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ORTHOGONAL STABILITY OF AN EULER-LAGRANGE-JENSEN (a, b)-CUBIC FUNCTIONAL EQUATION

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Abstract. In this paper, we introduce a new generalized (a, b)-cubic Euler-Lagrange-Jensen functional equation and obtain its general solution. Furthermore, we prove the Hyers-Ulam stability of the new generalized (a, b)-cubic Euler-Lagrange-Jensen functional equation in orthogonality normed spaces.

1. Introduction

The following question concerning the stability of homomorphisms was raised by Ulam [20].

Let G be a group and G' be a metric group with metric $\rho(. , .)$. Given $\epsilon > 0$ does there exist a $\delta > 0$ such that if a function $f : G \to G'$ satisfies the inequality $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then there exists a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

In 1941, Hyers [8] proved the following celebrated theorem as a partial solution to Ulam's question.

Theorem 1 ([8])**.** Assume that E_1 and E_2 are Banach spaces. If a mapping $f : E_1 \to E_2$ satisfies the inequality

$$||f(x + y) - f(x) - f(y)|| \leq \epsilon$$

for some $\epsilon \geq 0$ and for all $x, y \in E_1$, then the limit

$$a(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

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exists for each x in E_1 and $a : E_1 \to E_2$ is the unique additive mapping such that

$$||f(x) - a(x)|| \leq \epsilon$$

for all $x \in E_1$.

In 1978, Th. M. Rassias [16] provided a generalized solution to Ulam's problem where he used the controlled function as the sum of powers of norms. In 1982, J. M. Rassias [13] generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms. Later, J. M. Rassias *et al.* [17] discussed the stability of quadratic functional equation by using the mixed powers of norms. See [1, 2, 5, 7, 10, 11, 12, 19] for more information on functional equations and functional inequalities and their stability.

Ger and Sikorska discussed the orthogonal stability of the Cauchy functional equation in [6]. The orthogonally quadratic functional equation was generalized by Drljevic [3], Fochi [4] and Szabo [18].

Definition 1. A vector space X is called an *orthogonality vector space* if there is a relation $x \perp y$ on X such that
(i) totality of $\perp$ for zero: $x \perp 0,\ 0 \perp x$ for all $x \in X$;
(ii) independence: if $x \perp y$ and $x, y \neq 0$, then x, y are linearly independent;
(iii) homogeneity: if $x \perp y$, then $ax \perp by$ for all $a, b \in \mathbb{R}$;
(iv) the Thalesian property: if P is a two-dimensional subspace of X, then
(a) for every $x \in P$ there exists $0 \neq y \in P$ such that $x \perp y$;
(b) there exist vectors $x, y \neq 0$ such that $x \perp y$ and $x + y \perp x - y$.
Any vector space can be made into an orthogonality vector space if we define $x \perp 0, 0 \perp x$ for all x and for nonzero vector $x,\ y$ define $x \perp y$ if and only if $x,\ y$ are linearly independent. The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x,\ y \in X$.

Definition 2. The pair $(x, \perp)$ is called an *orthogonality space*. It becomes *orthogonality normed space* when the orthogonality space is equipped with a norm.

J. M. Rassias [14, 15] investigated the stability of Euler-Lagrange type quadratic functional equation

$$f(rx + sy) + f(sx - ry) = (r^2 + s^2)\,[f(x) + f(y)]$$

for fixed real numbers r, s with $r \neq 0, s \neq 0$.

In 2007, Jun and Kim [9] introduced the following generalized Euler-Lagrange type cubic functional equation

$$f(ax+by)+f(bx+ay)=(a+b)(a-b)^2\left[f(x)+f(y)\right]+ab(a+b)f(x+y) \tag{1.1}$$

for fixed integers a, b with $a \neq 0, b \neq 0, a \pm b \neq 0$.

In this paper, we investigate the various stabilities related to Ulam problem of the following (a, b)-cubic Euler-Lagrange-Jensen functional equation

$$\begin{aligned}\frac{a^3}{a+b}f\left(x+\frac{b}{a}y\right)+\frac{b^3}{a+b}f\left(x+\frac{a}{b}y\right)\\ =(a-b)^2\left(f(x)+f(y)\right)+\frac{ab}{8}f\left(2x+2y\right)\end{aligned} \tag{1.2}$$

for integers $a = 2^p$ or $a = 3^p$ and $b = 2^q$ or $b = 3^q$ with $a \neq b$ and $a + b \neq 1$ for integers p, q in the concept of orthogonality normed spaces.

