JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **35**, No. 2, May 2022
http://dx.doi.org/10.14403/jcms.2022.35.2.177

THE DOUBLE FUZZY ELZAKI TRANSFORM FOR SOLVING FUZZY PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The Elzaki Transform method is fuzzified to fuzzy Elzaki Transform by Rehab Ali Khudair. In this article, we propose a Double fuzzy Elzaki transform (DFET) method to solving fuzzy partial differential equations (FPDEs) and we prove some properties and theorems of DFET, fundamental results of DFET for fuzzy partial derivatives of the n^{th} order, construct the Procedure to find the solution of FPDEs by DFET, provide duality relation of Double Fuzzy Laplace Transform (DFLT) and Double Fuzzy Sumudu Transform(DFST) with proposed Transform. Also we solve the Fuzzy Poisson's equation and fuzzy Telegraph equation to show the DFET method is a powerful mathematical tool for solving FPDEs analytically.

1. Introduction

Fuzzy integral, differential, and integro-differential equations have gotten a lot of attention in recent years. There are important role of the fuzzy theory and play significant act in numerical analysis. Freshly, some fuzzy researchers have examined analytical and numerical solutions to FDEs[1, 2, 3, 4, 5, 6, 7]. Zadeh[8] introduce the theory of fuzzy numbers, fuzzy sets and arithmetical operations. Seikkala[9] defined the concept of fuzzified derivatives, Buckley proposed the main concept of FPDEs in [10]. Many authors have investigated FPDEs[1, 9, 11, 12]. Integral transforms are extremely advantageous for solving partial differential

Received October 08, 2021; Accepted April 20, 2022.
2010 Mathematics Subject Classification: Primary 94D05, 35R13; Secondary 35A22.
Key words and phrases: Fuzzy Laplace Transform, Double fuzzy Elzaki Transform, Fuzzy partial differential equations.
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equations because integral transforms acquire a difficult function and transform into a new function which is better for solving. The Fourier, Laplace, Mellin, and Hankel transforms are integral transforms[13, 14, 15].

Tarig Elzaki[16] introduced the Elzaki transform as a novel integral transform. "The problem is converted to an algebraic problem, which is significantly easier to answer while utilising DFET, which is a part of operational calculus, and fuzzy Elzaki transform(FET)"[17], along with the fuzzy Laplace transform[3](FLT), is considered among the most powerful methods in this category. Here we propose a new approach of solving FPDEs using DFET in this study, which deals with the fuzzy environment of PDEs.

This paper has outlined as follows: In section 2, we summarize some of the basic definitions of the fuzzy number. In section 3, we propose the definition of the Double fuzzy Elzaki Transform, as well as various related properties and theorems, along with proofs. In section 4, we explains the relationship between the DFLT[18] and the DFST[5]. In section 5, we introduce a comprehensive technique for solving a Fuzzy Partial Differential Equation using the DFET. Finally in section 6, we apply the procedure in section 5 to solve the Fuzzy Poission's equation [19] and Fuzzy Telegraph equations[20], using the DFET.

2. Basic Concepts

Definition 2.1. [6, 8, 21] A fuzzy membership function of bounded support $\tilde{\Omega}_* : \mathbb{R} \to [0, 1]$ is called a "fuzzy number" containing in $\mathbb{R}$ with a upper semi-continuous, normal, convex.

Definition 2.2. [22] "Let $\tilde{\Omega}_* \in E^1$ and a fuzzy number $\tilde{\Omega}_*$ has the parametric form $\tilde{\Omega}_*(r) = (\underline{\Omega_*}(r), \overline{\Omega_*}(r))$ of functions $\underline{\Omega_*}(r), \overline{\Omega_*}(r)$, $0 \leq r \leq 1$ if and only if following conditions holds:

1. $\underline{\Omega_*}(r)$, $\overline{\Omega_*}(r)$ is left continuous function on $(0, 1]$, bounded and right continuous at 0 with respect to r.
2. $\underline{\Omega_*}(r)$, $\overline{\Omega_*}(r)$ is a non-decreasing and non-increasing respectively.
3. for $0 \leq r \leq 1$, $\underline{\Omega_*}(r) \leq \overline{\Omega_*}(r)$".

if $r = \underline{\Omega_*}(r) = \overline{\Omega_*}(r)$ then r is called "**crisp number**."

Definition 2.3. [23] "Let $\tilde{\Omega}_* \in E^1$ and for any $r \in [0, 1]$. The following properties are true for all $r-$ level set of $\tilde{\Omega}_*(r) = \left[\underline{\Omega_*}(r), \overline{\Omega_*}(r)\right]$

, $\tilde{\mho}_*(r) = \left[\underline{\mho_*}(r), \overline{\mho_*}(r)\right]$ is the crisp set, and scalar k, the interval based fuzzy arithematic is as,

1. $\underline{\Omega_*}(r) = \underline{\mho_*}(r)$ and $\overline{\Omega_*}(r) = \overline{\mho_*}(r)$ is necessary for $\tilde{\Omega}_*(r) = \tilde{\mho}_*(r)$
2. $\tilde{\Omega}_* + \tilde{\mho}_* = [\underline{\Omega_*}(r) + \underline{\mho_*}(r), \overline{\Omega_*}(r) + \overline{\mho_*}(r)]$
3. $k \odot \tilde{\Omega}_*(r) = \begin{cases} \left[k\underline{\Omega_*}(r), k\overline{\Omega_*}(r)\right] & k \geq 0, \\ \left[k\overline{\Omega_*}(r), k\underline{\Omega_*}(r)\right] & k < 0." \end{cases}$

Definition 2.4. [24, 25] "The distance $D_T(\tilde{\Omega}_*, \tilde{\mho}_*)$ between two fuzzy numbers $\tilde{\Omega}_*$ and $\tilde{\mho}_*$ is defiend as follow,
$D_T : E^1 \times E^1 \to R_+ \cup \{0\}$ by

$$D_T(\tilde{\Omega}_*, \tilde{\mho}_*) = \underset{r\in[0,1]}{sup}\, d_H(\tilde{\Omega}_*(r), \tilde{\mho}_*(r)), \tag{2.1}$$

where,

$$d_H(\tilde{\Omega}_*(r), \tilde{\mho}_*(r)) = max\left\{|\underline{\Omega_*}(r) - \underline{\mho_*}(r)|, |\overline{\Omega_*}(r) - \overline{\mho_*}(r)|\right\} \tag{2.2}$$

is the Hausdroff distance between $\tilde{\Omega}_*(r)$ and $\tilde{\mho}_*(r)$.

Thus, D_T is a metric space and has the following properties:

1. $D_T(\tilde{\Omega}_*(r) \oplus \tilde{z}(r), \tilde{\mho}_*(r) \oplus \tilde{z}(r)) = D_T(\tilde{\Omega}_*, \tilde{\mho}_*), \forall \tilde{\Omega}_*, \tilde{\mho}_*, \tilde{z} \in E^1$,
2. $D_T(k \odot \tilde{\Omega}_*(r), k \odot \tilde{\mho}_*(r)) = |k| D_T(\tilde{\Omega}_*, \tilde{\mho}_*), \forall k \in \mathbb{R}, \tilde{\Omega}_*, \tilde{\mho}_* \in E^1$,
3. $D_T(\tilde{\Omega}_* \oplus \tilde{\mho}_*, \tilde{z} \oplus \tilde{w}) \leq D_T(\tilde{\Omega}_*, \tilde{z}) + D_T(\tilde{\mho}_*, \tilde{w}), \forall \tilde{\Omega}_*, \tilde{\mho}_*, \tilde{z}, \tilde{w} \in E^1$,
4. (D_T, E^1) is a complete metric space."

