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LINEAR ISOMORPHIC EULER FRACTIONAL DIFFERENCE SEQUENCE SPACES AND THEIR TOEPLITZ DUALS

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Abstract. In the present paper we introduce and study Euler sequence spaces of fractional difference and backward difference operators. We make an effort to prove that these spaces are $BK-$spaces and linearly isomorphic. Further, Schauder basis for Euler fractional difference sequence spaces $e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m)$ and $e^{\varsigma}_{c,p}(\Delta^{(\tilde{\beta})}, \nabla^m)$ are also elaborate. In addition to this, we determine the $\alpha-$, $\beta-$ and $\gamma-$ duals of these spaces.

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1. Introduction

Let ω be the space of all real or complex sequences. By $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ we denote the set of natural, real and complex numbers, respectively. Baliarsingh and Dutta [2] introduced fractional difference operators $\Delta^{\tilde{\beta}}, \Delta^{(\tilde{\beta})}, \Delta^{-\tilde{\beta}}$, $\Delta^{(-\tilde{\beta})}$ and studied some topological results among these operators. The generalized fractional difference operator $\Delta^{(\tilde{\beta})}$ for a positive proper fraction $\tilde{\beta}$ is defined as

$$\Delta^{(\tilde{\beta})}(x_\vartheta) = \sum_{\mu=0}^{\infty}(-1)^\mu \frac{\Gamma(\tilde{\beta}+1)}{\mu!\Gamma(\tilde{\beta}-\mu+1)} x_{\vartheta-\mu}.$$

For more details about the fractional difference operator (see [1, 3, 4, 9, 17]). By $\Gamma(\tilde{\beta})$, we denote the Euler gamma function of a real number $\tilde{\beta}$ or generalized factorial function. This series is convergent throughout the paper for $x \in \omega$. For $\tilde{\beta} \in I^+$, where I^+ denote the set of strictly positive integers, we define Euler gamma function as

$$\Gamma(\tilde{\beta}) \quad = \quad \int_0^\infty e^{-t} t^{\tilde{\beta}-1} dt.$$

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As a triangle the fractional difference operator can be expressed as

$$(\Delta^{(\tilde{\beta})})_{n\vartheta} = \begin{cases} (-1)^{n-\vartheta}\frac{\Gamma(\tilde{\beta}+1)}{(n-\vartheta)!\Gamma(\tilde{\beta}-n+\vartheta+1)}, & (0 \leq \vartheta \leq n) \\ 0, & (\vartheta > n). \end{cases}$$

The inverse of the difference matrix $(\Delta^{(\tilde{\beta})})_{n\vartheta}$ given by

$$(\Delta^{(-\tilde{\beta})})_{n\vartheta} = \begin{cases} (-1)^{n-\vartheta}\frac{\Gamma(-\tilde{\beta}+1)}{(n-\vartheta)!\Gamma(-\tilde{\beta}-n+\vartheta+1)}, & (0 \leq \vartheta \leq n) \\ 0, & (\vartheta > n). \end{cases}$$

For more detail about difference sequence spaces one may refer to [7, 12, 19, 20, 21, 22, 23, 24]. The difference operator of order m was introduced by Polat and Başar [18] to develop some new sequence spaces. For definition and results one can refer to [11, 18].

The Euler mean matrix $E^{\varsigma} = (e^{\varsigma}_{n\vartheta})$ of order ς, $(0 < \varsigma < 1)$ is given by

$$(e^{\varsigma}_{n\vartheta}) = \begin{cases} \binom{n}{\vartheta}(1-\varsigma)^{n-\vartheta}\varsigma^{\vartheta}, & (0 \leq \vartheta \leq n); \\ 0, & (\vartheta > n). \end{cases}$$

The Euler matrix can also be written as

$$(e^{\varsigma}_{n\vartheta}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1-\varsigma & \varsigma & 0 & 0 & \cdots \\ (1-\varsigma)^2 & 2(1-\varsigma)\varsigma & \varsigma^2 & 0 & \cdots \\ (1-\varsigma)^3 & 3(1-\varsigma)^2\varsigma & 3(1-\varsigma)\varsigma^2 & \varsigma^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The product matrix $(E^{\varsigma}(\Delta^{(\tilde{\beta})}))_{n\vartheta}$ can be represented by combining the Euler mean matrix of order ς and the fractional difference matrix of order $\tilde{\beta}$ as

$$(E^{\varsigma}(\Delta^{(\tilde{\beta})}))_{n\vartheta} =$$

$$\begin{cases} \sum_{\mu=\vartheta}^{n}(-1)^{\mu-\vartheta}\binom{n}{n-\mu}\frac{\Gamma(\tilde{\beta}+1)}{(\mu-\vartheta)!\Gamma(\tilde{\beta}-\mu+\vartheta+1)}\varsigma^{\mu}(1-\varsigma)^{n-\mu}, & (0 \leq \vartheta \leq n) \\ 0, & (\vartheta > n). \end{cases}$$

