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ON ORTHOGONAL REVERSE DERIVATIONS OF SEMIPRIME Γ-SEMIRINGS

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Abstract. In this paper, we introduce the notion of orthogonal reserve derivation on semiprime Γ-semirings. Some characterizations of semiprime Γ-semirimgs are obtained by means of orthogonal reverse derivations. We also investigate conditions for two reverse derivations on semiprime Γ-semiring to be orthogonal.

1. Introduction

The notion of semiring was first introduced in 1934 by H. S. Vandiver. A semiring is an algebraic structure consisting of a nonempty set S on which we have defined two associative binary operations addition (usually denoted by $+$) and multiplication (usually, $\cdot$) such that the multiplication is distributive over addition. The notion of rings with derivations is quite old and plays a significant role in the integration of analysis, algebraic geometry, and algebra. The study of derivations in rings though initiated long back, but got interested only after Posner who 1957 established two very striking results on derivations in prime rings. In this section, we review results on reverse derivations. The reverse derivations on semiprime rings has been studied by Samman and Alyamani [4]. Here the authors obtain some results of semiprime rings by reverse derivations. Also, Kalyan Kumar Dey, Akhil Chandra Paul and Isamiddin S. Rakhimov [3] studied orthogonal reverse derivations on semiprime Γ-rings and N. N. Sulaiman, and A. R. H. Majeed [5] studied orthogonal derivations on ideals of semiprime Γ-rings.

In this paper, we introduce the notion of orthogonal reserve derivation on semiprime Γ-semirings and generalized [3, 5]. Some characterizations of semiprime Γ-semirings are obtained by means of orthogonal reverse

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derivations. We also investigate conditions for two reverse derivations on semiprime Γ-semiring to be orthogonal.

2. Γ-Semirings

Let M and Γ be additive abelian groups with identity 0 and $0'$. If there exists a mapping $M \times \Gamma \times M \to M$ satisfying, for every $x, y, z \in M$ and $\alpha, \beta, \mu \in \Gamma$, the following conditions:

(ΓS1) $x\beta(y\mu z) = (x\beta y)\mu z$,
(ΓS2) $x\beta(y+z) = x\beta y + x\beta z$,
(ΓS3) $x(\beta+\mu)y = x\beta y + x\mu y$ and $(x+y)\beta z = x\beta z + y\beta z$,
(ΓS4) $x\alpha 0 = 0\alpha x = 0$ and $x0'y = 0$, then M is called a Γ-*semiring*.

Definition 2.1. A Γ-semiring M is called a *weak* Γ-*semiring* if it is also a M-semiring.

Example 2.2. Let S be a semiring with multiplicative identity and $M_{m\times n}(S)$ be the set all $m \times n$ matrices over S, clearly, $M_{m\times n}(S)$ is not closed with respect to usual multiplication of matrices.

Let $M = M_{m\times n}(S)$ and $\Gamma = M_{n\times m}(S)$. Then we can observe that M is a Γ-semiring and Γ is a M-semiring. Let $A \in M, \alpha \in \Gamma, B \in M$. Then $A\alpha B$ is a matrix of order $m \times n$ matrices over S, and so which implies that $A\alpha B \in M_{m\times n}(S) = M$. Similarly, if $\alpha, \beta \in M_{m\times n}(S) = \Gamma$ and $A \in M_{n\times m}(S) = M$, then $\alpha A\beta \in \Gamma$. The other axioms of Γ-semiring can also be observed easily. Similarly, Γ is a M-semiring, and hence (M, Γ) is also (Γ, M)-semiring. Clearly, M ane Γ are not semirings with respect to usual addition and multiplication of matrices.

Let S be a semiring with 0 and $M = M_{1\times 2}(S)$ and

$$\Gamma = \left\{ n \begin{pmatrix} 1 \\ 0 \end{pmatrix} : n \in Z^+ \right\}.$$

Then M is a Γ-semiring, but Γ is not a M-semiring. Let $M = M_{m\times n}(S)$ and $\Gamma = M_{n\times m}(Z^+) = M_{n\times m}(N)$. Then M is a Γ-semiring.