Definition 3. A mapping $f : A \to B$ is called *orthogonal Euler-Lagrange-Jensen type cubic* if it satisfies the functional equation (1.2) for all $x, y \in A$ with $x \perp y$ where A is an orthogonality space and B is a real Banach space.

Throughout this paper, let $(A, \perp)$ denote an orthogonality normed space with norm $\|\cdot\|_A$ and $(B, \|\cdot\|_B)$ be a Banach space. We define

$$\begin{aligned}E_f\left(x,y\right)=\frac{a^3}{a+b}f\left(x+\frac{b}{a}y\right)+\frac{b^3}{a+b}f\left(x+\frac{a}{b}y\right)\\ -(a-b)^2\left(f(x)+f(y)\right)-\frac{ab}{8}f\left(2x+2y\right)\end{aligned}$$

for all $x, y \in A$ with $x \perp y$. Assume that $a = 2^p$ or $a = 3^p$ and $b = 2^q$ or $b = 3^q$ with $a \neq b$ and $a + b \neq 1$ for integers p, q.

2. General Solution of (a, b)-cubic Euler-Lagrange-Jensen Functional Equation

Theorem 2. Let X and Y be real vector spaces. An odd mapping $f : X \to Y$ satisfies the (a, b)-cubic Euler-Lagrange-Jensen functional equation (1.2) if and only if it satisfies the functional equation (1.1).

Proof. Suppose that a mapping $f : X \to Y$ satisfies (1.2). Putting $x = y = 0$ in (1.2), we get $f(0) = 0$. Let $y = 0$ in (1.2), we obtain

$$\frac{a^3}{a+b}f\left(x\right)+\frac{b^3}{a+b}f\left(x\right)=(a-b)^2f(x)+\frac{ab}{8}f\left(2x\right)$$

and so

$$f(2x) = 8f(x) \tag{2.1}$$

for all $x \in X$. Replacing x by $2x$ in (2.1) and again using (2.1), we obtain

$$f(4x) = 64f(x) \tag{2.2}$$

for all $x \in X$. It follows from (2.1) and (2.2) that

$$f(2^p x) = 2^{3p} f(x) \tag{2.3}$$

for all $x \in X$ and all integers p. Replacing x by $3x$ in (2.1) and using (2.3), we obtain

$$f(3x) = 27f(x) \tag{2.4}$$

for all $x \in X$. It follows from (2.4) that

$$f(3^p x) = 3^{3p} f(x)$$

for all $x \in X$ and all integers p. So we conclude that

$$f(ax) = a^3 f(x)$$

for all $x \in X$ and all $a = 2^p$ or $a = 3^p$.

Similarly, we can obtain that

$$f(bx) = b^3 f(x)$$

for all $x \in X$ and all $b = 2^q$ or $b = 3^q$. Replacing y by aby in (1.2) and in the resultant again replacing x by y and y by x, we have

$$\begin{aligned} &a^3 f\left(b^2 x + y\right) + b^3 f\left(a^2 x + y\right) \\ &\quad = (a+b)(a-b)^2 (f(abx) + f(y)) + \frac{ab(a+b)}{8} f\left(2abx + 2y\right) \end{aligned} \tag{2.5}$$

for all $x, y \in X$. Replacing y by $-y$ and using the oddness of f, we obtain

$$\begin{aligned} &a^3 f\left(b^2 x - y\right) + b^3 f\left(a^2 x - y\right) \\ &\quad = (a+b)(a-b)^2 (f(abx) - f(y)) + \frac{ab(a+b)}{8} f\left(2abx - 2y\right) \end{aligned} \tag{2.6}$$

for all $x, y \in X$. Adding (2.5) and (2.6), we get

$$\begin{aligned} &a^3 \left(f\left(b^2 x + y\right) + f\left(b^2 x - y\right)\right) + b^3 \left(f\left(a^2 x + y\right) + f\left(a^2 x - y\right)\right) \\ &\quad = 2(a+b)(a-b)^2 f(abx) + \frac{ab(a+b)}{8} \left(f\left(2abx + 2y\right) + f\left(2abx - 2y\right)\right) \end{aligned} \tag{2.7}$$

for all $x, y \in X$. Replacing y by aby in (2.7), we get

$$f(ax+by)+f(ax-by)+f(bx+ay)+f(bx-ay) \\ = 2(a+b)(a-b)^2 f(x)+ab(a+b)\left[f(x+y)+f(x-y)\right] \tag{2.8}$$

for all $x, y \in X$. Replacing x by y and y by x in (2.8) and again adding the resultant with (2.8) and using the oddness of f, we obtain (1.1).