Moreover, $(E^{\varsigma}(\Delta^{(\tilde{\beta})}))_{n\vartheta}$ can also be written as

$$(E^{\varsigma}(\Delta^{(\tilde{\beta})}))_{n\vartheta} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ (1-\varsigma)-\tilde{\beta}\varsigma & \varsigma & 0 & 0 & \dots \\ (1-\varsigma)^2 - 2\tilde{\beta}(1-\varsigma)\varsigma + \frac{\tilde{\beta}(\tilde{\beta}-1)}{2!}\varsigma^2 & 2(1-\varsigma)\varsigma - \tilde{\beta}\varsigma^2 & \varsigma^2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Consider U and V be two sequence spaces. Let $\mathcal{A} = (a_{n\vartheta})$ be an infinite matrix of real or complex numbers for $n, \vartheta \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then, $\mathcal{A}$ defines a matrix transformation from U into V and it is denoted by $\mathcal{A} : U \to V$, if for every sequence $x = (x_\vartheta) \in U$, the sequence $\mathcal{A}x = \{\mathcal{A}_n(x)\}$ is in V, where

$$\mathcal{A}_n(x) = \sum_{\vartheta=0}^{\infty} a_{n\vartheta} x_\vartheta \quad (n \in \mathbb{N}_0). \tag{1}$$

By (U, V), we denote the class of all infinite matrices $\mathcal{A}$ such that $\mathcal{A} : U \to V$. For each $n \in \mathbb{N}_0$ and every $x \in U$, $\mathcal{A} \in (U, V)$ iff the series on the right-hand side of (1) converges. So, we have $\mathcal{A}x \in V$ for all $x \in U$. For a sequence space U, the matrix domain $U_\mathcal{A}$ of an infinite matrix $\mathcal{A}$ is defined as

$$U_\mathcal{A} = \{x = (x_\vartheta) \in \omega : \mathcal{A}x \in U\}$$

which is a sequence space. Recently, many mathematicians have defined sequence spaces by using matrix domain for a triangle infinite matrix (see [5, 8, 10, 15, 16]) and many others.

The multiplier space of U and V is denoted by $N(U, V)$ and is defined by

$$N(U, V) = \{v = (v_\vartheta) \in \omega : uv = (u_\vartheta v_\vartheta) \in V, \ \forall \ u = (u_\vartheta) \in U\}.$$

The $\alpha-, \beta-$ and $\gamma-$ duals of the sequence space U are defined by

$$U^\alpha = \{z = (z_\vartheta) \in \omega : zu = (z_\vartheta u_\vartheta) \in \ell_1, \forall \ u = (u_\vartheta) \in U\},$$

$$U^\beta = \{z = (z_\vartheta) \in \omega : zu = (z_v u_\vartheta) \in cs \text{ for all } u = (u_\vartheta) \in U\}$$

and

$$U^\gamma = \{z = (z_\vartheta) \in \omega : zu = (z_\vartheta u_\vartheta) \in bs \text{ for all } u = (u_\vartheta) \in U\},$$

respectively. That is $U^\alpha = N(U, \ell_1), U^\beta = N(U, cs)$ and $U^\gamma = N(U, bs)$.

A sequence space U with a linear topology is called a K-space, provided each of the maps $q_n : U \to \mathbb{R}$ defined by $q_n(x) = x_n$ is continuous $\forall n \in \mathbb{N}$. A K-space U is called an FK-space provided U is a complete linear metric space. An FK-space whose topology is normable is called BK-space. By c, c_0 and ℓ_∞, we denote the Banach spaces of convergent, null and bounded sequences $x = (x_\vartheta)$ with the usual norm $\|x\|_\infty = \sup_\vartheta |x_\vartheta|$. The spaces of all bounded and convergent series are denoted by bs and cs, respectively. Also, by ℓ_1 and ℓ_p, we denote the spaces of all absolutely and $p-$absolutely convergent series, respectively, which are BK spaces with the usual norm defined by

$$\|x\|_{\ell_p} = \left(\sum_{\vartheta=0}^{\infty} |x_\vartheta|^p \right)^{1/p}, \ \text{ for } \ 0 \le p < \infty.$$

A sequence $(x_\vartheta) \in X$ is called a Schauder basis for a normed space $(X, \|.\|)$, if for every $x \in X$, there is a unique scalar sequence (υ_ϑ) such that

$$\left\| x - \sum_{\vartheta=0}^{n} \upsilon_\vartheta x_\vartheta \right\| \to 0 \text{ as } n \to \infty.$$

Maddox [14] introduced the sequence spaces $\ell_\infty(p), c_0(p), c(p)$ as follows:

$$\ell_\infty(p) = \{x = (x_\vartheta) \in \omega : \sup_\vartheta |x_\vartheta|^{p_\vartheta} < \infty\},$$
$$c_0(p) = \{x = (x_\vartheta) \in \omega : \lim_{\vartheta \to \infty} |x_\vartheta|^{p_\vartheta} = 0\}$$

and

$$c(p) = \{x = (x_\vartheta) \in \omega : \lim_{\vartheta \to \infty} |x_\vartheta - l|^{p_\vartheta} = 0, \text{ for some } l \in \mathbb{R}\},$$

where $p = (p_\vartheta)$ denotes bounded sequence of positive real numbers with $\sup_\vartheta p_\vartheta = M$ and $R = \max\{1, M\}$.