Example 2.3. Every semiring S is a weak Γ-semiring with $\Gamma = S$.

Definition 2.4. A Γ-semiring $(M, +, \cdot)$ is said to be *additively commutative* if $(M, +)$ is a commutative semigroup. A Γ-semiring $(M, +, \cdot)$ is said to be *multiplicatively commutative* if $(M, \cdot)$ is a commutative semigroup. It is said to be commutative if both $(M, +)$ and $(M, \cdot)$ are commutative.

Definition 2.5. Let M be a Γ-semiring. An element a in M is said to be *additively left cancellative* if $a+b=a+c$ implies $b=c$, for every $b,c\in M$. It is said to be *additively right cancellative* if $b+a=c+a$ implies $b=c$. It is said to be *additively cancellative* if it is both left and right cancellative. A Γ-semiring M is said to be *additively cancellative* if all elements in M are additively cancellative.

Definition 2.6. Let M be a Γ-semiring. Then

(D1) M is said to be *prime* if $a\Gamma M\Gamma b=0$ implies $a=0$, or $b=0$ for all $a,b\in M$.

(D2) M is said to be *semiprime* if $a\Gamma M\Gamma a=0$ implies $a=0$, for all $a\in M$.

(D3) M is said to be 2-*torsion free* if $2a=0$ implies $a=0$ for all $a\in M$.

Definition 2.7. Let M be a Γ-semiring. An additive mapping $d: M\to M$ is called a *derivation* if $d(x\alpha y)=d(x)\alpha y+x\alpha d(y)$ for all $x,y\in M$ and $\alpha\in\Gamma$.

Example 2.8. Let S be a semiring and $d:S\to S$ be a derivation on S. Let $M=M_{1\times 2}(S)$ and $\Gamma=\left\{\begin{pmatrix} n\cdot 1\\ 0\end{pmatrix} : n\in Z^+\right\}$. Then M is a Γ-semiring. Now, define $D:M\to M$ by $D(x\ \ y)=(d(x)\ \ d(y))$. Since

$$(x\ \ y)\begin{pmatrix} n\cdot 1\\ 0\end{pmatrix}(a\ \ b)=(nxa\ \ nxb),$$

D is a Γ-derivation on M. Indeed,

$$\begin{aligned}
&D((x\ \ y)\begin{pmatrix} 0\ 0\\ b\ 0\end{pmatrix}(a\ \ b))\\
&=D(nxa\ \ nxb)\\
&=(nd(xa)\ \ nd(xb))\\
&=(nd(x)a+nxd(a)\ \ nd(x)b+nxd(b))\\
&=(nd(x)a\ \ nd(x)b)+(nxd(a)\ \ nxd(b))\\
&=(d(x)\ \ d(y))\begin{pmatrix} n\cdot 1\\ 0\end{pmatrix}(a\ \ b)+(x\ \ y)\begin{pmatrix} n\cdot 1\\ 0\end{pmatrix}(d(a)\ \ d(b))\\
&=D(x\ \ y)\begin{pmatrix} n\cdot 1\\ 0\end{pmatrix}(a\ \ b)+(x\ \ y)\begin{pmatrix} n\cdot 1\\ 0\end{pmatrix}D(a\ \ b)
\end{aligned}$$

for all $x,y,a,b\in S$ and $n\in Z^+$.

Definition 2.9. Let M be a Γ-semiring. Any nonempty subset I is called a *left ideal* of M if the following conditions are satisfied:

(1) $a,b\in I\Rightarrow a+b\in I$,

(2) $a \in I$ and $s \in M \Rightarrow s\alpha a \in I$, for every $\alpha \in \Gamma$.

Similarly, we can define right and two-sided ideal in a Γ-semiring.

