JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **35**, No. 2, May 2022
http://dx.doi.org/10.14403/jcms.2022.35.2.125

ON GENERALIZED SYMMETRIC BI-f-DERIVATIONS OF LATTICES

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Abstract. The goal of this paper is to introduce the notion of generalized symmetric bi-f-derivations in lattices and to study some properties of generalized symmetric f-derivations of lattice. Moreover, we consider generalized isotone symmetric bi-f-derivations and fixed sets related to generalized symmetric bi-f-derivations.

1. Introduction

Lattices play an important role in many fields such as information theory, information retrieval, information access controls and cryptanalysis. The properties of lattices were widely researched (for example, [1], [10], [14]). In the theory of rings and near rings, the properties of derivations are an important topic to study ([6], [12]). G. Szász [13] introduced the notion of derivation on a lattice and discussed some related properties, And then the notion of f-derivation, symmetric bi-derivations and permuting tri-derivations in lattices are introduced and proved some results (see to the reference [2], [3], [9], [7], [8]).

The goal of this paper is to introduce the notion of generalized symmetric bi-f-derivations in lattices and to study some properties of generalized symmetric f-derivations of lattice. Furthermore, we take into account generalized isotone symmetric bi-f-derivations and fixed sets related to generalized symmetric bi-f-derivations.

Received April 19, 2022; Accepted May 25, 2022.
2010 Mathematics Subject Classification: 16Y30, 03G25.
Key words and phrases: Lattice, generalized symmetric bi-f-derivation, isotone, $Fix_d(L)$, joinitive.
This was supported by Korea National University of Transportation in 2022.

2. Preliminary

DEFINITION 2.1. Let L be a nonempty set endowed with operations $\wedge$ and $\vee$. By a *lattice* $(L, \wedge, \vee)$, we mean a set L satisfying the following conditions:

(1) $x \wedge x = x$, $x \vee x = x$ for every $x \in L$.
(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$ for every $x, y \in L$.
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$ for every $x, y, z \in L$.
(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$ for every $x, y \in L$.

DEFINITION 2.2. Let $(L, \wedge, \vee)$ be a lattice. A binary relation $\leq$ is defined by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$ for every $x, y \in L$.

LEMMA 2.3. *Let $(L, \wedge, \vee)$ be a lattice. Define the binary relation $\leq$ as the Definition 2.2. Then $(L, \leq)$ is a poset and for any $x, y \in L$, $x \wedge y$ is the g.l.b. of $\{x, y\}$ and $x \vee y$ is the l.u.b. of $\{x, y\}$.*

DEFINITION 2.4. A lattice L is *distributive* if the identity (1) or (2) holds:

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for every $x, y, z \in L$.
(2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for every $x, y, z \in L$.

In any lattice, the conditions (1) and (2) are equivalent.

DEFINITION 2.5. A lattice L is *modular* if the following identity holds: If $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$ for every $x, y, z \in L$.

DEFINITION 2.6. A non-empty subset I is called an *ideal* if the following conditions hold:

(1) If $x \leq y$ and $y \in I$, then $x \in I$ for all $x, y \in L$.
(2) If $x, y \in I$ then $x \vee y \in I$.

DEFINITION 2.7. Let $(L, \wedge, \vee)$ be a lattice. Let $f : L \to M$ be a function from a lattice L to a lattice M.

(1) f is called a *meet-homomorphism* if $f(x \wedge y) = f(x) \wedge f(y)$ for every $x, y \in L$.
(2) f is called a *join-homomorphism* if $f(x \vee y) = f(x) \vee f(y)$ for every $x, y \in L$.
(3) f is called a *lattice-homomorphism* if f is a join-homomorphism and a meet-homomorphism.

DEFINITION 2.8. Let L be a lattice. A function $d : L \to L$ is called a f-*derivation* if there exists a function $f : L \to L$ such that

$$d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y))$$

for all $x, y \in L$.

Definition 2.9. ([11]) Let L be a lattice and $D(.,.) : L \times L \to L$ be a symmetric mapping. We call D a *symmetric bi-f-derivation* on L if there exists a function $f : L \to L$ such that

$$D(x \wedge y, z) = (D(x, z) \wedge f(y)) \vee (f(x) \wedge D(y, z))$$

for all $x, y \in L$.