Let $\tilde{\beta}$ be a positive proper fraction, $E^\varsigma = (e^\varsigma_{n\vartheta})$ denotes the Euler mean matrix, ∇^m denotes the backward difference operator of order m and $p = (p_n)$ be a bounded sequence of positive real numbers. Now, we define the following sequence spaces as follows:

$$e^\varsigma_p(\Delta^{(\tilde{\beta})}, \nabla^m) = \Big\{x = (x_\vartheta) : \sum_n \Big| \sum_{\vartheta=0}^{n} \sum_{\mu=\vartheta}^{n} (-1)^{\mu-\vartheta} \binom{n}{n-\mu} \frac{\Gamma(\tilde{\beta}+1)}{(\mu-\vartheta)!\Gamma(\tilde{\beta}-\mu+\vartheta+1)} \varsigma^\mu (1-\varsigma)^{n-\mu} (\nabla^m x_\vartheta) \Big|^p < \infty \Big\},$$

$$e^\varsigma_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m) = \Big\{x = (x_\vartheta) : \lim_{n \to \infty} \Big| \sum_{\vartheta=0}^{n} \sum_{\mu=\vartheta}^{n} (-1)^{\mu-\vartheta} \binom{n}{n-\mu} \frac{\Gamma(\tilde{\beta}+1)}{(\mu-\vartheta)!\Gamma(\tilde{\beta}-\mu+\vartheta+1)} \varsigma^\mu (1-\varsigma)^{n-\mu} (\nabla^m x_\vartheta) \Big|^{p_n} = 0 \Big\},$$

$$e^\varsigma_{c,p}(\Delta^{(\tilde{\beta})}, \nabla^m) = \Big\{x = (x_\vartheta) : \lim_{n \to \infty} \Big| \sum_{\vartheta=0}^{n} \sum_{\mu=\vartheta}^{n} (-1)^{\mu-\vartheta} \binom{n}{n-\mu} \frac{\Gamma(\tilde{\beta}+1)}{(\mu-\vartheta)!\Gamma(\tilde{\beta}-\mu+\vartheta+1)} \varsigma^\mu (1-\varsigma)^{n-\mu} (\nabla^m x_\vartheta) \Big|^{p_n} \text{ exists} \Big\}$$

and

$$e^\varsigma_{\infty,p}(\Delta^{(\tilde{\beta})}, \nabla^m) = \Big\{x = (x_\vartheta) : \sup_n \Big| \sum_{\vartheta=0}^{n} \sum_{\mu=\vartheta}^{n} (-1)^{\mu-\vartheta} \binom{n}{n-\mu} \frac{\Gamma(\tilde{\beta}+1)}{(\mu-\vartheta)!\Gamma(\tilde{\beta}-\mu+\vartheta+1)} \varsigma^\mu (1-\varsigma)^{n-\mu} (\nabla^m x_\vartheta) \Big|^{p_n} < \infty \Big\}.$$

By taking the $E^\varsigma(\Delta^{(\tilde{\beta})}, \nabla^m)$-transform of $x = (x_\vartheta)$ in the spaces $\ell_p, c_0(p), c(p)$ and $\ell_\infty(p)$ one can easily obtain the above defined spaces as

$$e^\varsigma_p(\Delta^{(\tilde{\beta})}, \nabla^m) = (\ell_p)_{E^\varsigma(\Delta^{(\tilde{\beta})}, \nabla^m)}, \quad e^\varsigma_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m) = (c_0(p))_{E^\varsigma(\Delta^{(\tilde{\beta})}, \nabla^m)}, \tag{2}$$

$$e^{\varsigma}_{c,p}(\Delta^{(\tilde{\beta})},\nabla^m)=(c(p))_{E^{\varsigma}(\Delta^{(\tilde{\beta})},\nabla^m)} \text{ and } e^{\varsigma}_{\infty,p}(\Delta^{(\tilde{\beta})},\nabla^m)=(\ell_\infty(p))_{E^{\varsigma}(\Delta^{(\tilde{\beta})},\nabla^m)}. \tag{3}$$

