{0,k}^2+c^2)^{\frac{3}{2}}}\right] \text{ for } n=1,2,\cdots,M-1.$$

$$f^a(t_M) = \frac{1}{\Delta t}[E(t_M)(1+\chi(t_M)) - E(t_{M-1})(1+\chi(t_{M-1}))] - \sum_{k=1}^{N}\lambda_k(t_M)\left[\frac{c^2}{(r_{0,k}^2+c^2)^{\frac{3}{2}}}\right].$$

Inverse Multiquadrics(IMQ)

$$\begin{aligned} u^a(x,t_{n+1}) &= u^a(x,t_n)+\sum_{k=1}^{N}\triangle t\lambda_k(t_n)\left[\frac{2r_k^2-c^2}{(r_k^2+c^2)^{\frac{5}{2}}}\right]+ \\ &\quad E(t_{n+1})(1+\chi(t_{n+1}))-E(t_n)(1+\chi(t_n))- \\ &\quad \sum_{k=1}^{N}\triangle t\lambda_k(t_n)\left[\frac{2r_{0,k}^2-c^2}{(r_{0,k}^2+c^2)^{\frac{5}{2}}}\right]. \\ f^a(t_n) &= \frac{1}{\Delta t}[E(t_{n+1})(1+\chi(t_{n+1}))-E(t_n)(1+\chi(t_n))] \\ &\quad -\sum_{k=1}^{N}\lambda_k(t_n)\left[\frac{2r_{0,k}^2-c^2}{(r_{0,k}^2+c^2)^{\frac{5}{2}}}\right] \text{ for } n=1,2,\cdots,M-1. \\ f^a(t_M) &= \frac{1}{\Delta t}[E(t_M)(1+\chi(t_M))-E(t_{M-1})(1+\chi(t_{M-1}))] \\ &\quad -\sum_{k=1}^{N}\lambda_k(t_M)\left[\frac{2r_{0,k}^2-c^2}{(r_{0,k}^2+c^2)^{\frac{5}{2}}}\right] \text{ for } n=1,2,\cdots,M-1. \end{aligned}$$

In order to illustrate the approximate effect, we define the root mean square error(RMSE) and maximum absolute error(MAE) for $u(x,t)$ and $f(t)$ as follows.

$$\begin{aligned} RMSE(u) &= \left[\frac{1}{MN}\sum_{i=1}^{M}\sum_{k=1}^{N}[u(x_k,t_i)-u^a(x_k,t_i)]^2\right]^{\frac{1}{2}}, \\ RMSE(f) &= \left[\frac{1}{M}\sum_{i=1}^{M}[f(t_i)-f^a(t_i)]^2\right]^{\frac{1}{2}}, \\ MAE(u) &= \max\left\{|u(x_k,t_i)-u^a(x_k,t_i)| : 1\le i\le M \text{ and } 1\le k\le N\right\}, \\ MAE(f) &= \max\left\{|f(t_i)-f^a(t_i)| : 1\le i\le M\right\}. \end{aligned}$$

3. Homotopy Perturbation Method(HPM)

Consider the linear differential equation

$$A(u(x,t))=f(t),\ x\in I \text{ and } t\in[0,T] \tag{18}$$

with boundary condition

$$B\left(u(x,t),\frac{\partial u(x,t)}{\partial n}\right)=0,\ x\in\partial I \text{ and } t\in[0,T], \tag{19}$$

where A is a general differential operator, B is a boundary operator, n is the outward unit vector at x, $f(t)$ is obtained from additional condition. Let L_1 and L_2 be two linear operators such that $A=L_1+L_2$. We write equation (18) as

$$L_1(u(x,t))+L_2(u(x,t))=f(t). \tag{20}$$

By a homotopy technique, we construct a homotopy defined as

$$H(v(x,t),p):\mathbb{R}\times[0,1]\to\mathbb{R}$$

which satisfies

$$H(v(x,t),p)=(1-p)[L_1(v(x,t))-L_1(u_0(x,t))]+p[(A(v(x,t))-f(t)]=0, \tag{21}$$

where $u_0(x,t)$ is the initial approximation of equation (20) which satisfies the boundary conditions equation (19). Hence, obviously we have