Proposition 2.10. ([11]) *Let L be a lattice and let d be a trace of symmetric bi-f-derivation D. Then*

(1) $D(x, y) \leq f(x)$ *and* $D(x, y) \leq f(y)$ *for every* $x, y \in L$.
(2) $D(x, y) \leq f(x) \wedge f(y)$ *for every* $x, y \in L$.
(3) $d(x) \leq f(x)$ *for every* $x, y \in L$.

3. Generalized symmetric bi-f-derivations of lattices

Throughout the paper, L denotes a lattice unless otherwise specified.

Definition 3.1. Let $D : L \to L$ be a symmetric bi-f-derivation on lattice L. A symmetric map $\Delta : L \times L \to L$ is called a *generalized symmetric bi-f- derivation* associated with D if

$$\Delta(x \wedge y, z) = (\Delta(x, z) \wedge f(y)) \vee (f(x) \wedge D(y, z))$$

for all $x, y, z \in L$. Obviously, a generalized symmetric bi-f-derivation Δ on L satisfies the relation

$$\Delta(x, y \wedge z) = (\Delta(x, y) \wedge f(z)) \vee (f(y) \wedge D(x, z))$$

for all $x, y, z \in L$.

Definition 3.2. Let L be a lattice. The mapping $\delta : L \to L$ defined by $\delta(x) = \Delta(x, x)$ for all $x \in L$, is called the *trace* of generalized symmetric bi-f-derivation Δ.

Example 3.3. Let L be a lattice with a least element 0 and let f be an endomorphism on L. The mapping $D(x, y) = 0$ for all $x, y \in L$, is a symmetric bi-f- derivation on L. Define a mapping on L by $\Delta(x, y) = f(x) \wedge f(y)$ for all $x, y \in L$. Then we can see that Δ is a generalized symmetric bi-f-derivation associated with D on L.

Example 3.4. Let L be a lattice with a least element 0 and let f be an endomorphism on L and let $a \in L$. The mapping on L defined by $\Delta(x, y) = (f(x) \wedge f(y)) \wedge a$, for all $x, y \in L$, is a generalized symmetric bi-f-derivation associated with $D(x, y) = 0$ on L.

EXAMPLE 3.5. Let $L = \{0, 1, 2\}$ be a lattice of following Figure 1 and define mappings D, f and Δ on L by

$$D(x, y) = \begin{cases} 1 & \text{if } (x, y) = (0, 0), (0, 1), (1, 0) \\ 0 & \text{if } (x, y) = (0, 2), (2, 0), (1, 1), (2, 2), (1, 2), (2, 1), \end{cases}$$

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 2 & \text{if } x = 1, 2 \end{cases}$$

and

$$\Delta(x, y) = \begin{cases} 1 & \text{if } x = (0, 0), (0, 1), (1, 0), (0, 2), (2, 0) \\ 2 & \text{if } x = (1, 1), (1, 2), (2, 1), (2, 2). \end{cases}$$

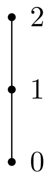

FIGURE 1

Then it is easily checked that Δ is a generalized symmetric bi-f-derivation of lattice L.

PROPOSITION 3.6. *Let Δ is a generalized symmetric bi-f-derivation associated with a symmetric bi-f-derivation D. Then the mapping $f_1 : L \to L$ defined by $f_1(x) = \Delta(x, z)$, for all $x, z \in L$, and $f_2 : L \to L$ defined by $f_2(y) = \Delta(x, y)$ for all $x, y \in L$, are generalized f-derivation on L.*

Proof. For every $x, y, z \in L$, we have

$$\begin{aligned} f_1(x \wedge y) &= \Delta(x \wedge y, z) \\ &= (f(x) \wedge D(y, z))) \vee (\Delta(x, z) \wedge f(y)) \\ &= (f(x) \wedge g_1(y)) \vee (f_1(x) \wedge f(y)). \end{aligned}$$

In this equation, the mapping $g_1 : L \to L$ defined by $g_1(y) = D(y, z)$ is a f- derivation on L, where D is a symmetric bi-f-derivation on L. Hence the mapping f_1 is a generalized symmetric bi-f-derivation associated with D. □

PROPOSITION 3.7. *Let Δ be a generalized symmetric bi-f-derivation associated with a symmetric bi-f-derivation D. If L is a distributive lattice, then we have*