Conversely, assume f satisfies the functional equation (1.1). Letting $y = 0$ in (1.1), we get

$$f(ax) = a^3 f(x) \text{ if and only if } f(bx) = b^3 f(x) \tag{2.9}$$

for all $x \in X$. Using (2.9) and (1.1), we have

$$\frac{a^3}{a+b} f\left(x+\frac{b}{a}y\right)+\frac{b^3}{a+b} f\left(x+\frac{a}{b}y\right) \\ = (a-b)^2\left(f(x)+f(y)\right)+abf\left(x+y\right) \tag{2.10}$$

for all $x, y \in X$. Replacing (x, y) by $(2x, 2y)$ in (2.10) and using (2.9), we obtain (1.2). □

3. Hyers-Ulam Stability of (a, b)-cubic Euler-Lagrange-Jensen Functional Equation

Theorem 3. Let μ and $s(s < 3)$ be nonnegative real numbers. Let $f : A \to B$ be a mapping fulfilling

$$\|E_f\left(x, y\right)\|_B \leq \mu\left\{\|x\|_A^s + \|y\|_A^s\right\} \tag{3.1}$$

for all $x, y \in A$ with $x \perp y$. Then there exists a unique orthogonally Euler-Lagrange-Jensen type cubic mapping $C : A \to B$ such that

$$\|f(x) - C(x)\|_B \leq \frac{8\mu}{ab(8-2^s)}\|x\|_A^s \tag{3.2}$$

for all $x \in A$. The mapping $C(x)$ is defined by

$$C(x) = \lim_{n\to\infty}\frac{f(2^n x)}{8^n}$$

for all $x \in A$.

Proof. Letting $x = y = 0$ in (3.1), we get $f(0) = 0$. Setting $y = 0$ in (3.1), we obtain

$$\left\|ab\left(\frac{f(2x)}{8} - f(x)\right)\right\|_B \leq \mu(\|x\|_A^s) \tag{3.3}$$

for all $x \in A$. Since $x \perp 0$, we have

$$\left\| \frac{f(2x)}{8} - f(x) \right\|_B \le \frac{\mu}{ab} \|x\|_A^s \tag{3.4}$$

for all $x \in A$. Now replacing x by $2x$ and dividing by 8 in (3.4) and summing the resulting inequality with (3.4), we obtain

$$\left\| \frac{f(2^2 x)}{8^2} - f(x) \right\|_B \le \frac{\mu}{ab} \left\{ 1 + \frac{2^s}{8} \right\} \|x\|_A^s$$

for all $x \in A$. In general, using induction on a positive integer n we obtain that

$$\left\| \frac{f(2^n x)}{8^n} - f(x) \right\|_B \le \frac{\mu}{ab} \sum_{k=0}^{n-1} \frac{2^{s\,k}}{8^k} \|x\|_A^s \le \frac{\mu}{ab} \sum_{k=0}^{\infty} \frac{2^{s\,k}}{8^k} \|x\|_A^s \tag{3.5}$$

for all $x \in A$. In order to prove the convergence of the sequence $\{\frac{f(2^n x)}{8^n}\}$, replacing x by $2^m x$ and dividing by 8^m in (3.5), for any $n, m > 0$, we obtain

$$\begin{aligned} \left\| \frac{f(2^n 2^m x)}{8^{(n+m)}} - \frac{f(2^m x)}{8^m} \right\|_B &= \frac{1}{8^m} \left\| \frac{f(2^n 2^m x)}{8^n} - f(2^m x) \right\|_B \\ &\le \frac{1}{8^m} \frac{\mu}{ab} \sum_{k=0}^{n-1} \frac{2^{s\,k}}{8^k} \|2^m x\|_A^s \\ &\le \frac{\mu}{ab} \sum_{k=0}^{\infty} \frac{1}{2^{(3-s)(k+m)}} \|x\|_A^s . \end{aligned} \tag{3.6}$$