$$H(v(x,t),0) = L_1(v(x,t)) - L_1(u_0(x,t)) \tag{22}$$
$$H(v(x,t),1) = A(v(x,t)) - f(t) \tag{23}$$

and the changing process of p from 0 to 1 is the same as changing $H(v(x,t),p)$ from $L_1(v(x,t)) - L_1(u_0(x,t))$ to $A(v(x,t)) - f(t)$. If, the embedding parameter $p;(0 \leq p \leq 1)$ is considered as a "small parameter", applying the classical perturbation technique, we can assume that the solution of equation (23) can be given as a power series in p, i.e.,

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots . \tag{24}$$

and setting $p = 1$ results in the approximate solution of equation (24) as;

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots . \tag{25}$$

For solving equation (1) together with equations (2), (3) and (4), we construct the Homotopy as follows:

$$\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} = p\left[\frac{\partial^2 u}{\partial x^2} + f(t) - \frac{\partial u_0}{\partial t}\right] \tag{26}$$

and from equation (24) the series solution is

$$u = v_0 + v_1 + v_2 + v_3 + \cdots . \tag{27}$$

In order to compute $v_0, v_1, v_2, \cdots$, substitute (27) into (26) we get

$$\left.\begin{array}{lll}
p^0: & \dfrac{\partial v_0}{\partial t} - \dfrac{\partial u_0}{\partial t} = 0, & v_0(x,0) = u(x,0) \\[2ex]
p^1: & \dfrac{\partial v_1}{\partial t} = -\dfrac{\partial u_0}{\partial t} + \dfrac{\partial^2 u_0}{\partial x^2} + E^{'}(t) - \dfrac{\partial^2 v_0(x_0,t)}{\partial x^2}, & v_1(x,0) = 0 \\[2ex]
p^2: & \dfrac{\partial v_2}{\partial t} = \dfrac{\partial^2 v_1}{\partial x^2} - \dfrac{\partial^2 v_1(x_0,t)}{\partial x^2}, & v_2(x,0) = 0 \\[2ex]
p^3: & \dfrac{\partial v_3}{\partial t} = \dfrac{\partial^2 v_2}{\partial x^2} - \dfrac{\partial^2 v_2(x_0,t)}{\partial x^2}, & v_3(x,0) = 0 \\
\vdots & \vdots & \vdots
\end{array}\right\} \tag{28}$$

4. Applications of Meshless and Homotopy Perturbation Method

Example 4.1. Consider the equation

$$u_t = u_{xx} + f(t), 0 < t \leq 2 \text{ and } 0 < x < 2$$

with conditions

$$u(x,0) = 2+\cos x,\ \frac{\partial u(x,t)}{\partial x}\Big|_{x=0} = 0,\ \frac{\partial u(x,t)}{\partial x}\Big|_{x=2} = -e^{-t}\sin 2,$$
$$u(1,t) = (2+t+\cos 1)e^{-t}.$$

We now use Homotopy perturbation method to obtain exact solution. So, here $x_0 = 1$ and $E(t) = (2+t+\cos 1)e^{-t}$. Using (28), we obtain

$$\begin{aligned} v_0(x,t) &= 2+\cos x, \\ v_1(x,t) &= -t\cos x + 2e^{-t} + te^{-t} + e^{-t}\cos 1 + t\cos 1 - \cos 1 - 2, \\ v_2(x,t) &= \tfrac{t^2}{2}\cos x - \tfrac{t^2}{2}\cos 1, \cdots . \end{aligned}$$