3. Orthogonal reverse derivations of semiprime Γ-semirings

Throughout this paper, we assume that M is a Γ-semiring with additive identity 0 and addition is commutative.

Definition 3.1. Let M be a Γ-semiring. An additive mapping $d : M \to M$ is a *reverse derivation* if $d(x\alpha y) = d(y)\alpha x + y\alpha d(x)$ for all $x, y \in M$ and $\alpha \in \Gamma$. Also, additive mapping $d : M \to M$ is called a *Jordan derivation* if $d(x\alpha x) = d(x)\alpha x + x\alpha d(x)$ for all $x \in M$ and $\alpha \in \Gamma$.

Obviously, if M is commutative Γ-semiring, then both reverse derivation and derivation of M are the same. It can be easily seen that the reverse derivation is not a derivation, in general, but it is a Jordan derivation.

Example 3.2. Let R be an associative ring with 1 and let $d : R \to R$ be a reverse derivation. Consider $M = M_{1\times 2}(R)$ and

$$\Gamma = \left\{ \begin{pmatrix} n \cdot 1 \\ 0 \end{pmatrix} : n \in Z^+ \right\}.$$

Then it is clear that M is a Γ-semiring. Let $N = \{(x \ \ x) | x \in R\} \subset M$. Then N is a subsemiring of M. Define a self-map $D : N \to N$ by

$$D(x \ \ x) = (d(x) \ \ d(x)).$$

Let $a = (x_1 \ \ x_1),\ b = (x_2 \ \ x_2)$ and $\alpha = \begin{pmatrix} n \cdot 1 \\ 0 \end{pmatrix} \in \Gamma$. Then we have

$$\begin{aligned} D(a \ \ b) &= D((x_1 \ \ x_1) \begin{pmatrix} n \cdot 1 \\ 0 \end{pmatrix} (x_2 \ \ x_2)) \\ &= D(x_1 n x_2 \ \ x_1 n x_2) \\ &= (d(x_2)nx_1 + x_2 n d(x_1) \ \ d(x_2)nx_1 + x_2 n d(x_1)) \\ &= (d(x_2)nx_1 + d(x_2)nx_1 \ \ x_2 n d(x_1) + x_2 n d(x_1)) \\ &= (d(x_2) \ \ d(x_2)) \begin{pmatrix} n \cdot 1 \\ 0 \end{pmatrix} (x_1 \ \ x_1) + (x_2 \ \ x_2) \begin{pmatrix} n \cdot 1 \\ 0 \end{pmatrix} (d(x_1) \ \ d(x_1)) \\ &= D((x_2 \ \ x_2))\alpha a + b\alpha D((x_1 \ \ x_1)) \\ &= D(b)\alpha a + b\alpha D(a). \end{aligned}$$

Hence D is a reverse derivation on M.

Proposition 3.3. *Let d be a reverse derivation of M. Then the following conditions hold:*

(1) *If M is a Γ-semiring with characteristic 2, then d^2 is a derivation of M.*

(2) *Let M is additively cancellative. If e is an idempotent element of M, then $e\alpha d(e)\alpha e = 0$.*

Proof. (1) Let d be a reverse derivation of M. Then we have

$$\begin{aligned} d^2(x\alpha y) &= d(d(x\alpha y)) = d(d(y)\alpha x + y\alpha d(x)) \\ &= d(x)\alpha d(y) + x\alpha d^2(y) + d^2(x)\alpha y + d(x)\alpha d(y) \\ &= d^2(x)\alpha y + x\alpha d^2(y), \end{aligned}$$

which implies that d^2 is a usual derivation of M, for every $x, y \in M$ and $\alpha \in \Gamma$.