Since $s < 3$, the right hand side of (3.6) tends to 0 as $m \to \infty$ for all $x \in A$. Thus $\{\frac{f(2^n x)}{8^n}\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $C : A \to B$ such that

$$C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{8^n} \quad \forall x \in A.$$

Letting $n \to \infty$ in (3.5), we get the formula (3.2) for all $x \in A$. To prove C satisfies (1.2), replacing (x, y) by $(2^n x, 2^n y)$ in (3.1) and dividing by 8^n, we obtain

$$\begin{aligned} \frac{1}{8^n} \Bigg\| \frac{a^3}{a+b} f\left(2^n\left(x + \frac{b}{a}y\right)\right) + \frac{b^3}{a+b} f\left(2^n\left(x + \frac{a}{b}y\right)\right) - (a-b)^2 \left(f(2^n x) + f(2^n y)\right) \\ - \frac{ab}{8} f\left(2^n(2x+2y)\right) \Bigg\|_B \le \frac{\mu}{8^n} \left\{ \|2^n x\|_A^s + \|2^n y\|_A^s \right\}. \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we get

$$\begin{aligned} \Bigg\| \frac{a^3}{a+b} C\left(x + \frac{b}{a}y\right) + \frac{b^3}{a+b} C\left(x + \frac{a}{b}y\right) \\ - (a-b)^2 \left(C(x) + C(y)\right) - \frac{ab}{8} C(2x+2y) \Bigg\|_B \le 0, \end{aligned}$$

which gives

$$\begin{aligned}&\frac{a^3}{a+b} C\left(x+\frac{b}{a} y\right)+\frac{b^3}{a+b} C\left(x+\frac{a}{b} y\right) \\ &\quad=(a-b)^2(C(x)+C(y))+\frac{a b}{8} C(2 x+2 y)\end{aligned}$$

for all $x, y \in A$ with $x \perp y$. Therefore $C: A \rightarrow B$ is an orthogonally Euler-Lagrange-Jensen type cubic mapping which satisfies (1.2).

Let C' be another orthogonally Euler-Lagrange-Jensen type cubic mapping satisfying (1.2) and the inequality (3.2). Then

$$\begin{aligned}\left\|C(x)-C'(x)\right\|_B &= \frac{1}{8^n}\left\|C\left(2^n x\right)-C'\left(2^n x\right)\right\|_B \\ &\leq \frac{1}{8^n}\left(\left\|C\left(2^n x\right)-f\left(2^n x\right)\right\|_B+\left\|f\left(2^n x\right)-C'\left(2^n x\right)\right\|_B\right) \\ &\leq \frac{16 \mu}{a b(8-2^s)} \frac{1}{2^{n(3-s)}}\|x\|_A^s \\ &\rightarrow 0 \text { as } n \rightarrow \infty\end{aligned}$$

for all $x \in A$. Therefore C is unique. This completes the proof of the theorem. □

Theorem 4. Let μ and $s(s>3)$ be nonnegative real numbers. Let $f: A \rightarrow B$ be a mapping satisfying (3.1) for all $x, y \in A$ with $x \perp y$. Then there exists a unique orthogonally Euler-Lagrange-Jensen type cubic mapping $C: A \rightarrow B$ such that

$$\|f(x)-C(x)\|_B \leq \frac{8 \mu}{a b(2^s-8)}\|x\|_A^s$$

for all $x \in A$. The mapping $C(x)$ is defined by

$$C(x)=\lim _{n \rightarrow \infty} 8^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$.

Proof. Replace x by $\frac{x}{2}$ in the inequality (3.3). Then the rest of the proof is similer to that of Theorem 3. □

4. Stability of (a, b)-cubic Euler-Lagrange-Jensen Functional Equation

Theorem 5. Let $f: A \rightarrow B$ be a mapping satisfying the inequality

$$\left\|E_f(x, y)\right\|_B \leq \mu\left\{\|x\|_A^{2 s}+\|y\|_A^{2 s}+\|x\|_A^s\|y\|_A^s\right\} \tag{4.1}$$

for all $x, y \in A$ where μ and s are constants with, $\mu, s > 0$ and $s < \frac{3}{2}$. Then the limit

$$C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{8^n} \tag{4.2}$$

exists for all $x \in A$ and $C : A \to B$ is the unique Euler-Lagrange-Jensen type cubic mapping such that

$$\|f(x) - C(x)\|_B \le \frac{8\mu}{ab(8 - 2^{2s})} \|x\|_A^{2s}$$

for all $x \in A$.