Definition 2.5. [7] "Let $\tilde{\Omega}_*, \tilde{\mho}_* \in E^1$. If $\tilde{\Omega}_* = \tilde{\mho}_* + \tilde{z}$ such that there exists $\tilde{z} \in E^1$, $\tilde{z}$ is the Hukuhara difference of $\tilde{\Omega}_*$ and $\tilde{\mho}_*$, and is denoted by $\tilde{z} = \tilde{\Omega}_* \ominus \tilde{\mho}_*$. Note that $\tilde{\Omega}_* + (-1)\tilde{\mho}_* \neq \tilde{\Omega}_* \ominus \tilde{\mho}_*$."

Definition 2.6. [5] A fuzzy valued function $\tilde{g} : D_T \to E^1$ is called continuous at $(\Omega_0, \mho_0) \in D_T$ if $\forall \epsilon > 0 \,\exists\, \delta > 0$ such that $d(\tilde{g}(\Omega, \mho), \tilde{g}(\Omega_0, \mho_0)) < \epsilon$ whenever $|\Omega - \Omega_0| + |\mho - \mho_0| < \delta$. $\tilde{g}$ is continuous on D_T, if $\tilde{g}$ be continuous $\forall (\Omega, \mho) \in D_T$.

Theorem 2.7. [11, 26] *"Let $\tilde{g} : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ be fuzzy valued function and for all $r \in [0, 1]$ the functions $\underline{g}(\Omega, \mho; r)$ and $\overline{g}(\Omega, \mho; r)$ are Riemann integrable on D_{T_r}. Then for every $r > 0$ there are constants $\underline{K}(r) > 0$ and $\overline{K}(r)$, such that*

$$\int\int_{D_{T_r}} |\underline{g}(\Omega, \mho; r)| d\Omega d\mho \leq \underline{K}(r), \int\int_{D_{T_r}} |\overline{g}(\Omega, \mho; r)| d\Omega d\mho \leq \overline{K}(r)$$

and the function $\tilde{g}(\Omega,\mho;r)$ *is improper fuzzy Riemann- integrable on* D_T *and*

$$(FR)\int_0^\infty (FR)\int_0^\infty \tilde{g}(\Omega,\mho;r)d\Omega d\mho = \left(\int_0^\infty\int_0^\infty \underline{g}(\Omega,\mho;r)d\Omega d\mho, \int_0^\infty\int_0^\infty \overline{g}(\Omega,\mho;r)d\Omega d\mho\right)."$$

Definition 2.8. [4, 5, 11, 27] "Let $\tilde{g}(\Omega,\mho;r)$ is strongly gH-differen–tiable of the n^{th} order at $\Omega_0\in(c,d)$ w.r.t. Ω, if $\exists\frac{\partial^n\tilde{g}(\Omega_0,\mho;r)}{\partial\Omega^n}\in E^1$ such that

1. $\forall\ \triangle>0$ sufficiently small the gH- differences

$$\frac{\partial^{n-1}\tilde{g}(\Omega_0+\triangle,\mho;r)}{\partial\Omega^{n-1}}\ominus_{gH}\frac{\partial^{n-1}\tilde{g}(\Omega_0,\mho;r)}{\partial\Omega^{n-1}},\ \frac{\partial^{n-1}\tilde{g}(\Omega_0,\mho;r)}{\partial\Omega^{n-1}}\ominus_{gH}\frac{\partial^{n-1}\tilde{g}(\Omega_0-\triangle,\mho;r)}{\partial\Omega^{n-1}}$$

exist and the following limits hold

$$\lim_{\triangle\to0+}\frac{\frac{\partial^{n-1}\tilde{g}(\Omega_0+\triangle,\mho;r)}{\partial\Omega^{n-1}}\ominus_{gH}\frac{\partial^{n-1}\tilde{g}(\Omega_0,\mho;r)}{\partial\Omega^{n-1}}}{\triangle}=\lim_{\triangle\to0+}\frac{\frac{\partial^{n-1}\tilde{g}(\Omega_0,\mho;r)}{\partial\Omega^{n-1}}\ominus_{gH}\frac{\partial^{n-1}\tilde{g}(\Omega_0-\triangle,\mho;r)}{\partial\Omega^{n-1}}}{\triangle}=\frac{\partial^n\tilde{g}(\Omega_0,\mho;r)}{\partial\Omega^n}.$$

or

2. $\forall\ \triangle>0$ sufficiently small the gH- differences

$$\frac{\partial^{n-1}\tilde{g}(\Omega_0,\mho;r)}{\partial\Omega^{n-1}}\ominus_{gH}\frac{\partial^{n-1}\tilde{g}(\Omega_0+\triangle,\mho;r)}{\partial\Omega^{n-1}},\ \frac{\partial^{n-1}\tilde{g}(\Omega_0-\triangle,\mho;r)}{\partial\Omega^{n-1}}\ominus_{gH}\frac{\partial^{n-1}\tilde{g}(\Omega_0,\mho;r)}{\partial\Omega^{n-1}}$$

exist and the following limits hold

$$\lim_{\triangle\to0+}\frac{\frac{\partial^{n-1}\tilde{g}(\Omega_0,\mho;r)}{\partial\Omega^{n-1}}\ominus_{gH}\frac{\partial^{n-1}\tilde{g}(\Omega_0+\triangle,\mho;r)}{\partial\Omega^{n-1}}}{-\triangle}=\lim_{\triangle\to0+}\frac{\frac{\partial^{n-1}\tilde{g}(\Omega_0-\triangle,\mho;r)}{\partial\Omega^{n-1}}\ominus_{gH}\frac{\partial^{n-1}\tilde{g}(\Omega_0,\mho;r)}{\partial\Omega^{n-1}}}{-\triangle}=\frac{\partial^n\tilde{g}(\Omega_0,\mho;r)}{\partial\Omega^n}".$$

Definition 2.9. [4, 5, 11, 27] "Let $\tilde{g}(\Omega,\mho;r)$ is strongly gH-differen–tiable of the n^{th} order at $\mho_0\in(c,d)$ w.r.t. $\mho$, if $\exists\frac{\partial^n\tilde{g}(\Omega,\mho_0;r)}{\partial\mho^n}\in E^1$ such that

1. $\forall\ \triangle>0$ sufficiently small the gH- differences

$$\frac{\partial^{n-1}\tilde{g}(\Omega,\mho_0+\triangle;r)}{\partial\mho^{n-1}}\ominus_{gH}\frac{\partial^{n-1}\tilde{g}(\Omega,\mho_0;r)}{\partial\mho^{n-1}},\ \frac{\partial^{n-1}\tilde{g}(\Omega,\mho_0;r)}{\partial\mho^{n-1}}\ominus_{gH}\frac{\partial^{n-1}\tilde{g}(\Omega,\mho_0-\triangle;r)}{\partial\mho^{n-1}}$$

exist and the following limits hold

$$\lim_{\triangle\to0+}\frac{\frac{\partial^{n-1}\tilde{g}(\Omega,\mho_0+\triangle;r)}{\partial\mho^{n-1}}\ominus_{gH}\frac{\partial^{n-1}\tilde{g}(\Omega,\mho_0;r)}{\partial\mho^{n-1}}}{\triangle}=\lim_{\triangle\to0+}\frac{\frac{\partial^{n-1}\tilde{g}(\Omega,\mho_0;r)}{\partial\mho^{n-1}}\ominus_{gH}\frac{\partial^{n-1}\tilde{g}(\Omega,\mho_0-\triangle;r)}{\partial\mho^{n-1}}}{\triangle}=\frac{\partial^n\tilde{g}(\Omega,\mho_0;r)}{\partial\mho^n}$$