We observe that

$$\begin{aligned} u &= v_0 + v_1 + v_2 + v_3 + \cdots \\ &= \left(1 - t + \tfrac{t^2}{2!} - \tfrac{t^3}{3!} - \cdots\right)\cos x + \left(e^{-t} - \left[1 - t + \tfrac{t^2}{2!} - \tfrac{t^3}{3!} - \cdots\right]\right)\cos 1 \\ &\quad +2e^{-t} + te^{-t} \\ &= (2+t+\cos x)e^{-t}. \end{aligned}$$

So, the exact solutions are $u(x,t) = (2+t+\cos x)e^{-t}$ and $f(t) = -(1+t)e^{-t}$. Figure (2) describes numerical(with no noisy data) and exact solution of u at $x = 0, 0.5, 1, 1.5, 2$ for $\Delta t = 0.001$. Figure (3) shows numerical(with no noisy data) and exact solution of f for $\Delta t = 0.001$. Figure(4) represents numerical(with no noisy data) and exact solutions of u with $\Delta x = 0.5$ and $\Delta t = 0.001$. Table (3) describes RMSE and MAE for $\chi(t) = 0, 0.01, 0.01(t-1)$ at $\Delta t = 0.001$.

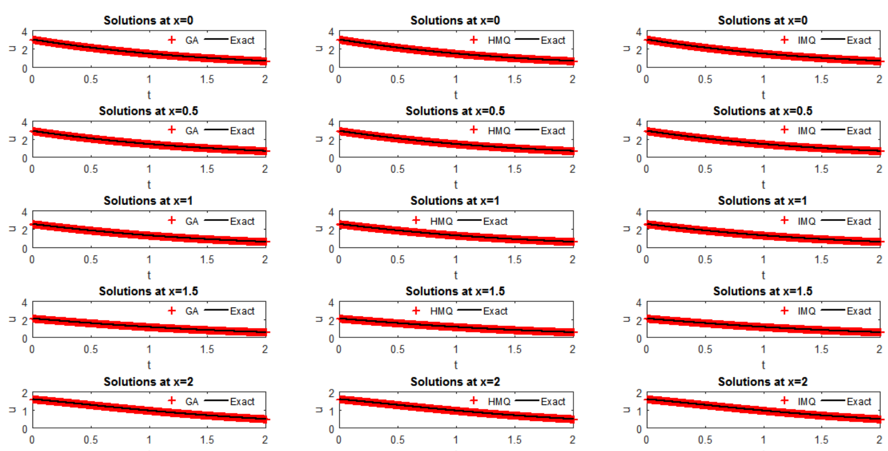

FIGURE 2. Numerical($\chi(t) = 0$) and exact solutions of u with $\Delta x = 0.5$ and $\Delta t = 0.001$

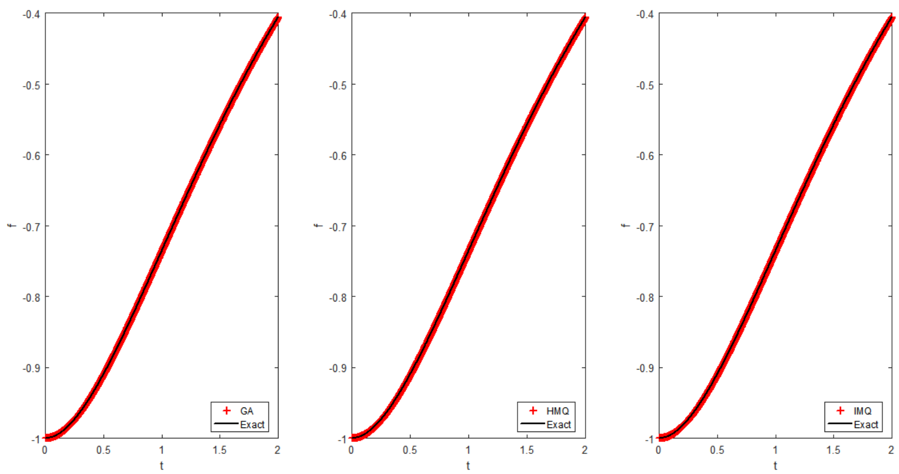

FIGURE 3. Numerical($\chi(t) = 0$) and exact solutions of f with $\Delta x = 0.5$ and $\Delta t = 0.001$