(1) *$D(x,y) \leq \Delta(x,y)$ for every $x,y \in L$.*
(2) *$\Delta(x,y) \leq f(x)$ and $\Delta(x,y) \leq f(y)$ for every $x,y \in L$.*
(3) *$\Delta(x \wedge w, y) \leq \Delta(x,y) \vee \Delta(w,y)$ for every $x,y \in L$.*
(4) *$\Delta(x \wedge w, y) \leq f(x) \vee f(w)$ for every $w,x,y \in L$.*
(5) *If L has a least element 0, $f(0) = 0$ implies $\Delta(0,y) = 0$ for every $y \in L$.*

Proof. (1) For every $x,y \in L$, we obtain

$$\begin{aligned}\Delta(x,y) &= \Delta(x \wedge x, y) = (\Delta(x,y) \wedge f(x)) \vee (f(x) \wedge D(x,y)) \\ &= (\Delta(x,y) \vee f(x)) \vee D(x,y).\end{aligned}$$

This implies $D(x,y) \leq \Delta(x,y)$ for every $x,y \in L$.

(2) Since $x \wedge x = x$ for all $x \in L$, we have by Proposition 2.10,

$$\begin{aligned}\Delta(x,y) &= \Delta(x \wedge x, y) = (\Delta(x,y) \wedge f(x)) \vee (f(x) \wedge D(x,y)) \\ &= (\Delta(x,y) \wedge f(x)) \vee D(x,y) \\ &= (\Delta(x,y) \vee D(x,y)) \wedge (f(x) \vee D(x,y)) \\ &= \Delta(x,y) \wedge f(x).\end{aligned}$$

Therefore $\Delta(x,y) \leq f(x)$ for all $x,y \in L$. Similarly, we can check $\Delta(x,y) \leq f(y)$ for all $x,y \in L$.

(3) Since $f(x) \wedge D(w,y) \leq D(w,y) \leq \Delta(x,y)$ and $\Delta(x,y) \wedge f(w) \leq \Delta(x,y)$ for every $w,x,y \in L$, we obtain

$$(\Delta(x,y) \wedge f(w)) \vee (f(x) \wedge D(w,y)) \leq \Delta(x,y) \vee \Delta(w,y).$$

That is, $\Delta(x \wedge w, y) \leq \Delta(x,y) \vee \Delta(w,y)$.

(4) Since $\Delta(x,y) \wedge f(w) \leq f(w)$ and $f(x) \wedge D(w,y) \leq f(x)$, we get

$$(\Delta(x,y) \wedge f(w)) \vee (f(x) \wedge D(w,y)) \leq f(x) \vee f(w)$$

for every $w,x,y \in L$. That is, $D(x \wedge w, y) \leq f(x) \vee f(w)$ for every $w,x,y \in L$.

(5) Since 0 is the least element of L, we have

$$\begin{aligned}\Delta(0,y) &= \Delta(0 \wedge 0, y) = (\Delta(0,y) \wedge f(0)) \vee (f(0) \wedge D(0,y)) \\ &= 0 \vee 0 = 0\end{aligned}$$

for all $x,y \in L$. □

Corollary 3.8. *Let Δ be a generalized symmetric bi-f-derivation associated with a symmetric bi-f-derivation D and let δ be a trace of Δ and let d be a trace of D. If L is a distributive lattice,, then the following conditions hold.*

(1) *$\Delta(x,y) \leq f(x) \wedge f(y)$ for every $x,y \in L$.*

(2) $d(x) \leq \delta(x) \leq f(x)$ *for every* $x, y \in L$.
(3) $d(x) = x$ *implies* $\delta(x) = x$ *for* $x, y \in L$.

THEOREM 3.9. *Let* L *be a distributive lattice and let* Δ *be a generalized symmetric bi-*f*-derivation associated with a symmetric bi-*f*-derivation* D *and let* δ *a trace of* Δ *and let* d *be a trace of* D*. Then we have*

$$\delta(x \wedge y) = D(x, y) \vee (f(x) \wedge d(y)) \vee (f(y) \wedge \delta(x))$$

for all $x, y \in L$.