or

2. $\forall\ \triangle>0$ sufficiently small the gH- differences

$$\frac{\partial^{n-1}\tilde{g}(\Omega,\mho_0;r)}{\partial\mho^{n-1}}\ominus_{gH}\frac{\partial^{n-1}\tilde{g}(\Omega,\mho_0+\triangle;r)}{\partial\mho^{n-1}},\ \frac{\partial^{n-1}\tilde{g}(\Omega,\mho_0-\triangle;r)}{\partial\mho^{n-1}}\ominus_{gH}\frac{\partial^{n-1}\tilde{g}(\Omega,\mho_0;r)}{\partial\mho^{n-1}}$$

exist and the following limits hold

$$\lim_{\Delta\to 0+}\frac{\frac{\partial^{n-1}\tilde{g}(\Omega,\mho_0;r)}{\partial\mho^{n-1}}\ominus_{gH}\frac{\partial^{n-1}\tilde{g}(\Omega,\mho_0+\Delta;r)}{\partial\mho^{n-1}}}{-\Delta}=\lim_{\Delta\to 0+}\frac{\frac{\partial^{n-1}\tilde{g}(\Omega,\mho_0-\Delta;r)}{\partial\mho^{n-1}}\ominus_{gH}\frac{\partial^{n-1}\tilde{g}(\Omega,\mho_0;r)}{\partial\mho^{n-1}}}{-\Delta}$$
$$=\frac{\partial^n\tilde{g}(\Omega,\mho_0;r)}{\partial\mho^n}\text{”}.$$

THEOREM 2.10. [2, 4, 5, 27] *“Let $\tilde{g}(\Omega,\mho;r):\mathbb{R}_+\times\mathbb{R}_+\to E^1$ be a gH differentiable fuzzy valued function. Then*

1. *If $\tilde{g}(\Omega,\mho;r)$ is First differentiable of the n^{th} order with respect to Ω then $\underline{g}(\Omega,\mho;r)$ and $\overline{g}(\Omega,\mho;r)$ are differentiable of the n^{th} order with respect to Ω and*

$$\frac{\partial^n\tilde{g}(\Omega,\mho;r)}{\partial\Omega^n}=\left(\frac{\partial^n\underline{g}(\Omega,\mho;r)}{\partial\Omega^n},\frac{\partial^n\overline{g}(\Omega,\mho;r)}{\partial\Omega^n}\right).\tag{2.3}$$

2. *If $\tilde{g}(\Omega,\mho;r)$ is Second differentiable of the n^{th} order with respect to Ω then $\underline{g}(\Omega,\mho;r)$ and $\overline{g}(\Omega,\mho;r)$ are differentiable of the n^{th} order with respect to Ω and*

$$\frac{\partial^n\tilde{g}(\Omega,\mho;r)}{\partial\Omega^n}=\left(\frac{\partial^n\overline{g}(\Omega,\mho;r)}{\partial\Omega^n},\frac{\partial^n\underline{g}(\Omega,\mho;r)}{\partial\Omega^n}\right)\text{”}.\tag{2.4}$$

3. Main Results

In this section, we present DFET, its inverse, some properties and theorems associated with DFET.

DEFINITION 3.1. Let $\tilde{g}:\mathbb{R}_+\times\mathbb{R}_+\to E^1$ be the continuous fuzzy valued function. Suppose that,

$$Exp\left(-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)\right)\odot\tilde{g}(\Omega,\mho;r)$$

is improper fuzzy Riemann integrable on D then,

$$(pq)(FR)\int_0^\infty(FR)\int_0^\infty Exp\left(-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)\right)\odot\tilde{g}(\Omega,\mho;r)d\Omega d\mho\tag{3.1}$$

is known as DFET and for $\Omega, \mho > 0, p, q \in \mathbb{C}$ is denoted by

(3.2)
$$\begin{aligned}\varepsilon(p,q) &= \mathcal{E}\left[\tilde{g}(\Omega,\mho;r);p,q\right] \\ &= (pq)(FR)\int_0^\infty (FR)\int_0^\infty Exp\left(-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)\right)\odot \tilde{g}(\Omega,\mho;r)d\Omega d\mho\end{aligned}$$

where $D \subseteq \mathbb{R}_+ \times \mathbb{R}_+$.

DFET also can be represented parametrically as follow

(3.3)
$$\varepsilon(p,q) = \mathcal{E}\left[\tilde{g}(\Omega,\mho;r);p,q\right] = \left(e\left[\underline{g}(\Omega,\mho;r);p,q\right], e\left[\overline{g}(\Omega,\mho;r);p,q\right]\right)$$

where

$$e\left[\underline{g}(\Omega,\mho;r);p,q\right] = (pq)\int_0^\infty\int_0^\infty e^{-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)}\underline{g}(\Omega,\mho;r)d\Omega d\mho \tag{3.4}$$

$$e\left[\overline{g}(\Omega,\mho;r);p,q\right] = (pq)\int_0^\infty\int_0^\infty e^{-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)}\overline{g}(\Omega,\mho;r)d\Omega d\mho \tag{3.5}$$

we can rewrite (3.2) in different form as:

(3.6)
$$\varepsilon(p,q) = (p^2q^2)(FR)\int_0^\infty (FR)\int_0^\infty Exp\left(-\left(\Omega+\mho\right)\right)\odot \tilde{g}(\Omega p,\mho q;r)d\Omega d\mho.$$

Definition 3.2. Let $\varepsilon(p,q) = \mathcal{E}\left[\tilde{g}(\Omega,\mho;r);p,q\right]$ be a Double Fuzzy Elzaki Transform (DFET) then inverse of Double Fuzzy Elzaki Transform is defined as

$$\mathcal{E}^{-1}\left[\varepsilon(p,q)\right] = \tilde{g}(\Omega,\mho;r) = \left(e^{-1}\left[\underline{\varepsilon}(p,q)\right], e^{-1}\left[\overline{\varepsilon}(p,q)\right]\right) \tag{3.7}$$

where

$$e^{-1}\left[\underline{\varepsilon}(p,q)\right] = \frac{1}{2\pi i}\odot\int_{c-i\infty}^{c+i\infty} e^{\frac{\Omega}{p}}dp\frac{1}{2\pi i}\odot\int_{d-i\infty}^{d+i\infty} e^{\frac{\mho}{q}}\underline{\varepsilon}(p,q)dq \tag{3.8}$$

$$e^{-1}\left[\overline{\varepsilon}(p,q)\right] = \frac{1}{2\pi i}\odot\int_{c-i\infty}^{c+i\infty} e^{\frac{\Omega}{p}}dp\frac{1}{2\pi i}\odot\int_{d-i\infty}^{d+i\infty} e^{\frac{\mho}{q}}\overline{\varepsilon}(p,q)dq \tag{3.9}$$

where, $\Re(p) \geq c$, $\Re(q) \geq d$ and $c, d \in \mathbb{R}$.