(2) Let e is an idempotent element of M. Then we have $d(e) = d(e\alpha e) = d(e)\alpha e + e\alpha d(e)$. Multiplying by e in equation on left, we obtain $e\alpha d(e) = e\alpha d(e)\alpha e + e\alpha e\alpha d(e)$. Also, multiplying by e in equation on right, we have $e\alpha d(e)\alpha e = e\alpha d(e)\alpha e\alpha e + e\alpha e\alpha d(e)\alpha e = e\alpha d(e)\alpha e + e\alpha d(e)\alpha e$, i.e., $e\alpha d(e)\alpha e = 0$. □

Proposition 3.4. *Let d be a reverse derivation of a prime Γ-semiring M and let $a \in M$. If $a\alpha d(x) = 0$, for every $x \in M$ and $\alpha \in \Gamma$, then $a = 0$ or d is zero.*

Proof. Let $a\alpha d(x) = 0$, for every $x \in M$ and $\alpha, \beta \in \Gamma$. Then replacing x by $x\beta y$, we have

$$\begin{aligned} 0 &= a\alpha(d(x\beta y)) = a\alpha(d(y)\beta x + y\beta d(x) \\ &= a\alpha d(y)\beta x + a\alpha y\beta d(x) = a\alpha y\beta d(x), \end{aligned}$$

for every $x, y \in M$ and $\alpha, \beta \in \Gamma$. Since M is a prime Γ-semiring, if $d(x) \neq 0$, for some $x \in M$, then $a = 0$. □

Now, we give the derivation of orthogonality of two reverse derivations.

Definition 3.5. Let d and g be two reverse derivations on M. Then d and g are said to be *orthogonal* if $d(x)\Gamma M\Gamma g(y) = 0 = g(x)\Gamma M\Gamma d(y)$, for all $x, y \in M$.

Example 3.6. Let M_1 be a Γ_1-semiring and M_2 be a Γ_2-semiring. Consider $M = M_1 \times M_2$ and $\Gamma = \Gamma_1 \times \Gamma_2$. The addition and multiplication on M and Γ are defined as follows:

$$(a, b) + (c, d) = (a + c, b + d) \text{ and } (a, b)(\alpha, \beta)(c, d) = (a\alpha c, b\beta d),$$

for every $a, c \in M_1, b, d \in M_2, \alpha \in \Gamma_1$ and $\beta \in \Gamma_2$. Under these operations, M_1 is a Γ-semiring. Let d_1 be a reverse derivation on M. Define a derivation d on M by $d((a,b)) = (d_1(a), 0)$. Then d is a reverse derivation on M. Let d_2 be a reverse derivation on M_2. Define a derivation g on M by $g((a,b)) = (0, d_2(b))$. Then g is a reverse derivation on M. It is clear that d and g are orthogonal reverse derivation on M.

We start this section by some observations which are useful in proving our main results.

Lemma 3.7. *Let M be a 2-torsion free semiprime Γ-semiring and $a, b \in M$. Then the following conditions are equivalent:*

(1) $a\Gamma x\Gamma b = 0$,
(2) $b\Gamma x\Gamma a = 0$,
(3) $a\Gamma x\Gamma b + b\Gamma x\Gamma a = 0$, *for every $x \in M$.*

Also, if one of these conditions is fulfilled, then $a\Gamma b = b\Gamma a = 0$.

Proof. (1) $\Rightarrow$ (2). Let $a\Gamma x\Gamma b = 0$, for every $a, b, x \in M$. Multiplying by $b\Gamma x$ on left side and multiplying by $x\Gamma a$ on right side, we have $b\Gamma x\Gamma a\Gamma x\Gamma b\Gamma x\Gamma a = 0$. Since M is semiprime, we have $b\Gamma x\Gamma a = 0$.

(2) $\Rightarrow$ (3). Let $b\Gamma x\Gamma a = 0$. Multiplying by $a\Gamma x$, on left side and multiplying by $x\Gamma b$, on right side, we have $a\Gamma x\Gamma b\Gamma x\Gamma a\Gamma x\Gamma b = 0$. Since M is semiprime, we have $a\Gamma x\Gamma b = 0$, which implies $a\Gamma x\Gamma b + b\Gamma x\Gamma a = 0$.