Proof. Using Proposition 3.7 (1), we have

$$\begin{aligned}
\delta(x \wedge y) &= \Delta(x \wedge y, x \wedge y) \\
&= (\Delta(x \wedge y, x) \wedge f(y)) \vee (D(x \wedge y, y) \wedge f(x)) \\
&= (((\Delta(x, x) \wedge f(y)) \vee (f(x) \wedge D(x, y))) \wedge f(y)) \\
&\quad \vee (f(x) \wedge ((D(x, y) \wedge f(y)) \vee (f(x) \wedge D(y, y)))) \\
&= (((\delta(x) \wedge f(y) \vee D(x, y)) \wedge f(y)) \vee (f(x) \wedge (D(x, y)) \\
&\quad \vee (f(x) \wedge d(y)))) \\
&= ((\delta(x) \wedge f(y) \vee D(x, y)) \vee (D(x, y) \vee (f(x) \wedge d(y))) \\
&= (\delta(x) \wedge f(y)) \vee (d(y) \wedge f(x)) \vee D(x, y)
\end{aligned}$$

for every $x, y \in L$. □

THEOREM 3.10. *Let* L *be a distributive lattice and let* Δ_1 *and* Δ_2 *be generalized symmetric bi-*f*-derivations associated with a same symmetric bi-*f*-derivation* D*. Then the mapping* $\Delta_1 \wedge \Delta_2$ *defined by*

$$(\Delta_1 \wedge \Delta_2)(x, y) = \Delta_1(x, y) \wedge \Delta_2(x, y)$$

for every $x, y \in L$ *is a generalized symmetric bi-*f*-derivations associated with a symmetric bi-*f*-derivation* D.

Proof. For every $x, y, z \in L$, we have

$$\begin{aligned}
(\Delta_1 \wedge \Delta_2)(x \wedge y, z) &= \Delta_1(x \wedge y, z) \wedge \Delta_2(x \wedge y, z) \\
&= ((\Delta_1(x, z) \wedge f(y)) \vee (f(x) \wedge D(y, z))) \\
&\quad \wedge ((\Delta_2(x, z) \wedge f(y)) \vee (f(x) \wedge D(y, z))) \\
&= (((\Delta_1(x, z) \wedge f(y)) \wedge (\Delta_2(x, z) \wedge f(y)))) \\
&\quad \vee (f(x) \wedge D(y, z)) \\
&= (\Delta_1(x, z) \wedge \Delta_2(x, z) \wedge f(y)) \vee (f(x) \wedge D(y, z)) \\
&= ((\Delta_1 \wedge \Delta_2)(x, z) \wedge f(y)) \vee (f(x) \wedge D(y, z)).
\end{aligned}$$

This completes the proof. □

THEOREM 3.11. *Let L be a distributive lattice and let Δ_1 and Δ_2 be generalized symmetric bi-f-derivations associated with a same symmetric bi-f-derivation D. Then the mapping $\Delta_1 \vee \Delta_2$ defined by*

$$(\Delta_1 \vee \Delta_2)(x, y) = \Delta_1(x, y) \vee \Delta_2(x, y)$$

for every $x, y \in L$ is a generalized symmetric bi-f-derivations associated with a symmetric bi-f-derivation D.

Proof. For every $x, y, z \in L$, we have

$$\begin{aligned}(\Delta_1 \vee \Delta_2)(x \wedge y, z) &= \Delta_1(x \wedge y, z) \vee \Delta_2(x \wedge y, z) \\ &= ((\Delta_1(x, z) \wedge f(y)) \vee (f(x) \wedge D(y, z))) \\ &\quad \vee ((\Delta_2(x, z) \wedge f(y)) \vee (f(x) \wedge D(y, z))) \\ &= (((\Delta_1(x, z) \wedge f(y)) \vee (\Delta_2(x, z) \wedge f(y)))) \\ &\quad \vee (f(x) \wedge D(y, z)) \\ &= (\Delta_1(x, z) \vee \Delta_2(x, z) \wedge f(y)) \vee (f(x) \wedge D(y, z)) \\ &= ((\Delta_1 \vee \Delta_2)(x, z) \wedge f(y)) \vee (f(x) \wedge D(y, z)).\end{aligned}$$

This completes the proof. □

DEFINITION 3.12. Let L be a distributive lattice and let Δ be a generalized symmetric bi-f-derivations associated with a symmetric bi-f-derivation D and let δ be a trace of Δ. If $x \leq y$ implies $\delta(x) \leq \delta(y)$ for every $x, y \in L$, then δ is called an *isotone mapping.*