Now, we define the $E^{\varsigma}(\Delta^{(\tilde{\beta})},\nabla^m)$-transform of the sequence $x=(x_\vartheta)$ i.e., $y=(y_\nu)$ as follows:

$$y_\nu=\sum_{\vartheta=0}^{\nu}\sum_{\mu=\vartheta}^{\nu}(-1)^{\mu-\vartheta}\binom{\nu}{\nu-\mu}\frac{\Gamma(\tilde{\beta}+1)}{(\mu-\vartheta)!\Gamma(\tilde{\beta}-\mu+\vartheta+1)}\varsigma^{\mu}(1-\varsigma)^{\nu-\mu}(\nabla^m x_\vartheta),$$

for each $\nu\in\mathbb{N}$, where

$$\nabla^m x_\vartheta=\sum_{\mu=0}^{m}(-1)^{\mu}\binom{m}{\mu}x_{\vartheta-\mu}=\sum_{\mu=\max\{0,\vartheta-m\}}(-1)^{\vartheta-\mu}\binom{m}{\vartheta-\mu}x_\mu.$$

2. Main results

Theorem 2.1. *Suppose $\tilde{\beta}$ be a positive proper fraction. Then the Euler difference sequence space $e^{\varsigma}_p(\Delta^{(\tilde{\beta})},\nabla^m)$ is a BK space with the norm*

$$\|x\|_{e^{\varsigma}_p(\Delta^{(\tilde{\beta})},\nabla^m)}=\|E^{\varsigma}(\Delta^{(\tilde{\beta})},\nabla^m)x\|_p \text{ for } (1\le p<\infty).$$

Proof. The sequence spaces ℓ_p,ℓ_∞,c_0,c are $BK-$spaces with their natural norms. Also $(\Delta^{(\tilde{\beta})})$ is a triangle matrix, (2) and (3) holds. By using Theorem 4.3.12 of Wilansky [25], we conclude that Euler sequence space $e^{\varsigma}_p(\Delta^{(\tilde{\beta})},\nabla^m)$ is a $BK-$space. □

Theorem 2.2. *The Euler difference sequence spaces $e^{\varsigma}_p(\Delta^{(\tilde{\beta})},\nabla^m), e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})},\nabla^m)$, $e^{\varsigma}_{c,p}(\Delta^{(\tilde{\beta})},\nabla^m)$ and $e^{\varsigma}_{\infty,p}(\Delta^{(\tilde{\beta})},\nabla^m)$ are linearly isomorphic to $\ell_p, c_0(p), c(p)$ and $\ell_\infty(p)$ spaces, respectively.*

Proof. We only give the proof for the space $e^{\varsigma}_{\infty,p}(\Delta^{(\tilde{\beta})},\nabla^m)$. To prove $e^{\varsigma}_{\infty,p}(\Delta^{(\tilde{\beta})},\nabla^m)\cong\ell_\infty(p)$, we need to show the existence of linear bijection between $e^{\varsigma}_{\infty,p}(\Delta^{(\tilde{\beta})},\nabla^m)$ and $\ell_\infty(p)$. Define a mapping $Q: e^{\varsigma}_{\infty,p}(\Delta^{(\tilde{\beta})},\nabla^m)\to\ell_\infty(p)$ by $x\mapsto y=Qx$. The linearity of Q is obvious. Moreover, $x=\theta$ whenever $Qx=\theta=(0,0,0,\cdots)$. Therefore, Q is injective. Consider $y=(y_\nu)\in\ell_\infty(p)$. Now, define a sequence $x=(x_\vartheta)$ by

$$x_\vartheta=\sum_{\mu=0}^{\vartheta}\sum_{j=\mu}^{\vartheta}(-1)^{\vartheta-j}\binom{m+\vartheta-j-1}{\vartheta-j}\binom{j}{\mu}\frac{\Gamma(-\tilde{\beta}+1)}{(\vartheta-j)!\Gamma(-\tilde{\beta}-\vartheta+j+1)}\varsigma^{-j}(\varsigma-1)^{j-\mu}y_\mu. \tag{4}$$

So, we get

$$\sup_n\left|\sum_{\vartheta=0}^{n}\sum_{\mu=\vartheta}^{n}(-1)^{\mu-\vartheta}\binom{n}{n-\mu}\frac{\Gamma(\tilde{\beta}+1)}{(\mu-\vartheta)!\Gamma(\tilde{\beta}-\mu+\vartheta+1)}(1-\varsigma)^{n-\mu}\varsigma^{\mu}(\nabla^m x_\vartheta)\right|^{p_n}$$

$$= \sup_n |y_n|^{p_n} = \|y\|_{\infty,p} < \infty,$$

which implies that for $x \in e^{\varsigma}_{\infty,p}(\Delta^{(\tilde{\beta})}, \nabla^m)$. Hence, Q is surjective. Thus, $e^{\varsigma}_{\infty,p}(\Delta^{(\tilde{\beta})}, \nabla^m) \cong \ell_\infty(p)$. □