(3) $\Rightarrow$ (1). Let $a\Gamma x\Gamma b + b\Gamma x\Gamma a = 0$, for every $x, a, b \in M$. Multiplying by $b\Gamma x$ on left side, we have $b\Gamma x\Gamma(a\Gamma x\Gamma b) + b\Gamma x\Gamma(b\Gamma x\Gamma a) = 0$. Also, multiplying by $a\Gamma x$ on left side, we get

$$(a\Gamma x\Gamma b)\Gamma x\Gamma(a\Gamma x\Gamma b) + (a\Gamma x\Gamma b)\Gamma x\Gamma(b\Gamma x\Gamma a) = 0. \qquad \text{(a)}$$

Furthermore, the equation $a\Gamma x\Gamma b = 0$ multiplication by $x\Gamma a$ on right side, we have $(a\Gamma x\Gamma b)\Gamma x\Gamma a + (b\Gamma x\Gamma a)\Gamma x\Gamma a = 0$. Also, multiplying by $x\Gamma b$, on right side, we get

$$(a\Gamma x\Gamma b)\Gamma x\Gamma(a\Gamma x\Gamma b) + (b\Gamma x\Gamma a)\Gamma x\Gamma(a\Gamma x\Gamma b) = 0. \qquad \text{(b)}$$

Adding equation (a) to (b) and using (3), we have

$$2((a\Gamma x\Gamma b)\Gamma x\Gamma(a\Gamma x\Gamma b)) = 0.$$

Since M is 2-torsion free and M is semiprime, we have $a\Gamma x\Gamma b = 0$, for all $x \in M$. □

Remark 3.8. Let $a\Gamma x\Gamma b = 0$. Multiplying by b on left side and multiplying by a on right side, we have $(b\Gamma a)\Gamma x\Gamma(b\Gamma a) = 0$. Since M is semiprime, we have $b\Gamma a = 0$. Similarly, from $b\Gamma x\Gamma a = 0$, we can prove that $a\Gamma b = 0$.

Lemma 3.9. *Let M be a 2-torsion free semiprime Γ-semiring. Suppose that additive mappings of d and g of M into itself satisfy*

$$d(x)\Gamma M\Gamma g(x) = 0,$$

for any $x \in M$. Then $d(x)\Gamma M\Gamma g(y) = 0$, for every $x, y \in M$.

Proof. Suppose that $d(x)\alpha m\beta g(x) = 0$, for any $x, m \in M$ and $\alpha, \beta \in \Gamma$. Replacing x by $x + y$, we have

$$\begin{aligned} 0 &= d(x+y)\alpha m\beta g(x+y) \\ &= d(x)\alpha m\beta g(x) + d(x)\alpha m\beta g(y) + d(y)\alpha m\beta g(x) + d(y)\alpha m\beta g(y) \\ &= d(x)\alpha m\beta g(y) + d(y)\alpha m\beta g(x). \end{aligned}$$

Also, multiplying $d(x)\alpha m\beta g(y)$ on right side of the last equation, we have

$$\begin{aligned} 0 = {} & (d(x)\alpha m\beta g(y))\gamma n\delta(d(x)\alpha m\beta g(y)) \\ & + d(x)\alpha m\beta(g(y)\gamma n\delta(d(y))\alpha m\beta g(x). \end{aligned}$$

By Remark 3.8, we have $(d(x)\alpha m\beta g(y))\gamma n\delta(d(x)\alpha m\beta g(y)) = 0$. Since M is semiprime, we get $d(x)\alpha m\beta g(y) = 0$, for every $x, y, m, n \in M$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. □

Theorem 3.10. *Let M be a 2-torsion free semiprime Γ-semiring and d and g be reverse derivations. Then for all $x, y \in M$,*

$$d(x)\Gamma g(y) + g(x)\Gamma d(y) = 0, \qquad \text{(c)}$$

if and only if d and g are orthogonal.