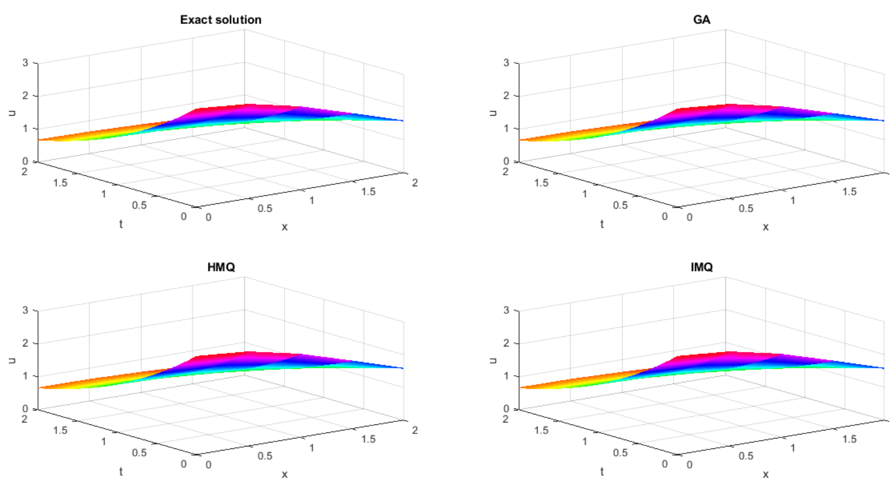

FIGURE 4. Numerical($\chi(t) = 0$) and exact solutions of u with $\Delta x = 0.5$ and $\Delta t = 0.001$

TABLE 3. Numerical errors for $\triangle t = 0.001$

	$\chi(t) = 0$	$\chi(t) = 0.01$	$\chi(t) = 0.01(t-1)$
GA:RMSE(u)	1.9991×10^{-3}	1.3921×10^{-2}	2.2933×10^{-2}
GA:MAE(u)	6.2926×10^{-3}	2.5534×10^{-2}	3.1596×10^{-2}
GA:RMSE(f)	1.2795×10^{-4}	1.022×10^{-2}	1.9400×10^{-2}
GA:MAE(f)	4.2184×10^{-4}	1.4986×10^{-2}	4.1420×10^{-2}
HMQ:RMSE(u)	3.8838×10^{-3}	1.5346×10^{-2}	2.1908×10^{-2}
HMQ:MAE(u)	1.1226×10^{-2}	3.0488×10^{-2}	3.1613×10^{-2}
HMQ:RMSE(f)	5.0319×10^{-4}	1.0584×10^{-2}	1.9056×10^{-2}
HMQ:MAE(f)	8.8240×10^{-4}	1.5210×10^{-2}	4.1420×10^{-2}
IMQ:RMSE(u)	3.1941×10^{-3}	1.4799×10^{-2}	2.2287×10^{-2}
IMQ:MAE(u)	9.5576×10^{-3}	2.8813×10^{-2}	3.1613×10^{-2}
IMQ:RMSE(f)	3.5736×10^{-4}	1.0443×10^{-2}	1.9186×10^{-2}
IMQ:MAE(f)	6.3691×10^{-4}	1.5079×10^{-2}	4.1340×10^{-2}

Example 4.2. Consider the equation

$$u_t = u_{xx} + f(t), 0 < t \leq 2 \text{ and } 0 < x < 2$$

with conditions

$$u(x,0) = x^2, \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} + u(0,t) = 2t + \sin(\pi t),$$
$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=2} + u(2,t) = 8 + 2t + \sin(\pi t), u(1,t) = 1 + 2t + \sin(\pi t).$$