THEOREM 3.13. *Let L be a distributive lattice with greatest element 1 and let f be a meet-homomorphism on L and let δ be a trace of generalized symmetric bi-f-derivation Δ associated with a symmetric bi-f-derivation D. Then the following conditions are equivalent.*

(1) *δ is an isotone mapping on L.*
(2) *$\delta(x) = f(x) \wedge \delta(1)$ for every $x \in L$.*
(3) *$\delta(x \wedge y) = \delta(x) \wedge \delta(y)$ for every $x, y \in L$.*
(4) *$\delta(x) \vee \delta(y) \leq \delta(x \vee y)$ for every $x, y \in L$.*

Proof. (1) $\Rightarrow$ (2). Since d is isotone and $x \leq 1$, we have $\delta(x) \leq \delta(1)$. By Proposition 3.7(1), we obtain $\delta(x) \leq f(x)$, and so $\delta(x) \leq f(x) \wedge \delta(1)$. By Corollary 3.8(2), we have $f(x) \wedge \delta(1) \leq \delta(x)$. Hence we obtain $\delta(x) = f(x) \wedge \delta(1)$ for all $x \in L$.

(2) $\Rightarrow$ (3). Let $\delta(x) = f(x) \wedge \delta(1)$ for all $x \in L$. Then we have

$$\begin{aligned}\delta(x \wedge y) &= f(x \wedge y) \wedge \delta(1) = (f(x) \wedge f(y)) \wedge (\delta(1) \wedge \delta(1)) \\ &= (f(x) \wedge \delta(1)) \wedge (f(y) \wedge \delta(1)) = \delta(x) \wedge \delta(y)\end{aligned}$$

for all $x, y \in L$.

(3) $\Rightarrow$ (1). Let $\delta(x \wedge y) = \delta(x) \wedge \delta(y)$ and $x \leq y$. Then $\delta(x) = \delta(x \wedge y) = \delta(x) \wedge \delta(y)$. Hence $\delta(x) \leq \delta(y)$ for every $x, y \in L$.

(1) $\Rightarrow$ (4). Let δ be isotone. Since $x \leq x \vee y$ and $y \leq x \vee y$, $\delta(x) \leq \delta(x \vee y)$ and $\delta(y) \leq \delta(x \vee y)$. Hence $\delta(x) \vee \delta(y) \leq \delta(x \vee y)$ for every $x, y \in L$.

(4) $\Rightarrow$ (1). Let $x \leq y$. Since $\delta(x) \leq \delta(x \vee y) = \delta(y)$, which implies $\delta(x) \leq \delta(y)$ for every $w, x, y \in L$. Hence δ is isotone. □

Definition 3.14. Let L be a lattice and Δ be a generalized symmetric bi-f-derivation associated with a symmetric bi-f-derivation D. If $x \leq w$ implies $\Delta(x, y) \leq \Delta(w, y)$ for every $w, x, y \in L$, then Δ is called a *generalized isotone symmetric bi-f-derivation* of L.

Theorem 3.15. *Let L be a lattice with greatest element* 1, *Δ a generalized symmetric bi-f-derivation associated with a symmetric bi-f-derivation D and let f be a meet-homomorphism on L. The following conditions are equivalent.*

(1) *Δ is a generalized isotone symmetric bi-f-derivation of L.*
(2) *$\Delta(x, y) \vee \Delta(w, y) \leq \Delta(x \vee w, y)$ for every $w, x, y \in L$.*
(3) *$\Delta(x, y) = f(x) \wedge \Delta(1, y)$ for every $x, y \in L$.*
(4) *$\Delta(x \wedge w, y) = \Delta(x, y) \wedge \Delta(w, y)$ for every $w, x, y \in L$.*

Proof. (1) $\Rightarrow$ (2). Suppose that Δ is a generalized isotone symmetric bi-f-derivation. Since $x \leq x \vee w$ and $w \leq x \vee w$ for every $w, x, y \in L$, we obtain $\Delta(x, y) \leq \Delta(x \vee w, y)$ and $\Delta(w, y) \leq \Delta(x \vee w, y)$. Therefore, $\Delta(x, y) \vee \Delta(w, y) \leq \Delta(x \vee w, y)$.