Theorem 2.3. *Let* $\xi_\vartheta = (E^{\varsigma}(\Delta^{(\tilde{\beta})}, \nabla^m)x)_\vartheta \ \forall \ \mu, \vartheta \in \mathbb{N}_0$, *define the sequence* $g^{(\vartheta)} = \{g^{(\vartheta)}_\mu\}_{\mu \in \mathbb{N}_0}$ *by*

$$g^{(\vartheta)}_\mu = \begin{cases} \sum_{j=\vartheta}^{\mu} (-1)^{\mu-j} \binom{m+\mu-j-1}{\mu-j} \binom{j}{\vartheta} \\ \frac{\Gamma(-\tilde{\beta}+1)}{(\mu-j)!\Gamma(-\tilde{\beta}-\mu+j+1)} \varsigma^{-j}(\varsigma-1)^{j-\vartheta}, & (0 \leq \vartheta \leq \mu) \\ 0, & (\vartheta > \mu). \end{cases}$$

Then
(i) The sequence $\{g^{(\vartheta)}_\mu\}_{\mu \in \mathbb{N}_0}$ *is a basis for the space* $e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m)$ *and* $x \in e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m)$ *has a unique representation in the form*

$$x = \sum_\vartheta \xi_\vartheta g^{(\vartheta)}. \tag{5}$$

(ii) The set $\{w, g^{(\vartheta)}\}$ *is a basis for the space* $e^{\varsigma}_{c,p}(\Delta^{(\tilde{\beta})}, \nabla^m)$ *and* $x \in e^{\varsigma}_{c,p}(\Delta^{(\tilde{\beta})}, \nabla^m)$ *has a unique representation in the form*

$$x = \varphi w + \sum_\vartheta (\xi_\vartheta - \varphi) g^{(\vartheta)},$$

where $\varphi = \lim_{\vartheta \to \infty} \xi_\vartheta$ *and* $w = (w_\nu)$ *defined by*

$$w_\nu = \sum_{\vartheta=0}^{\nu} \sum_{j=\vartheta}^{\nu} (-1)^{\nu-j} \binom{j}{\vartheta} \binom{m+\nu-j-1}{\nu-j} \frac{\Gamma(-\tilde{\beta}+1)}{(\nu-j)!\Gamma(-\tilde{\beta}-\nu+j+1)} \varsigma^{-j}(\varsigma-1)^{j-\vartheta}.$$

Proof. (i) Clearly, $E^{\varsigma}(\Delta^{(\tilde{\beta})}, \nabla^m) g^{(\vartheta)}_\mu = (e_\vartheta) \in c_0$, where (e_ϑ) is the sequence with 1 in the ϑ^{th} place and zeros elsewhere for each $\vartheta \in \mathbb{N}$. Now for $x \in e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m)$ and $l \in \mathbb{N}$, we define

$$x^{(l)} = \sum_{\vartheta=0}^{l} \xi_\vartheta g^{(\vartheta)}. \tag{6}$$

By applying $E^{\varsigma}(\Delta^{(\tilde{\beta})}, \nabla^m)$ to (6) with (5), we have

$$(E^{\varsigma}(\Delta^{(\tilde{\beta})}, \nabla^m) x^{(l)}_\mu) = \sum_{\vartheta=0}^{l} \xi_\vartheta (E^{\varsigma}(\Delta^{(\tilde{\beta})}, \nabla^m) g^{(\vartheta)}_\mu) = \sum_{\vartheta=0}^{l} \xi_\vartheta e_\vartheta.$$

Also,

$$(E^{\varsigma}(\Delta^{(\tilde{\beta})},\nabla^m)(x_\mu - x_\mu^{(l)}))_\vartheta = \begin{cases} 0, & 0 \le \vartheta \le l; \\ (E^{\varsigma}(\Delta^{(\tilde{\beta})},\nabla^m)x_\mu)_\vartheta, & \vartheta \ge l. \end{cases}$$

Let $\epsilon > 0$ be arbitrary. We choose $l_0 \in \mathbb{N}$, such that

$$\left|(E^{\varsigma}(\Delta^{(\tilde{\beta})},\nabla^m)x_\mu)_\vartheta\right| < \frac{\epsilon}{2},\ \forall\, \vartheta \ge l_0.$$

Then, we have

$$\begin{aligned} \|x - x^{(l)}\|_{e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})},\nabla^m)} &= \sup_{\vartheta \ge l} |(E^{\varsigma}(\Delta^{\tilde{\beta}},\nabla^m)x_\mu)_\vartheta| \\ &\le \sup_{\vartheta \ge l_0} |(E^{\varsigma}(\Delta^{(\tilde{\beta})},\nabla^m)x_\mu)_\vartheta| \\ &< \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