Results of some special function in the term of classical double Elzaki transform
for $\Omega > 0, \mho > 0$ and $\alpha, \beta \in \mathbb{R}$

- If $g(\Omega,\mho) = 1$ then $e\left[g(\Omega,\mho)\right] = p^2q^2$.
- If $g(\Omega,\mho) = \Omega^{\alpha}\mho^{\beta}$ then $e\left[g(\Omega,\mho)\right] = (\alpha!)(\beta!)p^{\alpha+2}q^{\beta+2}$, where α, β are positive integers.
- If $g(\Omega,\mho) = e^{(\alpha\Omega+\beta\mho)}$ then $e\left[g(\Omega,\mho)\right] = \frac{p^2q^2}{(1-\alpha p)(1-\beta q)}$.
- If $g(\Omega,\mho) = \cos(\alpha\Omega+\beta\mho)$ then $e\left[g(\Omega,\mho)\right] = \frac{p^2q^2}{(1+\alpha^2p^2)(1+\beta^2q^2)}$.
- If $g(\Omega,\mho) = \sin(\alpha\Omega+\beta\mho)$ then $e\left[g(\Omega,\mho)\right] = \frac{(\alpha\beta)(p^3q^3)}{(1+\alpha^2p^2)(1+\beta^2q^2)}$.
- If $g(\Omega,\mho) = \cosh(\alpha\Omega+\beta\mho)$ then $e\left[g(\Omega,\mho)\right] = \frac{p^2q^2}{(1-\alpha^2p^2)(1-\beta^2q^2)}$.
- If $g(\Omega,\mho) = \sinh(\alpha\Omega+\beta\mho)$ then $e\left[g(\Omega,\mho)\right] = \frac{(\alpha\beta)(p^3q^3)}{(1-\alpha^2p^2)(1-\beta^2q^2)}$.

We give some theorems and properties related with DFET.

THEOREM 3.3. *let $\tilde{g}_1, \tilde{g}_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ be continuous fuzzy valued functions then*

$$\mathcal{E}[c_1\odot\tilde{g}_1(\Omega,\mho;r)\oplus c_2\odot\tilde{g}_2(\Omega,\mho;r);p,q] = c_1\odot\mathcal{E}[\tilde{g}_1(\Omega,\mho;r);p,q]\oplus c_2\odot\mathcal{E}[\tilde{g}_2(\Omega,\mho;r);p,q] \tag{3.10}$$

where, c_1, c_2 are abitrary constants.

Proof: Let $\tilde{g}_1, \tilde{g}_2$ be continuous fuzzy valued functions.
So, we prove for the lower bound of $\tilde{g}_1(\Omega,\mho;r)$ and $\tilde{g}_2(\Omega,\mho;r)$

$$\begin{aligned} e\left[c_1\underline{g_1}(\Omega,\mho;r) + c_2\underline{g_2}(\Omega,\mho;r)\right] &= (pq)\int_0^\infty\int_0^\infty e^{-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)}\left(c_1\underline{g_1}(\Omega,\mho;r) + c_2\underline{g_2}(\Omega,\mho;r)\right)d\Omega d\mho \\ &= \Bigg(c_1(pq)\int_0^\infty\int_0^\infty e^{-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)}\underline{g_1}(\Omega,\mho;r)d\Omega d\mho \\ &\quad + c_2(pq)\int_0^\infty\int_0^\infty e^{-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)}\underline{g_2}(\Omega,\mho;r)d\Omega d\mho\Bigg) \end{aligned} \tag{3.11}$$

simillarly we prove,

(3.12)

$$e\left[c_1\overline{g_1}(\Omega,\mho;r)+c_2\overline{g_2}(\Omega,\mho;r)\right]=\left(c_1(pq)\int_0^\infty\int_0^\infty e^{-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)}\overline{g_1}(\Omega,\mho;r)d\Omega d\mho\right.$$
$$\left.+c_2(pq)\int_0^\infty\int_0^\infty e^{-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)}\overline{g_2}(\Omega,\mho;r)d\Omega d\mho\right)$$

from (3.11) and (3.12) produces (3.10).

THEOREM 3.4. ***First Translation Property***[11]
"If $\varepsilon(p,q)=\mathcal{E}\left[\tilde{g}(\Omega,\mho,r);p,q\right]$ is DFET and α,β are arbitrary constants, where, $\tilde{g}(\Omega,\mho;r)$ be a continuous fuzzy valued function then

(3.13)

$$\mathcal{E}\left[e^{(\alpha\Omega+\beta\mho)}\odot\tilde{g}(\Omega,\mho;r)\right]=(1-\alpha p)(1-\beta q)\varepsilon\left(\frac{p}{(1-\alpha p)},\frac{q}{(1-\beta q)}\right)\text{''}.$$

Proof: Let $\tilde{g}$ be continuous fuzzy valued functions, from the definition of DFET for lower bound we obtain,

$$e\left[e^{(\alpha\Omega+\beta\mho)}\underline{g}(\Omega,\mho;r)\right]=(pq)\int_0^\infty\int_0^\infty e^{(\alpha\Omega+\beta\mho)}e^{-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)}\underline{g}(\Omega,\mho;r)d\Omega d\mho$$
$$=(pq)\int_0^\infty\int_0^\infty e^{-\left[\left(\frac{1}{p}-\alpha\right)\Omega+\left(\frac{1}{q}-\beta\right)\mho\right]}\underline{g}(\Omega,\mho;r)d\Omega d\mho$$
$$=(1-\alpha p)(1-\beta q)\varepsilon\left(\frac{p}{(1-\alpha p)},\frac{q}{(1-\beta q)}\right)\tag{3.14}$$

and simillarly we can prove for upper bound

$$e\left[e^{(\alpha\Omega+\beta\mho)}\overline{g}(\Omega,\mho;r)\right]=(1-\alpha p)(1-\beta q)\varepsilon\left(\frac{p}{(1-\alpha p)},\frac{q}{(1-\beta q)}\right)\tag{3.15}$$

Hence, equation (3.14) and (3.15) produces equation (3.13).

THEOREM 3.5. ***Second Translation Property:*** *If the DFET of function $\tilde{g}(\Omega,\mho;r)$ exists then*

(3.16)

$$\mathcal{E}\left[\tilde{g}(\Omega-\eta,\mho-\zeta;r)H(\Omega-\eta,\mho-\zeta)\right]=e^{-\left(\frac{\eta}{p}+\frac{\zeta}{q}\right)}\mathcal{E}\left[\tilde{g}(\Omega-\eta,\mho-\zeta;r)\right]$$

where $H(\Omega,\mho)$ is Heaviside unit step function defined by

$$H(\Omega-\eta,\mho-\zeta)=\begin{cases}1, & \text{when }\ \Omega>\eta\ \text{ and }\ \mho>\zeta\\ 0, & \text{when }\ \Omega<\eta\ \text{ and }\ \mho<\zeta\end{cases}\tag{3.17}$$

Proof: Let $\tilde{g}$ be continuous fuzzy valued functions, from the definition of DFET for lower bound ($\Omega > \eta$ and $\mho > \zeta$) we obtain,

$$e\left[\underline{g}(\Omega-\eta,\mho-\zeta;r)H(\Omega-\eta,\mho-\zeta)\right]=(pq)\int_0^\infty\int_0^\infty e^{-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)}\underline{g}(\Omega-\eta,\mho-\zeta;r)d\Omega d\mho$$

By Putting $\alpha=\Omega-\eta,\ \beta=\mho-\zeta \implies d\alpha=d\Omega,\ d\beta=d\mho$

$$=(pq)\int_0^\infty\int_0^\infty e^{-\left(\frac{\alpha+\eta}{p}+\frac{\beta+\zeta}{q}\right)}\underline{g}(\alpha,\beta;r)d\Omega d\mho$$

$$=(pq)e^{-\left(\frac{\eta}{p}+\frac{\zeta}{q}\right)}\int_0^\infty\int_0^\infty e^{-\left(\frac{\alpha}{p}+\frac{\beta}{q}\right)}\underline{g}(\alpha,\beta;r)d\Omega d\mho$$

$$=e^{-\left(\frac{\eta}{p}+\frac{\zeta}{q}\right)}\mathcal{E}\left[\underline{g}(\alpha,\beta;r)\right]$$

$$=e^{-\left(\frac{\eta}{p}+\frac{\zeta}{q}\right)}\mathcal{E}\left[\underline{g}(\Omega-\eta,\mho-\zeta;r)\right]$$

simillarly we can prove for upper bound and we get desired result.