(2) $\Rightarrow$ (1). Suppose that $\Delta(x, y) \vee \Delta(w, y) \leq \Delta(x \vee w, y)$ and $x \leq w$ for all $w, x, y \in L$. Then we have

$$\Delta(x, y) \leq \Delta(x, y) \vee \Delta(w, y) \leq \Delta(x \vee w, y) = \Delta(w, y).$$

Hence Δ is a generalized isotone symmetric bi-f-derivation on L.

(1) $\Rightarrow$ (3). Suppose that Δ is a generalized isotone symmetric bi-f-derivation. Since $\Delta(x, y) \leq \Delta(1, y)$, we have $\Delta(x, y) \leq f(x) \wedge \Delta(1, y)$ by Proposition 3.6 (1). Hence we have

$$\Delta(x, y) = (\Delta(1, y) \wedge f(x)) \vee D(x, y) = \Delta(1, y) \wedge f(x)$$

for every $x, y \in L$.

(3) $\Rightarrow$ (4). Suppose that $\Delta(x, y) = f(x) \wedge \Delta(1, y)$. Then we have

$$\begin{aligned}\Delta(x \wedge w, y) &= f(x \wedge w) \wedge \Delta(1, y)\\ &= f(x) \wedge f(w) \wedge D(1, y)\\ &= (f(x) \wedge D(1, y)) \wedge (f(w) \wedge D(1, y))\\ &= D(x, y) \wedge D(w, y)\end{aligned}$$

for every $w, x, y \in L$.

(4) $\Rightarrow$ (1). Let $\Delta(x \wedge w, y) = \Delta(x, y) \wedge \Delta(w, y)$ and $x \leq w$. Then we have $\Delta(x, y) = \Delta(x \wedge w, y) = \Delta(x, y) \wedge \Delta(w, y)$. Therefore, $\Delta(x, y) \leq \Delta(w, y)$ for every $w, x, y \in L$. □

Let Δ be a generalized symmetric bi-f-derivation associated with a symmetric bi-f-derivation D and let δ be a trace of Δ and let d be a trace of D. For each $a \in L$ and define sets $Fix_d(L)$ and $Fix_\delta(L)$ by

$$Fix_D(L) = \{x \in L \mid D(x, a) = f(x)\}$$

and

$$Fix_\Delta(L) = \{x \in L \mid \Delta(x, a) = f(x)\}.$$

Lemma 3.16. *Let L be a lattice and let Δ be a generalized symmetric bi-f-derivation associated with a symmetric bi-f-derivation D. Then we have $Fix_D(L) \subseteq Fix_\Delta(L)$.*

Proof. Let $x \in Fix_D(L)$. Then we have $D(x, a) = f(x)$ for $a \in L$. Hence

$$\begin{aligned}\Delta(x, a) = \Delta(x \wedge x, a) &= (\Delta(x, a) \wedge f(x)) \vee (f(x) \wedge D(x, a))\\ &= (\Delta(x, a) \wedge f(x)) \vee (f(x) \wedge f(x))\\ &= (\Delta(x, a) \wedge f(x)) \vee f(x) = f(x)\end{aligned}$$

This implies $x \in Fix_\Delta(L)$, that is, $Fix_D(L) \subseteq Fix_\Delta(L)$. □

Proposition 3.17. *Let L be a distributive lattice and let Δ be a generalized symmetric bi-f-derivation associated with a symmetric bi-f-derivation D. If f is isotone, $x \leq y$ and $y \in Fix_D(L)$, then $x \in Fix_\Delta(L)$ for all $x, y \in L$.*

Proof. Let $y \in Fix_D(L)$. Then we get $D(y, a) = f(y)$. Hence we have

$$\begin{aligned}\Delta(x, a) = \Delta(x \wedge y, a) &= (\Delta(x, a) \wedge f(x)) \vee (f(x) \wedge D(y, a))\\ &= (\Delta(x, a) \wedge f(x)) \vee (f(x) \wedge f(y)) = (\Delta(x, a) \wedge f(x)) \vee f(x)\\ &= (\Delta(x, a) \vee f(x)) \wedge (f(x) \vee f(x)) = f(x) \vee f(x) = f(x).\end{aligned}$$

This implies $x \in Fix_\Delta(L)$. □

Definition 3.18. Let L be a lattice. The mapping Δ satisfying $\Delta(x \vee y, z) = \Delta(x, z) \vee \Delta(y, z)$ for all $x, y, z \in L$, is called a *joinitive mapping* on L.