This implies $x = \sum_\vartheta \xi_\vartheta g^{(\vartheta)}$. Now we should show the uniqueness of this representation. Let us assume that there exists

$$x = \sum_\vartheta \lambda_\vartheta g^{(\vartheta)}.$$

By using the continuity of Q transformation defined in the proof of Theorem 2.2, we get

$$\begin{aligned} (E^{\varsigma}(\Delta^{(\tilde{\beta})},\nabla^m)x_\mu)_\vartheta &= \sum_\vartheta \lambda_\vartheta (E^{\varsigma}(\Delta^{(\tilde{\beta})},\nabla^m)g_\mu^{(\vartheta)})_\vartheta \\ &= \sum_\vartheta \lambda_\vartheta (e_\vartheta)_\vartheta = \lambda_\vartheta \end{aligned}$$

which is a contradiction with the assumption that $\xi_\vartheta = (E^{\varsigma}(\Delta^{(\tilde{\beta})},\nabla^m)x_\mu)_\vartheta$ for each $\vartheta \in \mathbb{N}$. Hence, the representation

$$x = \sum_\vartheta \xi_\vartheta g^{(\vartheta)}$$

is unique.
(ii) In a similar manner as in (i), one can easily show that $\{w, g^{(\vartheta)}\}$ is a basis for the Euler difference sequence space $e^{\varsigma}_{c,p}(\Delta^{(\tilde{\beta})},\nabla^m)$ and $x \in e^{\varsigma}_{c,p}(\Delta^{(\tilde{\beta})},\nabla^m)$ has a unique representation in the form $x = \varphi w + \sum_\vartheta (\xi_\vartheta - \varphi) g^{(\vartheta)}$. □

Lemma 2.4. [6] *Let $H = (h_{\nu\vartheta})$ be an infinite matrix, A be a positive integer and $\mathcal{G}$ be a collection of all finite subsets of $\mathbb{N}$. Then, following conditions hold:*
(i) $H = (h_{\nu\vartheta}) \in (c_0(p) : \ell(q))$ iff

$$\sup_{K \in \mathcal{G}} \sum_\nu \left| \sum_{\vartheta \in K} h_{\nu\vartheta} A^{-1/p_\vartheta} \right|^{q_\nu} < \infty. \tag{7}$$

(ii) $H = (h_{\nu\vartheta}) \in (c(p) : \ell(q))$ *iff* (7) *holds and*

$$\sum_{\nu} \left| \sum_{\vartheta} h_{\nu\vartheta} \right|^{q_\nu} < \infty. \tag{8}$$

(iii) $H = (h_{\nu\vartheta}) \in (c_0(p) : c(q))$ *iff*

$$\sup_{\nu \in \mathbb{N}} \sum_{\vartheta} \left| h_{\nu\vartheta} \right| A^{-1/p_\vartheta} < \infty, \tag{9}$$

$$\lim_{\nu \to \infty} \left| h_{\nu\vartheta} - c_\vartheta \right|^{q_\nu} = 0, \quad \forall\, \vartheta \in \mathbb{N} \tag{10}$$

and

$$\sup_{\nu \in \mathbb{N}} \sum_{\vartheta} \left| h_{\nu\vartheta} - c_\vartheta \right| A^{-1/p_\vartheta} < \infty, \text{ where } c_\vartheta \in \mathbb{R}. \tag{11}$$

(iv) $H = (h_{\nu\vartheta}) \in (c(p) : c(q))$ *iff* (9), (10), (11) *hold and*

$$\lim_{\nu \to \infty} \left| \sum_{\vartheta} h_{\nu\vartheta} - c \right|^{q_\nu} = 0, \text{ where } c \in \mathbb{R}. \tag{12}$$

(v) $H = (h_{\nu\vartheta}) \in (\ell(p) : \ell_1)$ *iff*
(a) Let $0 < p_\vartheta \le 1, \ \forall\, \vartheta \in \mathbb{N}$. *Then*

$$\sup_{N \in \mathcal{G}} \sup_{\vartheta \in \mathbb{N}} \left| \sum_{\nu \in \mathbb{N}} h_{\nu\vartheta} \right|^{p_\vartheta} < \infty. \tag{13}$$

(b) Let $1 < p_\vartheta \le M \le \infty, \ \forall\, \vartheta \in \mathbb{N}$. *Then*

$$\sup_{N \in \mathcal{G}} \sum_{\vartheta} \left| \sum_{\nu \in \mathbb{N}} h_{\nu\vartheta} A^{-1} \right|^{p'_\vartheta} < \infty, \quad \text{where } p'_\vartheta = p_\vartheta/(p_\vartheta - 1). \tag{14}$$

Lemma 2.5. [13] *The following statements hold.*
(i) Let $1 < p_\vartheta \le M \le \infty$. *Then* $H = (h_{\nu\vartheta}) \in (\ell(p) : \ell_\infty)$ *iff* $\exists$ *an integer* $A > 1$ *such that*

$$\sup_{n} \sum_{\vartheta} \left| h_{\nu\vartheta} A^{-1} \right|^{p'_\vartheta} < \infty. \tag{15}$$