THEOREM 3.6. *Let $\tilde{g}:\mathbb{R}_+\times\mathbb{R}_+\to E^1$ be the continuous fuzzy valued function. The functions $Exp\left(-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)\right)\odot\tilde{g}(\Omega,\mho;r)$ and $Exp\left(-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)\right)\odot\frac{\partial^m\tilde{g}(\Omega,\mho;r)}{\partial\Omega^m}$ are improper fuzzy Riemann integral on D, then*

$$\mathcal{E}\left[\frac{\partial^m\tilde{g}(\Omega,\mho;r)}{\partial\Omega^m}\right]=\frac{\partial^m}{\partial\Omega^m}\mathcal{E}\left[\tilde{g}(\Omega,\mho;r)\right]. \tag{3.18}$$

Proof: Let $\tilde{g}(\Omega,\mho;r)$ be First- differentiable from definition of DFET we have

$$\mathcal{E}\left[\frac{\partial^m\tilde{g}(\Omega,\mho;r)}{\partial\Omega^m}\right]=(pq)(FR)\int_0^\infty(FR)\int_0^\infty e^{\left(-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)\right)}\odot\frac{\partial^m\tilde{g}(\Omega,\mho;r)}{\partial\Omega^m}d\Omega d\mho$$

$$=\frac{\partial^m}{\partial\Omega^m}\left((pq)(FR)\int_0^\infty(FR)\int_0^\infty e^{\left(-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)\right)}\odot\tilde{g}(\Omega,\mho;r)d\Omega d\mho\right)$$

$$=\frac{\partial^m}{\partial\Omega^m}\mathcal{E}\left[\tilde{g}(\Omega,\mho;r)\right].$$

THEOREM 3.7. *Let $\tilde{g}:\mathbb{R}_+\times\mathbb{R}_+\to E^1$ be the continuous fuzzy valued function. The functions $Exp\left(-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)\right)\odot\tilde{g}(\Omega,\mho;r)$ and $Exp\left(-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)\right)\odot\frac{\partial^m\tilde{g}(\Omega,\mho;r)}{\partial\Omega^m}$ are improper fuzzy Riemann integral on*

D for all $\Omega > 0$ *and* $m \in \mathbb{N}$ $\exists$ *to continuous partial gH-derivatives to* $(m-1)^{th}$ *order w.r.t.* Ω *and* $\exists$ $\frac{\partial^m \tilde{g}(\Omega,\mho;r)}{\partial \Omega^m}$ *with,*

a) If the function $\tilde{g}(\Omega,\mho;r)$ is First- differentiable then

$$\mathcal{E}\left[\frac{\partial^m \tilde{g}(\Omega,\mho;r)}{\partial\Omega^m}\right] = \left(e\left[\frac{\partial^m \underline{g}(\Omega,\mho;r)}{\partial\Omega^m}\right], e\left[\frac{\partial^m \overline{g}(\Omega,\mho;r)}{\partial\Omega^m}\right]\right)$$

b) If the function $\tilde{g}(\Omega,\mho;r)$ is Second- differentiable then

$$\mathcal{E}\left[\frac{\partial^m \tilde{g}(\Omega,\mho;r)}{\partial\Omega^m}\right] = \left(e\left[\frac{\partial^m \overline{g}(\Omega,\mho;r)}{\partial\Omega^m}\right], e\left[\frac{\partial^m \underline{g}(\Omega,\mho;r)}{\partial\Omega^m}\right]\right)$$

where,

(3.19)
$$e\left[\frac{\partial^m \underline{g}(\Omega,\mho;r)}{\partial\Omega^m}\right] = \frac{1}{p^m} e\left[\underline{g}(\Omega,\mho;r)\right] - \sum_{k=0}^{m-1} p^{2-m+k} e\left[\frac{\partial^k \underline{g}(\Omega,\mho;r)}{\partial\Omega^k}\right]_{\Omega=0}$$

(3.20)
$$e\left[\frac{\partial^m \overline{g}(\Omega,\mho;r)}{\partial\Omega^m}\right] = \frac{1}{p^m} e\left[\overline{g}(\Omega,\mho;r)\right] - \sum_{k=0}^{m-1} p^{2-m+k} e\left[\frac{\partial^k \overline{g}(\Omega,\mho;r)}{\partial\Omega^k}\right]_{\Omega=0}.$$

Proof: If the function $\tilde{g}(\Omega,\mho;r)$ is First- differentiable then by mathematical induction we proof the equation (3.19).
The equation (3.19) is true for $m = 1$, as from parametric form of DFET
,

$$\mathcal{E}\left[\frac{\partial \tilde{g}(\Omega,\mho;r)}{\partial\Omega}\right] = \left(e\left[\frac{\partial \underline{g}(\Omega,\mho;r)}{\partial\Omega}\right], e\left[\frac{\partial \overline{g}(\Omega,\mho;r)}{\partial\Omega}\right]\right)$$

now using integration by parts with respect to Ω , we get

$$\begin{aligned} e\left[\frac{\partial \underline{g}(\Omega,\mho;r)}{\partial\Omega}\right] &= (pq)\int_0^\infty\int_0^\infty e^{-\left(\frac{\Omega}{p}+\frac{\mho}{q}\right)} \frac{\partial \underline{g}(\Omega,\mho;r)}{\partial\Omega} d\Omega d\mho \\ &= \frac{1}{p} e\left[\underline{g}(\Omega,\mho;r)\right] - pe\left[\underline{g}(\Omega,\mho;r)\right]_{\Omega=0}. \end{aligned}$$

Assume that, equation (3.19) is true for $m = j$.

$$e\left[\frac{\partial^j \underline{g}(\Omega,\mho;r)}{\partial\Omega^j}\right] = \frac{1}{p^j} e\left[\underline{g}(\Omega,\mho;r)\right] - \sum_{k=0}^{j-1} p^{2-j+k} e\left[\frac{\partial^k \underline{g}(\Omega,\mho;r)}{\partial\Omega^k}\right]_{\Omega=0}$$

Now to show that the equation (3.19) is also true for $m = j+1$, we get

$$\begin{aligned}
e\left[\frac{\partial^{j+1}\underline{g}(\Omega,\mho;r)}{\partial\Omega^{j+1}}\right] &= \frac{\partial}{\partial\Omega}e\left[\frac{\partial^{j}\underline{g}(\Omega,\mho;r)}{\partial\Omega^{j}}\right]\\
&= \frac{\partial}{\partial\Omega}\left[\frac{1}{p^j}e\left[\underline{g}(\Omega,\mho;r)\right]-\sum_{k=0}^{j-1}p^{2-j+k}e\left[\frac{\partial^{k}\underline{g}(\Omega,\mho;r)}{\partial\Omega^{k}}\right]_{\Omega=0}\right]\\
&= \frac{\partial}{\partial\Omega}\left(\frac{1}{p^j}e\left[\underline{g}(\Omega,\mho;r)\right]\right)-\frac{\partial}{\partial\Omega}\left(\sum_{k=0}^{j-1}p^{2-j+k}e\left[\frac{\partial^{k}\underline{g}(\Omega,\mho;r)}{\partial\Omega^{k}}\right]_{\Omega=0}\right)\\
&= \frac{1}{p^j}e\left[\frac{\partial\underline{g}(\Omega,\mho;r)}{\partial\Omega}\right]-\sum_{k=1}^{j-1}p^{2-j+k}e\left[\frac{\partial^{k+1}\underline{g}(\Omega,\mho;r)}{\partial\Omega^{k+1}}\right]_{\Omega=0}\\
&= \frac{1}{p^{j+1}}e\left[\underline{g}(\Omega,\mho;r)\right]-pe\left[\underline{g}(\Omega,\mho;r)\right]_{\Omega=0}-\sum_{k=1}^{j-1}p^{2-j+k}e\left[\frac{\partial^{k+1}\underline{g}(\Omega,\mho;r)}{\partial\Omega^{k+1}}\right]_{\Omega=0}\\
&= \frac{1}{p^{j+1}}e\left[\underline{g}(\Omega,\mho;r)\right]-\sum_{k=0}^{j}p^{1-j+k}e\left[\frac{\partial^{k}\underline{g}(\Omega,\mho;r)}{\partial\Omega^{k}}\right]_{\Omega=0}
\end{aligned}$$

similarly, we can prove equation (3.20).