Proof. Suppose that $d(x)\alpha g(y) + g(x)\alpha d(y) = 0$, for every $x, y \in M$ and $\alpha \in \Gamma$. Replacing y by $x\beta y$ in (c), we have

$$\begin{aligned} 0 &= d(x)\alpha g(x\beta y) + g(x)\alpha d(x\beta y) \\ &= d(x)\alpha g(y)\beta x + y\beta g(x) + g(x)\alpha d(y)\beta x + y\beta d(x) \\ &= (d(x)\alpha g(y) + g(x)\alpha d(y))\beta x + d(x)\alpha y\beta g(x) + g(x)\alpha y\beta d(x) \end{aligned}$$

for every $x, y \in M$ and $\alpha, \beta \in \Gamma$. By hypothesis,

$$d(x)\alpha y\beta g(x) + g(x)\alpha y\beta d(x) = 0,$$

and so by Lemma 3.7, we have $d(x)\alpha y\beta g(x) = 0 = g(x)\alpha y\beta d(x)$, for every $x \in M$ and $\alpha, \beta \in \Gamma$. Hence, by Lemma 3.9, we get $d(x)\Gamma M\Gamma g(z) = 0 = g(x)\Gamma M\Gamma d(z)$, which implies $d(x)\Gamma M\Gamma g(y) = g(x)\Gamma M\Gamma d(y) = 0$ for any $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. This proves that d and g are orthogonal.

Conversely, assume that d and g are orthogonal. Then we have $d(x)\Gamma g(y) = g(y)\Gamma d(x) = 0$. By Remark 3.8, $d(x)\Gamma g(y) = g(x)\Gamma d(y) = 0$, which implies that $d(x)\Gamma g(y) + g(x)\Gamma d(y) = 0$, for all $x, y \in M$. □

Remark 3.11. Suppose that d and g are reverse derivations of a Γ-semiring M. Then the following identities are immediate from the definition of reverse derivations.

$$\begin{aligned}(dg)(x\alpha y) &= d(g(x\alpha y)) = d(g(y)\alpha x + y\alpha g(x)\\ &= (dg)(x)\alpha y + d(x)\alpha g(y) + g(x)\alpha d(y) + x\alpha(dg)(y), \qquad \text{(d)}\end{aligned}$$

for any $x, y \in M$.

Similarly, we have

$$\begin{aligned}(gd)(x\alpha y) &= g(d(x\alpha y)) = g(d(y)\alpha x + y\alpha d(x)\\ &= (gd)(x)\alpha y + g(x)\alpha d(y) + d(x)\alpha g(y) + x\alpha(gd)(y), \qquad \text{(e)}\end{aligned}$$

for any $x, y \in M$ and $\alpha \in \Gamma$.

The following theorem gives a few criteria on the orthogonality of reverse derivations.

Theorem 3.12. *Let M be a 2-torsion free semiprime Γ-semiring and d and g be reverse derivations. Then d and g are orthogonal if and only if $dg = 0$.*

Proof. Suppose that $dg = 0$. Then by using the identity (c) in the Remark 3.11, we obtain $d(x)\alpha g(y) + g(x)\alpha d(y) = 0$, for every $x, y \in M$ and $\alpha \in \Gamma$. Therefore, by Theorem 3.10, d and g are orthogonal.