(ii) Let $0 < p_\vartheta \le 1$, *for every* $\vartheta \in \mathbb{N}$. *Then* $H = (h_{\nu\vartheta}) \in (\ell(p) : \ell_\infty)$ *iff*

$$\sup_{\nu,\vartheta} \left| h_{\nu\vartheta} \right|^{p_\vartheta} < \infty. \tag{16}$$

Lemma 2.6. [13] *Let* $0 < p_\vartheta \le M < \infty$, *for every* $\vartheta \in \mathbb{N}$. *Then* $H = (h_{\nu\vartheta}) \in (\ell(p) : c)$ *iff* (15) *and* (16) *hold along with there is* $\beta_\vartheta \in \mathbb{C}$ *such that* $\lim_{\nu} h_{\nu\vartheta} = \beta_\vartheta$, *for every natural number* ϑ.

Theorem 2.7. *Let $\tilde{\beta}$ be a positive proper fraction. Then, $\{e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m)\}^{\alpha} = L_1(p)$ and $\{e^{\varsigma}_{c,p}(\Delta^{(\tilde{\beta})}, \nabla^m)\}^{\alpha} = L_1(p) \cap L_2$, where the sets $L_1(p)$and L_2 are defined below:*

$$L_1(p) = \left\{ r = (r_\vartheta) \in \omega : \sup_{K \in \mathcal{G}} \sum_{\vartheta} \left| \sum_{\mu \in K} \lambda_{\vartheta\mu} A^{-1/p_\mu} \right| < \infty \right\}$$

and

$$L_2 = \left\{ r = (r_\vartheta) \in \omega : \sum_{\vartheta} \left| \sum_{\mu=0}^{\vartheta} \lambda_{\vartheta\mu} \right| \textit{ exists for each } \vartheta \in \mathbb{N} \right\},$$

where

$$\Lambda = \lambda_{\vartheta\mu} = \begin{cases} \sum_{j=\mu}^{\vartheta} (-1)^{\vartheta-j} \binom{j}{\mu} \binom{m+\vartheta-j-1}{\vartheta-j} & \\ \frac{\Gamma(-\tilde{\beta}+1)}{(\vartheta-j)!\Gamma(-\tilde{\beta}-\vartheta+j+1)} (\varsigma-1)^{j-\mu} \varsigma^{-j} r_\vartheta, & \textit{if } 0 \le \mu \le \vartheta; \\ 0, & \textit{if } \mu > \vartheta. \end{cases}$$

Proof. Let $r = (r_\vartheta) \in \omega$. From (4) we can see that

$$r_\vartheta x_\vartheta = \sum_{j=\mu}^{\vartheta} (-1)^{\vartheta-j} \binom{j}{\mu} \binom{m+\vartheta-j-1}{\vartheta-j} \frac{\Gamma(-\tilde{\beta}+1)}{(\vartheta-j)!\Gamma(-\tilde{\beta}-\vartheta+j+1)} (\varsigma-1)^{j-\mu} \varsigma^{-j} r_\vartheta y_\mu.$$

This implies

$$r_\vartheta x_\vartheta = (\Lambda y)_\vartheta, \ \forall \mu, \vartheta \in \mathbb{N}. \tag{17}$$

Also, from (17) one can easily get that $rx = (r_\vartheta x_\vartheta) \in \ell_1$, whenever $x \in e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m)$ iff $\Lambda y \in \ell_1$, whenever $y \in c_0(p)$. Therefore, $r = (r_\vartheta) \in \{e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m)\}^{\alpha}$ iff $\Lambda \in (c_0(p) : \ell_1)$. Thus, from (7) and for $q_\vartheta = 1, \ \forall \ \vartheta \in \mathbb{N}$ gives $\{e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m)\}^{\alpha} = L_1(p)$. By using (8) with $q_\vartheta = 1, \ \forall \ \vartheta \in \mathbb{N}$ and (17) the proof of the $\{e^{\varsigma}_{c,p}(\Delta^{(\tilde{\beta})}, \nabla^m)\}^{\alpha} = L_1(p) \cap L_2$ can be obtained in a similar manner. □