Proof. Letting $x = y = 0$ in (4.1), we get $f(0) = 0$. Letting $y = 0$ in (4.1), we obtain

$$\left\| \frac{f(2x)}{8} - f(x) \right\|_B \le \frac{\mu}{ab} \|x\|_A^{2s} \tag{4.3}$$

for all $x \in A$. Now replacing x by $2x$ and dividing by 8 in (4.3) and summing the resulting inequality with (4.3), we obtain

$$\left\| \frac{f(2^2 x)}{8^2} - f(x) \right\|_B \le \frac{\mu}{ab} \left\{ 1 + \frac{2^{2s}}{8} \right\} \|x\|_A^{2s}$$

for all $x \in A$. Using induction on positive integers n, we obtain that

$$\left\| \frac{f(2^n x)}{8^n} - f(x) \right\|_B \le \frac{\mu}{ab} \sum_{k=0}^{n-1} \left(\frac{2^{2s}}{8} \right)^k \|x\|_A^{2s} \le \frac{\mu}{ab} \sum_{k=0}^{\infty} \left(\frac{2^{2s}}{8} \right)^k \|x\|_A^{2s} \tag{4.4}$$

for all $x \in A$. In order to prove the convergence of the sequence $\{\frac{f(2^n x)}{8^n}\}$, replacing x by $2^m x$ and dividing by 8^m in (4.4), for any $n, m > 0$, we obtain

$$\begin{aligned} \left\| \frac{f(2^n 2^m x)}{8^{(n+m)}} - \frac{f(2^m x)}{8^m} \right\|_B &= \frac{1}{8^m} \left\| \frac{f(2^n 2^m x)}{8^n} - f(2^m x) \right\|_B \\ &\le \frac{1}{8^m} \frac{\mu}{ab} \sum_{k=0}^{n-1} \left(\frac{2^{2s}}{8} \right)^k \|2^m x\|_A^{2s} \\ &\le \frac{\mu}{ab} \sum_{k=0}^{\infty} \frac{1}{2^{(3-2s)(k+m)}} \|x\|_A^{2s} \qquad (4.5) \end{aligned}$$

Since $s < \frac{3}{2}$, the right hand side of (4.5) tends to 0 as $m \to \infty$ for all $x \in A$. Thus $\{\frac{f(2^n x)}{8^n}\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $C : A \to B$ such that

$$C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{8^n} \quad \forall x \in A.$$

Letting $n \to \infty$ in (4.4), we get the formula (4.2) for all $x \in A$.

The rest of the proof is similar to that of Theorem 3. $\square$

Theorem 6. Let $f : A \to B$ be a mapping satisfying the inequality (4.1) for all $x, y \in A$ where μ and s are constants with, $\mu, s > 0$ and $s > \frac{3}{2}$. Then the limit

$$C(x) = \lim_{n\to\infty} 8^n \, f\left(\frac{x}{2^n}\right)$$

exists for all $x \in A$ and $C : A \to B$ is the unique Euler-Lagrange-Jensen type cubic mapping such that

$$\|f(x) - C(x)\|_B \leq \frac{8\mu}{ab(2^{2s} - 8)} \|x\|_A^{2s}$$

for all $x \in A$.

Proof. Replace x by $\frac{x}{2}$ in (4.3). Then the proof is similar to that of Theorem 5. $\square$

5. Conclusion

In this paper, we have introduced a new generalized (a, b)-cubic Euler-Lagrange-Jensen functional equation and obtained its general solution. Furthermore, we have proved the Hyers-Ulam of the generalized (a, b)-cubic Euler-Lagrange-Jensen functional equation in orthogonality normed spaces.

Declarations

Availablity of data and materials
Not applicable.

Human and animal rights
We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Conflict of interest
The authors declare that they have no competing interests.

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