Conversely, since d and g are orthogonal, we have $d(x)\Gamma M\Gamma g(z) = 0$. Hence we get

$$\begin{aligned}0 &= d(d(x)\alpha y\beta g(z)) = d(y\beta g(z))\alpha d(x) + y\beta g(z)\alpha d(d(x))\\ &= (dg)(z)\beta y\alpha d(x) + g(z)\beta d(y)\alpha d(x) + y\beta g(z)\alpha d(d(x))\\ &= (dg)(z)\beta y\alpha d(x)\end{aligned}$$

for every $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Replacing x by $g(z)$, we have $(dg)(z)\beta y\alpha(dg)(z) = 0$, for any $z \in M$. Since M is semiprime, we obtain $(dg)(z) = 0$, for every $z \in M$, that is $dg = 0$. □

Theorem 3.13. *Let M be a 2-torsion free semiprime Γ-semiring and d and g be reverse derivations. Then d and g are orthogonal if and only if $dg + gd = 0$.*

Proof. Suppose that $dg + gd = 0$. Then we have

$$\begin{aligned}0 &= (dg + gd)(x\alpha y)\\ &= (dg)(x)\alpha y + d(x)\alpha g(y) + g(x)\alpha d(y) + \alpha(dg)(y) + (gd)(x)\alpha y\\ &\quad + g(x)\alpha d(x) + d(x)\alpha g(y) + x\alpha(gd)(y)\\ &= (dg + gd)(x)\alpha y + 2d(x)\alpha g(y) + 2g(x)\alpha d(y) + x\alpha((dg)(y) + (gd)(y)),\end{aligned}$$

for every $x, y \in M$ and $\alpha \in \Gamma$. Since M is 2-torsion free, we obtain $d(x)\alpha g(y) + g(x)\alpha d(y) = 0$, and so by Theorem 3.10, d and g are orthogonal.

Conversely, let d and g be orthogonal reverse derivations. By Theorem 3.10, $dg = gd = 0$. Hence $dg + gd = 0$. □

Theorem 3.14. *Let M be a 2-torsion free semiprime Γ-semiring and d and g be reverse derivations. Then d and g are orthogonal if and only if dg is a derivation of M.*

Proof. Suppose that dg is a derivation on M. Then we have

$$(dg)(x\alpha y) = (dg)(x)\alpha y + x\alpha(dg)(y).$$

Comparing this expression with (d) of Remark 3.11, we obtain

$$d(x)\alpha g(y) + g(x)\alpha d(y) = 0,$$

and so by Theorem 3.10, d and g are orthogonal.

Conversely, if d and g are orthogonal, by Theorem 3.12, $dg = 0$. Thus dg is a derivation of M. □

Theorem 3.15. *Let M be a 2-torsion free semiprime, additively cancellative Γ-semiring and d be a reverse derivation of M. If d^2 is a derivation of M, then $d = 0$.*

Proof. Since d^2 is a derivation of M, we have $d^2(x\alpha y) = d^2(x)\alpha y + x\alpha d^2(y)$ and

$$\begin{aligned} d^2(x\alpha y) &= d(d(x\alpha y)) = d(d(y)\alpha x + y\alpha d(x)) \\ &= d(x)\alpha d(y) + d(x)\alpha d(y) + x\alpha d^2(y) + d^2(x)\alpha y \\ &= 2d(x)\alpha d(y) + d^2(x)\alpha y + x\alpha d^2(y). \end{aligned}$$

Hence we have $2d(x)\alpha d(y) = 0$. Since M is 2-torsion free semiprime, we get $d(x)\alpha d(y) = 0$, for any $x, y \in M$ and $\alpha \in \Gamma$. Replacing x by $s\beta x$, we have

$$0 = d(s\beta x)\alpha d(y) = (d(x)\beta s + x\beta d(s))\alpha d(y) = d(x)\beta s\alpha d(y).$$

for all $x, s \in M, \beta \in \Gamma$. Replacing y by $x + y$, we have

$$\begin{aligned} 0 &= d(x)\beta s\alpha d(x+y) \\ &= d(x)\beta s\alpha(d(x) + d(y)) \\ &= d(x)\beta s\alpha d(x) + d(x)\beta s\alpha d(y) = d(x)\beta s\alpha d(x). \end{aligned}$$

Since M is semiprime, we obtain $d(x) = 0$, for any $x \in M$ i.e., $d = 0$. □

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