Theorem 2.8. *Let $\tilde{\beta}$ be a positive proper fraction. Define the sets $L_3(p), L_4, L_5(p), L_6$ by*

$$L_3(p) = \bigcup_{A>1} \left\{ r = (r_\vartheta) \in \omega : \sup_{K \in \mathbb{N}} \sum_{\vartheta=0}^{\nu} |\kappa_{\nu\vartheta}| A^{-1/p_\vartheta} < \infty \right\},$$

$$L_4 = \left\{ r = (r_\vartheta) \in \omega : \lim_{\nu \to \infty} |\kappa_{\nu\vartheta}| \textit{ exists for each } \vartheta \in \mathbb{N} \right\},$$

$$L_5(p) = \bigcup_{A>1} \left\{ r = (r_\vartheta) \in \omega : \sup_{K \in \mathbb{N}} \sum_{\vartheta=0}^{\nu} |\kappa_{\nu\vartheta} - c_\vartheta| A^{-1/p_\vartheta} < \infty \right\}$$

and

$$L_6 = \left\{ r = (r_\vartheta) \in \omega : \lim_{\nu \to \infty} \sum_{\vartheta=0}^{\nu} |\kappa_{\nu\vartheta}| \textit{ exists} \right\},$$

where the matrix $\tau = \kappa_{\nu\vartheta}$ *is given by*

$$\kappa_{\nu\vartheta} = \begin{cases} \sum_{\mu=\vartheta}^{\nu} \sum_{j=\vartheta}^{\mu} (-1)^{\mu-j} \binom{j}{\vartheta} \binom{m+\mu-j-1}{\mu-j} & \\ \frac{\Gamma(-\tilde{\beta}+1)}{(\mu-j)!\Gamma(1-\tilde{\beta}-\mu+j)} (\varsigma-1)^{j-\vartheta} \varsigma^{-j} r_\mu, & \textit{if } 0 \le \vartheta \le \nu; \\ 0, & \textit{if } \vartheta > \nu. \end{cases} \tag{18}$$

Then, we have

$$\{e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m)\}^{\beta} = L_3(p) \cap L_4 \cap L_5(p)$$

and

$$\{e^{\varsigma}_{c,p}(\Delta^{(\tilde{\beta})}, \nabla^m)\}^{\beta} = \{e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m)\}^{\beta} \cap L_6.$$

Proof. Consider the equality

$$\begin{aligned} \sum_{\vartheta=0}^{\nu} r_\vartheta x_\vartheta &= \sum_{\vartheta=0}^{\nu} \Bigg[\sum_{j=\mu}^{\vartheta} (-1)^{\vartheta-j} \binom{j}{\mu} \binom{m+\vartheta-j-1}{\vartheta-j} \\ &\quad \frac{\Gamma(-\tilde{\beta}+1)}{(\vartheta-j)!\Gamma(1-\tilde{\beta}-\vartheta+j)} (\varsigma-1)^{j-\mu} \varsigma^{-j} y_\mu \Bigg] r_\vartheta \\ &= \sum_{\vartheta=0}^{\nu} \Bigg[\sum_{\mu=\vartheta}^{\nu} \sum_{j=\vartheta}^{\mu} (-1)^{\mu-j} \binom{j}{\vartheta} \binom{m+\mu-j-1}{\mu-j} \\ &\quad \frac{\Gamma(-\tilde{\beta}+1)}{(\mu-j)!\Gamma(1-\tilde{\beta}-\mu+j)} (\varsigma-1)^{j-\vartheta} \varsigma^{-j} r_\mu \Bigg] y_\vartheta. \end{aligned}$$

This implies

$$\sum_{\vartheta=0}^{\nu} r_\vartheta x_\vartheta = (\tau y)_\nu, \tag{19}$$

where $\tau = \kappa_{\nu\vartheta}$ is defined by (18). Hence, from (19) we have $rx = (r_\vartheta x_\vartheta) \in cs$, whenever $x = (x_\vartheta) \in \{e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m)\}^{\beta}$ iff $\tau y \in c$, whenever $y = (y_\vartheta) \in c_0(p)$. Thus, by using (9), (10) and (11) for $q_\vartheta = 1$, $\forall\, \vartheta \in \mathbb{N}$, we get $\{e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m)\}^{\beta} = L_3(p) \cap L_4 \cap L_5(p)$. In the similar manner one can obtain the proof of $\{e^{\varsigma}_{c,p}(\Delta^{(\tilde{\beta})}, u, \nabla^m)\}^{\beta} = \{e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m)\}^{\beta} \cap L_6$ by using (9), (10), (11) and (12) with $q_\vartheta = 1$, $\forall \vartheta \in \mathbb{N}$. □

Theorem 2.9. *Let* $\tilde{\beta}$ *be a positive proper fraction. Then the* γ *dual of spaces* $e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m)$ *is* $L_3(p)$ *and that of* $e^{\varsigma}_{\infty,p}(\Delta^{(\tilde{\beta})}, \nabla^m)$ *is* $e^{\varsigma}_{0,p}(\Delta^{(\tilde{\beta})}, \nabla^m) \cap L_7$*, where* *the set L_7 is defined as:*

$$L_7 = \left\{ r = (r_\vartheta) \in \omega : \sup_\nu |\sum_\vartheta \kappa_{\nu\vartheta}| < \infty \right\}.$$

Proof. In a similar manner as in the above theorem one can easily get the proof of this theorem. □

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