4. Duality of Double Fuzzy Elzaki Transform

THEOREM 4.1. ***Duality of Double fuzzy Elzaki-Laplace Transform:*** *If DFET and DFLT*[18] *of* $\tilde{g}(\Omega,\mho;r)$ *are* $\varepsilon(\alpha,\beta)$ *and* $\mathfrak{L}(\alpha,\beta)$ *respectively then*

$$\begin{aligned}
\varepsilon(\alpha,\beta) &= (\alpha\beta)\mathfrak{L}\left(\frac{1}{\alpha},\frac{1}{\beta}\right)\\
\mathfrak{L}(\alpha,\beta) &= (\alpha\beta)\varepsilon\left(\frac{1}{\alpha},\frac{1}{\beta}\right).
\end{aligned} \tag{4.1}$$

THEOREM 4.2. ***Duality of Double fuzzy Elzaki-Sumudu Transform:*** *If DFET and DFST*[5] *of* $\tilde{g}(\Omega,\mho;r)$ *are* $\varepsilon(\alpha,\beta)$ *and* $\mathfrak{S}(\alpha,\beta)$ *respectively then*

$$\begin{aligned}
\varepsilon(\alpha,\beta) &= (\alpha\beta)^2\mathfrak{S}\left(\alpha,\beta\right)\\
\mathfrak{S}(\alpha,\beta) &= \frac{1}{(\alpha\beta)^2}\varepsilon\left(\alpha,\beta\right).
\end{aligned} \tag{4.2}$$

5. Procedure to find the solution of Fuzzy Partial Differential Equation by Double Fuzzy Elzaki Transform

In this section, we present the DFET method for solving FPDE. consider the following general FPDE

(5.1)
$$\sum_{i=1}^{m} A_i \odot \frac{\partial^i \tilde{g}(\Omega, \mho; r)}{\partial \Omega^i} \oplus \sum_{j=1}^{n} B_j \odot \frac{\partial^j \tilde{g}(\Omega, \mho; r)}{\partial \mho^j} \oplus C \odot \tilde{g}(\Omega, \mho; r) = \tilde{f}(\Omega, \mho; r),$$

with initial conditions

$$\frac{\partial^j \tilde{g}(\Omega, 0; r)}{\partial \mho^j} = \Psi_j(\Omega), \qquad j = 0 \to k-1 \tag{5.2}$$

and boundary conditions

$$\frac{\partial^i \tilde{g}(0, \mho; r)}{\partial \Omega^i} = \Phi_i(\mho), \qquad i = 0 \to m-1 \tag{5.3}$$

where $\tilde{f}, \tilde{g} : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$, $\Phi_i : \mathbb{R}_+ \to E^1$, $\Psi_j : \mathbb{R}_+ \to E^1$ are continuous fuzzy functions and $A_i, i = 0 \to m-1$, $B_j, j = 0 \to k-1$ and C are constants. Applying DFET on both side of equation (5.1) we get,

(5.4)
$$\mathcal{E}\left[\sum_{i=1}^{m} A_i \odot \frac{\partial^i \tilde{g}(\Omega, \mho; r)}{\partial \Omega^i}\right] \oplus \mathcal{E}\left[\sum_{j=1}^{n} B_j \odot \frac{\partial^j \tilde{g}(\Omega, \mho; r)}{\partial \Omega^j}\right] \oplus \mathcal{E}\left[\tilde{g}(\Omega, \mho; r)\right] = \mathcal{E}\left[\tilde{f}(\Omega, \mho; r)\right]$$

Using above Theorem 3.7

a) if $\tilde{g}(\Omega, \mho; r)$ First- differentiable then

$$\sum_{i=1}^{m} A_i e\left(\frac{\partial^i \underline{g}(\Omega, \mho; r)}{\partial \Omega^i}\right) + \sum_{j=1}^{n} B_j e\left(\frac{\partial^j \underline{g}(\Omega, \mho; r)}{\partial \Omega^j}\right) + Ce\left(\underline{g}(\Omega, \mho; r)\right) = e\left(\underline{f}(\Omega, \mho; r)\right)$$

and

$$\sum_{i=1}^{m} A_i e\left(\frac{\partial^i \overline{g}(\Omega, \mho; r)}{\partial \Omega^i}\right) + \sum_{j=1}^{n} B_j e\left(\frac{\partial^j \overline{g}(\Omega, \mho; r)}{\partial \Omega^j}\right) + Ce\left(\overline{g}(\Omega, \mho; r)\right) = e\left(\overline{f}(\Omega, \mho; r)\right)$$

From equation (3.19) and equation (3.20) we have,

$$\left(\sum_{i=1}^{m}\frac{A_i}{p^i}+\sum_{j=1}^{n}\frac{B_j}{q^j}+C\right)e\left[\underline{g}(\Omega,\mho;r)\right]$$
$$=e\left(\underline{f}(\Omega,\mho;r)\right)+\sum_{i=1}^{m}A_i\sum_{k=0}^{i-1}p^{2-i+k}e\left[\frac{\partial^k\underline{g}(0,\mho;r)}{\partial\Omega^k}\right]$$
$$+\sum_{j=1}^{n}B_j\sum_{k=0}^{j-1}p^{2-j+k}e\left[\frac{\partial^k\underline{g}(\Omega,0;r)}{\partial\mho^k}\right]$$

and

$$\left(\sum_{i=1}^{m}\frac{A_i}{p^i}+\sum_{j=1}^{n}\frac{B_j}{q^j}+C\right)e\left[\overline{g}(\Omega,\mho;r)\right]$$
$$=e\left(\overline{f}(\Omega,\mho;r)\right)+\sum_{i=1}^{m}A_i\sum_{k=0}^{i-1}p^{2-i+k}e\left[\frac{\partial^k\overline{g}(0,\mho;r)}{\partial\Omega^k}\right]$$
$$+\sum_{j=1}^{n}B_j\sum_{k=0}^{j-1}p^{2-j+k}e\left[\frac{\partial^k\overline{g}(\Omega,0;r)}{\partial\mho^k}\right]$$

and using ICs equation (5.2) and BCs equation (5.3) we get,

$$\left(\sum_{i=1}^{m}\frac{A_i}{p^i}+\sum_{j=1}^{n}\frac{B_j}{q^j}+C\right)e\left[\underline{g}(\Omega,\mho;r)\right]$$
$$=\left\{\begin{array}{c} e\left(\underline{f}(\Omega,\mho;r)\right)+\sum_{i=1}^{m}A_i\sum_{k=0}^{i-1}p^{2-i+k}e\left[\underline{g}_k^k(0,\mho;r)\right] \\ +\sum_{j=1}^{n}B_j\sum_{k=0}^{j-1}p^{2-j+k}e\left[\underline{g}_k^k(\Omega,0;r)\right]\end{array}\right\}$$