We now use Homotopy perturbation method to obtain exact solution. So, here $x_0 = 1$ and $E(t) = 1 + 2t + \sin \pi t$. Using (28), we obtain

$$\begin{aligned} v_0(x,t) &= x^2, \\ v_1(x,t) &= 2t + \sin \pi t, \\ v_2(x,t) &= 0, v_3(x,t) = 0, \cdots . \end{aligned}$$

We observe that

$$\begin{aligned} u &= v_0 + v_1 + v_2 + v_3 + \cdots \\ &= x^2 + 2t + \sin \pi t. \end{aligned}$$

So, the exact solutions are $u(x,t) = x^2 + 2t + \sin(\pi t)$ and $f(t) = \pi \cos(\pi t)$. Figure (5) describes numerical(with no noisy data) and exact solution of u at $x = 0, 0.5, 1, 1.5, 2$ for $\Delta t = 0.001$. Figure (6) shows numerical(with no noisy data) and exact solution of f for $\Delta t = 0.001$. Figure(7) represents numerical(with no noisy data) and exact solutions of u with $\Delta x = 0.5$ and $\Delta t = 0.001$.Table (4) describes RMSE and MAE for $\chi(t) = 0, 0.01, 0.01(t-1)$ at $\Delta t = 0.001$.

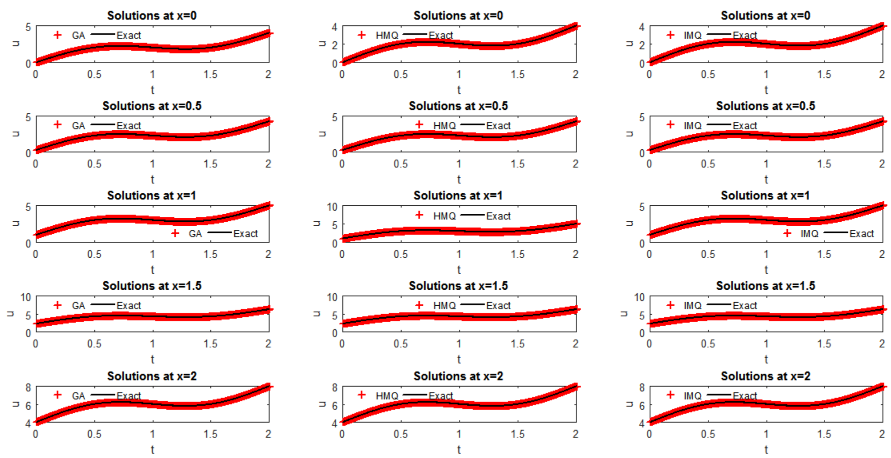

FIGURE 5. Numerical($\chi(t) = 0$) and exact solutions of u with $\Delta x = 0.5$ and $\Delta t = 0.001$

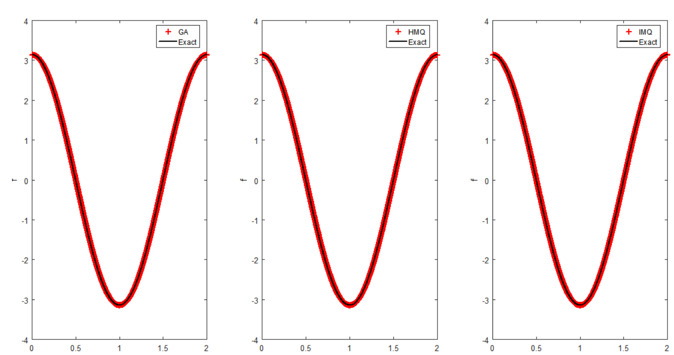

FIGURE 6. Numerical($\chi(t) = 0$) and exact solutions of u with $\Delta x = 0.5$ and $\Delta t = 0.001$