Theorem 3.19. *Let L be a lattice and let Δ be a generalized symmetric bi-f-derivation associated with a symmetric bi-f-derivation D. If f is a join-homomorphism on L and let Δ is joinitive, then $x, y \in Fix_\Delta(L)$ implies $x \vee y \in Fix_\Delta(L)$.*

Proof. Let $x, y \in Fix_\Delta(L)$. Then $\Delta(x, a) = f(x)$ and $\Delta(y, a) = f(y)$. Hence $\Delta(x \vee y, a) = \Delta(x, a) \vee \Delta(y, a) = f(x) \vee f(y) = f(x \vee y)$, which implies $x \vee y \in Fix_\Delta(L)$. □

Proposition 3.20. *Let L be a lattice and let Δ be a generalized symmetric bi-f-derivation associated with a symmetric bi-f-derivation D. If f is a meet-homomorphism on L and $x \in Fix_\Delta(L)$ and $y \in Fix_D(L)$, we have $x \wedge y \in Fix_\Delta(L)$ for all $x, y \in L$.*

Proof. Let $x \in Fix_\Delta(L)$ and $y \in Fix_D(L)$. Then $f(x) = \Delta(x, a)$ and $f(y) = D(y, a)$. Hence we have

$$\begin{aligned} \Delta(x \wedge y, a) &= (\Delta(x, a) \wedge f(y)) \vee (f(x) \wedge D(y, a)) \\ &= (f(x) \wedge f(y)) \vee (f(x) \wedge f(y)) \\ &= f(x) \wedge f(y) = f(x \wedge y). \end{aligned}$$

Hence $x \wedge y \in Fix_\Delta(L)$. □

Proposition 3.21. *Let L be a lattice and let Δ be a generalized symmetric bi-f-derivation associated with a symmetric bi-f-derivation D. Then, for every $w, x, y \in L$, the following identities hold.*

(1) *If Δ is a generalized isotone symmetric bi-f-derivation, then*

$$\Delta(x, y) = D(x, y) \vee (\Delta(x \vee w, y) \wedge f(x))$$

for every $w, x, y \in L$.

(2) *If f is a join-homomorphism on L, then*

$$\Delta(x, y) = D(x, y) \vee (\Delta(x \vee w, y) \wedge f(x))$$

for every $w, x, y \in L$.

(3) *If $f(x)$ is an increasing function, then*

$$\Delta(x, y) = D(x, y) \vee (f(x) \wedge \Delta(x \vee w, y))$$

for every $w, x, y \in L$.

Proof. (1) Let Δ be a generalized isotone symmetric bi-f-derivation. Then we have

$$\begin{aligned}\Delta(x,y) &= \Delta((x\vee w)\wedge x, y)\\ &= (\Delta(x\vee w,y)\wedge f(x))\vee(f(x\vee w)\wedge D(x,y))\\ &= (\Delta(x\vee w,y)\wedge f(x))\vee D(x,y)\end{aligned}$$

since $D(x,y)\leq \Delta(x,y)\leq \Delta(x\vee w,y)\leq f(x\vee w)$ for every $w,x,y\in L$.

(2) Since $D(x,y)\leq f(x)\leq f(x)\vee f(w)$ and $f(x\vee w)=f(x)\vee f(w)$, we obtain

$$\begin{aligned}\Delta(x,y) &= \Delta((x\vee w)\wedge x, y)\\ &= (\Delta(x\vee w,y)\wedge f(x))\vee(f(x\vee w)\wedge D(x,y))\\ &= (\Delta(x\vee w,y)\wedge f(x))\vee D(x,y)\end{aligned}$$

for every $w,x,y\in L$.

(3) Since f is an increasing function and $x\leq x\vee w$, we have $D(x,y)\leq f(x)\leq f(x\vee w)$ and so,

$$\begin{aligned}\Delta(x,y) &= \Delta((x\vee w)\wedge x, y)\\ &= (\Delta(x\vee w,y)\wedge f(x))\vee(f(x\vee w)\wedge D(x,y))\\ &= (\Delta(x\vee w,y)\wedge f(x))\vee D(x,y)\end{aligned}$$

for every $w,x,y\in L$. □

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