and

$$\left(\sum_{i=1}^{m}\frac{A_i}{p^i}+\sum_{j=1}^{n}\frac{B_j}{q^j}+C\right)e\left[\overline{g}(\Omega,\mho;r)\right]$$
$$=\left\{\begin{array}{c} e\left(\overline{f}(\Omega,\mho;r)\right)+\sum_{i=1}^{m}A_i\sum_{k=0}^{i-1}p^{2-i+k}e\left[\overline{g}_k^k(0,\mho;r)\right] \\ +\sum_{j=1}^{n}B_j\sum_{k=0}^{j-1}p^{2-j+k}e\left[\overline{g}_k^k(\Omega,0;r)\right]\end{array}\right\}$$

then

(5.5)

$$e\left[\underline{g}(\Omega,\mho;r)\right]=\frac{\left\{\begin{array}{c} e\left(\underline{f}(\Omega,\mho;r)\right)+\sum\limits_{i=1}^{m}A_i\sum\limits_{k=0}^{i-1}p^{2-i+k}e\left[\underline{g}_k^k(0,\mho;r)\right]+\\ \sum\limits_{j=1}^{n}B_j\sum\limits_{k=0}^{j-1}p^{2-j+k}e\left[\underline{g}_k^k(\Omega,0;r)\right]\end{array}\right\}}{\sum\limits_{i=1}^{m}\frac{A_i}{p^i}+\sum\limits_{j=1}^{n}\frac{B_j}{q^j}+C}$$

$$e\left[\overline{g}(\Omega,\mho;r)\right]=\frac{\left\{\begin{array}{c} e\left(\overline{f}(\Omega,\mho;r)\right)+\sum\limits_{i=1}^{m}A_i\sum\limits_{k=0}^{i-1}p^{2-i+k}e\left[\overline{g}_k^k(0,\mho;r)\right]+\\ \sum\limits_{j=1}^{n}B_j\sum\limits_{k=0}^{j-1}p^{2-j+k}e\left[\overline{g}_k^k(\Omega,0;r)\right]\end{array}\right\}}{\sum\limits_{i=1}^{m}\frac{A_i}{p^i}+\sum\limits_{j=1}^{n}\frac{B_j}{q^j}+C}.$$

b) If $\tilde{g}(\Omega,\mho;r)$ Second- differentiable then

(5.6)

$$e\left[\overline{g}(\Omega,\mho;r)\right]=\frac{\left\{\begin{array}{c} e\left(\overline{f}(\Omega,\mho;r)\right)+\sum\limits_{i=1}^{m}A_i\sum\limits_{k=0}^{i-1}p^{2-i+k}e\left[\overline{g}_k^k(0,\mho;r)\right]+\\ \sum\limits_{j=1}^{n}B_j\sum\limits_{k=0}^{j-1}p^{2-j+k}e\left[\overline{g}_k^k(\Omega,0;r)\right]\end{array}\right\}}{\sum\limits_{i=1}^{m}\frac{A_i}{p^i}+\sum\limits_{j=1}^{n}\frac{B_j}{q^j}+C}$$

$$e\left[\underline{g}(\Omega,\mho;r)\right]=\frac{\left\{\begin{array}{c} e\left(\underline{f}(\Omega,\mho;r)\right)+\sum\limits_{i=1}^{m}A_i\sum\limits_{k=0}^{i-1}p^{2-i+k}e\left[\underline{g}_k^k(0,\mho;r)\right]+\\ \sum\limits_{j=1}^{n}B_j\sum\limits_{k=0}^{j-1}p^{2-j+k}e\left[\underline{g}_k^k(\Omega,0;r)\right]\end{array}\right\}}{\sum\limits_{i=1}^{m}\frac{A_i}{p^i}+\sum\limits_{j=1}^{n}\frac{B_j}{q^j}+C}$$

by using inverse of DFET we obtain $\tilde{g}(\Omega,\mho;r)=\left(\underline{g}(\Omega,\mho;r),\overline{g}(\Omega,\mho;r)\right).$

6. Application

In this section, we evaluate the solution of Fuzzy Poission's equation and Fuzzy Telegraph equations using DFET.

EXAMPLE 6.1. *Consider the fuzzy Poission's equation*

$$\frac{\partial^2\tilde{\Lambda}(\Omega,\mho;r)}{\partial\Omega^2}\oplus\frac{\partial^2\tilde{\Lambda}(\Omega,\mho;r)}{\partial\mho^2}=\tilde{k}(r)\odot(\Omega^2\oplus\mho^2),\quad \tilde{k}(r)=[1+r,3-r] \tag{6.1}$$

with ICs

$$\tilde{\Lambda}(\Omega,0;r)=\tilde{0}=\tilde{\Lambda}_\Omega(0,\mho;r) \tag{6.2}$$

and BCs

$$\tilde{\Lambda}(0,\mho;r)=\tilde{0}=\tilde{\Lambda}_\mho(\Omega,0;r). \tag{6.3}$$

Applying DFET to equation (6.1) on both sides, we get

$$\mathcal{E}\left[\frac{\partial^2\tilde{\Lambda}(\Omega,\mho;r)}{\partial\Omega^2}\right]\oplus\mathcal{E}\left[\frac{\partial^2\tilde{\Lambda}(\Omega,\mho;r)}{\partial\mho^2}\right]=\mathcal{E}\left[\Omega^2\oplus\mho^2\right].$$

Let $\tilde{\Lambda}(\Omega,\mho;r)$ be First- differentiable. In this situation $m=2=n, A_1=0=B_1, A_2=1=B_2, C=1$ and $\tilde{f}(\Omega,\mho;r)=\tilde{k}(r)(\Omega^2\oplus\mho^2)=(1+r,3-r)(\Omega^2\oplus\mho^2)$

$$e\left[\frac{\partial^2\underline{\Lambda}(\Omega,\mho;r)}{\partial\Omega^2}\right]+e\left[\frac{\partial^2\underline{\Lambda}(\Omega,\mho;r)}{\partial\mho^2}\right]=e\left[(1+r)(\Omega^2+\mho^2)\right]$$
$$e\left[\frac{\partial^2\overline{\Lambda}(\Omega,\mho;r)}{\partial\Omega^2}\right]+e\left[\frac{\partial^2\overline{\Lambda}(\Omega,\mho;r)}{\partial\mho^2}\right]=e\left[(3-r)(\Omega^2+\mho^2)\right]$$

from equation (3.19) and (3.20) of Theorem 3.7 and using equation (6.2) and equation (6.3) we have,

$$\left(\frac{1}{p^2}+\frac{1}{q^2}\right)e\left[\underline{\Lambda}(\Omega,\mho;r)\right]=2(1+r)(p^4q^2+q^4p^2)$$
$$\left(\frac{1}{p^2}+\frac{1}{q^2}\right)e\left[\overline{\Lambda}(\Omega,\mho;r)\right]=2(3-r)(p^4q^2+q^4p^2).$$

On simplification we get,

$$e\left[\underline{\Lambda}(\Omega,\mho;r)\right]=2(1+r)(p^4q^2+q^4p^2)\frac{p^2q^2}{p^2+q^2}$$
$$e\left[\overline{\Lambda}(\Omega,\mho;r)\right]=2(3-r)(p^4q^2+q^4p^2)\frac{p^2q^2}{p^2+q^2}.$$

Taking inverse Double Fuzzy Elzaki Transform we obtain the desired solution of equation (6.1).