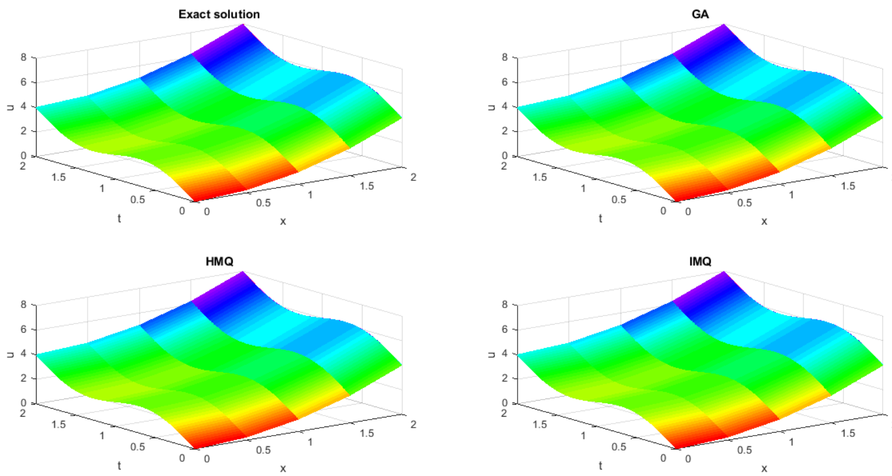

FIGURE 7. Numerical($\chi(t) = 0$) and exact solutions of u with $\Delta x = 0.5$ and $\Delta t = 0.001$

TABLE 4. Numerical errors for $\triangle t = 0.001$

	$\chi(t) = 0$	$\chi(t) = 0.01$	$\chi(t) = 0.01(t-1)$
GA:RMSE(u)	1.2545×10^{-3}	3.2332×10^{-2}	2.8390×10^{-2}
GA:MAE(u)	4.9342×10^{-3}	1.0302×10^{-1}	1.2596×10^{-1}
GA:RMSE(f)	3.5418×10^{-3}	2.5864×10^{-2}	3.3452×10^{-2}
GA:MAE(f)	5.0226×10^{-3}	5.1441×10^{-2}	5.8321×10^{-2}
HMQ:RMSE(u)	3.4693×10^{-3}	3.1143×10^{-2}	2.7336×10^{-2}
HMQ:MAE(u)	1.3090×10^{-2}	9.6997×10^{-2}	1.2021×10^{-1}
HMQ:RMSE(f)	3.5307×10^{-3}	2.6052×10^{-2}	3.2845×10^{-2}
HMQ:MAE(f)	5.4849×10^{-3}	5.1550×10^{-2}	5.8321×10^{-2}
IMQ:RMSE(u)	2.6013×10^{-3}	3.1694×10^{-2}	2.7794×10^{-2}
IMQ:MAE(u)	1.0078×10^{-2}	9.9957×10^{-2}	1.2304×10^{-1}
IMQ:RMSE(f)	3.5350×10^{-3}	2.5953×10^{-2}	3.3149×10^{-2}
IMQ:MAE(f)	5.2470×10^{-3}	5.1502×10^{-2}	5.8916×10^{-2}

5. Conclusion

Meshless and Homotopy perturbation method are successfully applied to compute the solution of inverse heat conduction problem. Meshless method which is based on radial basis functions GA,HMQ and IMQ provides numerical solution where as Homotopy perturbation method gives exact solution. Two examples have been considered, and for each example several computations are carried out, namely u, f, RMSE and MAE. GA, HMQ and IMQ best approximates exact solutions with no noisy data. If the data is noisy, the result is worse. However, the noisy parameter close to zero for $t \in [0, T]$, the result close to the exact solution. Finally, we suggest that the problem can be extended to higher dimensions with different boundary conditions.

6. Abbreviations

IC = Initial condition
BC = Boundary condition
AC = Additional condition
RBF = Radial basis function
GA = Gaussian
HMQ = Hardy multiquadratics
IMQ = Inverse multiquadratics
HPM = Homotopy Perturbation Method
PDE = Partial differential equation
RMSE = Root mean square error
MAE = Maximum absolute error

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