$$\begin{aligned}\underline{\Lambda}(\Omega,\mho;r)&=(1+r)\frac{\Omega^2\mho^2}{2}\\ \overline{\Lambda}(\Omega,\mho;r)&=(3-r)\frac{\Omega^2\mho^2}{2}\end{aligned} \tag{6.4}$$

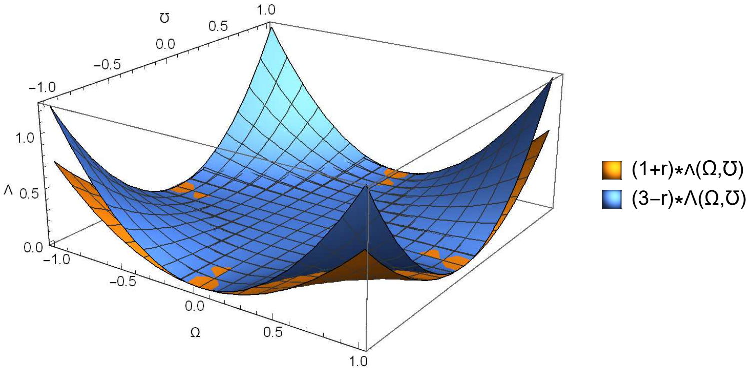

FIGURE 1. Solution of Fuzzy Poisson eqaution $\underline{\Lambda}(\Omega,\mho;r)$ and $\overline{\Lambda}(\Omega,\mho;r)$ at value of r =0.5

EXAMPLE 6.2. *Consider the fuzzy Telegraph equation*

$$\frac{\partial^2\tilde{\Lambda}(\Omega,\mho;r)}{\partial\Omega^2}=\frac{\partial^2\tilde{\Lambda}(\Omega,\mho;r)}{\partial\mho^2}\oplus\frac{\partial\tilde{\Lambda}(\Omega,\mho;r)}{\partial\mho}\oplus\tilde{\Lambda}(\Omega,\mho;r) \tag{6.5}$$

with ICs

$$\tilde{\Lambda}(\Omega,0;r)=\tilde{\delta}(r)\odot e^{\Omega},\qquad \tilde{\Lambda}_{\mho}(\Omega,0;r)=\tilde{\delta}(r)\odot -e^{\Omega} \tag{6.6}$$

and BCs

$$\tilde{\Lambda}(0,\mho;r)=\tilde{\delta}(r)\odot e^{-\mho},\qquad \tilde{\Lambda}_{\Omega}(0,\mho;r)=\tilde{\delta}(r)\odot e^{-\mho} \tag{6.7}$$

where, $\tilde{\delta}(r) = [1 + r, 3 - r]$.
Applying DFET to equation (6.5) *on both sides, we get*

$$\mathcal{E}\left[\frac{\partial^2\tilde{\Lambda}(\Omega,\mho;r)}{\partial\Omega^2}\right] = \mathcal{E}\left[\frac{\partial^2\tilde{\Lambda}(\Omega,\mho;r)}{\partial\mho^2} \oplus \frac{\partial\tilde{\Lambda}(\Omega,\mho;r)}{\partial\mho} \oplus \tilde{\Lambda}(\Omega,\mho;r)\right]$$

Let $\tilde{\Lambda}(\Omega,\mho;r)$ *be First- differentiable. In this situation* $m = 2 = n, A_1 = 0, B_1 = -1 = B_2, A_2 = 1, C = -1$ *and* $\tilde{f}(\Omega,\mho;r) = \tilde{0}$.

$$e\left[\frac{\partial^2\underline{\Lambda}(\Omega,\mho;r)}{\partial\Omega^2}\right] = e\left[\frac{\partial^2\underline{\Lambda}(\Omega,\mho;r)}{\partial\mho^2} + \frac{\partial\underline{\Lambda}(\Omega,\mho;r)}{\partial\mho} + \underline{\Lambda}(\Omega,\mho;r)\right]$$
$$e\left[\frac{\partial^2\overline{\Lambda}(\Omega,\mho;r)}{\partial\Omega^2}\right] = e\left[\frac{\partial^2\overline{\Lambda}(\Omega,\mho;r)}{\partial\mho^2} + \frac{\partial\overline{\Lambda}(\Omega,\mho;r)}{\partial\mho} + \overline{\Lambda}(\Omega,\mho;r)\right]$$

from equation (3.19) *and equation* (3.20) *of Theorem 3.7 and using equation* (6.6) *and equation* (6.7) *we have,*

$$(q^2 - p^2 - p^2q - p^2q^2)e\left[\underline{\Lambda}(\Omega,\mho;r)\right] = (1 + r)\begin{Bmatrix} \frac{q^3p^4}{1-p} - \frac{q^3p^4}{1-p} - \frac{q^2p^4}{1-p} \\ +\frac{p^2q^4}{1+q} + \frac{p^3q^4}{1+q} \end{Bmatrix}$$

$$(q^2 - p^2 - p^2q - p^2q^2)e\left[\overline{\Lambda}(\Omega,\mho;r)\right] = (3 - r)\begin{Bmatrix} \frac{q^3p^4}{1-p} - \frac{q^3p^4}{1-p} - \frac{q^2p^4}{1-p} \\ +\frac{p^2q^4}{1+q} + \frac{p^3q^4}{1+q} \end{Bmatrix}.$$

On simplification we get,

$$e\left[\underline{\Lambda}(\Omega,\mho;r)\right] = (1 + r)\frac{p^2q^2}{(1 + q)(1 - p)}$$
$$e\left[\overline{\Lambda}(\Omega,\mho;r)\right] = (3 - r)\frac{p^2q^2}{(1 + q)(1 - p)}$$

Taking inverse Double Fuzzy Elzaki Transform we obtain the desired solution of equation (6.5).

$$\underline{\Lambda}(\Omega,\mho;r) = (1 + r)e^{\Omega-\mho}$$
$$\overline{\Lambda}(\Omega,\mho;r) = (3 - r)e^{\Omega-\mho}$$

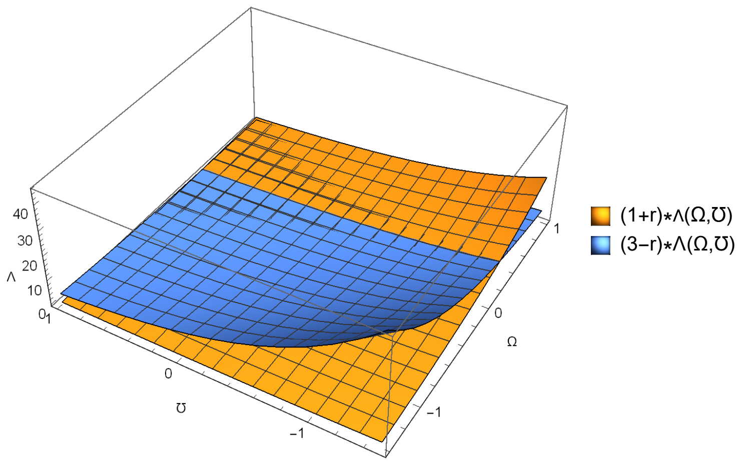

FIGURE 2. Solution of Fuzzy Telegraph eqaution $\underline{\Lambda}(\Omega, \mho; r)$ and $\overline{\Lambda}(\Omega, \mho; r)$ at value of r= 0.5

7. Conclusion

The double fuzzy Elzaki transform is presented in this article, and the DFET technique is used to solve FPDEs sequentially. New DFET findings for the n^{th} order Fuzzy partial H-derivative have been proposed. Also postulated is a duality connection between DFLT, DFST and DFET. Under gH-differentiability, DFET has been effectively used to the solutions of the fuzzy Poissons equation and the fuzzy Telegraph equation. The result is shown by providing two examples.

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