J. Korean Math. Soc. **59** (2022), No. 3, pp. 469–494
https://doi.org/10.4134/JKMS.j210115
pISSN: 0304-9914 / eISSN: 2234-3008

BOUNDEDNESS OF CALDERÓN-ZYGMUND OPERATORS ON INHOMOGENEOUS PRODUCT LIPSCHITZ SPACES

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Abstract. In this paper, we study the boundedness of a class of inhomogeneous Journé's product singular integral operators on the inhomogeneous product Lipschitz spaces. The consideration of such inhomogeneous Journé's product singular integral operators is motivated by the study of the multi-parameter pseudo-differential operators. The key idea used here is to develop the Littlewood-Paley theory for the inhomogeneous product spaces which includes the characterization of a special inhomogeneous product Besov space and a density argument for the inhomogeneous product Lipschitz spaces in the weak sense.

1. Introduction

Classical Calderón-Zygmund theory may be observed to center around singular integrals associated with the Hardy-Littlewood maximal operator that commutes with the usual dilations on $\mathbb{R}^n$, $\delta x = (\delta x_1, \dots, \delta x_n)$ for $\delta > 0$. This theory has been extensively studied and is by now well understood; see for example the monograph [36]. If we consider more general non-isotropic groups of dilations, such as the family of product dilations defined by $\delta(x_1, \dots, x_n) = (\delta_1 x_1, \dots, \delta_n x_n)$, $\delta_i > 0$, $i = 1, \dots, n$, then we see many non-isotropic variants of classical theories have been developed. Understanding of the various function spaces associated with multi-parameter structures and the boundedness of Fourier multipliers, as well as singular integral operators on such spaces, has been greatly advanced in recent decades.

To be more precise, in [13] R. Fefferman and Stein first initiated a class of product convolution singular integral operators which satisfy analogous conditions enjoyed by the double Hilbert transform defined on $\mathbb{R} \times \mathbb{R}$. They established the L^p boundedness of these singular integral operators for $1 < p < \infty$.

Received February 17, 2021; Revised July 29, 2021; Accepted January 6, 2022.
2020 *Mathematics Subject Classification.* Primary 42B20; Secondary 42B25, 46E30.
Key words and phrases. Calderón-Zygmund operator, inhomogeneous product Lipschitz space, Littlewood-Paley theory.
The first author was supported by Zhejiang Provincial Natural Science Foundation of China (Grant No. LQ22A010018) and the second author was supported by National Natural Science Foundation of China (Grant Nos. 11626213, 11771399).

Chang and R. Fefferman [1–3] developed a comprehensive theory of product Hardy spaces initially introduced by Gundy-Stein [15], including the atomic decompositions and their dual spaces, namely, the product BMO spaces. Journé in [28] introduced non-convolution product singular integral operators, established the product $T1$ theorem and proved the L^∞-BMO boundedness for such operators. Subsequently, more and more results on L^p, $1 < p < \infty$, boundedness and H^p boundedness for operators in Journé's class were obtained [10–12, 18, 19, 34]. As demonstrated by Journé, the remarkable boundedness criterion of R. Fefferman [11] in two parameter case does not apply to the setting of three or more parameters. To this end, Pipher [34] proved a Journé type covering lemma in higher dimensions and established the $H^p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ to $L^p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ boundedness for singular integral operators in Journé's class by considering directly their actions on the atoms supported in arbitrary open sets. More recently, the authors of [19] have established the necessary and sufficient conditions of the product $H^p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ boundedness of Journé's type singular integrals.

A more recent breakthrough is due to Müller, Ricci and Stein [29, 30]. They introduced a new type of multi-parameter structure, called a flag structure, and studied the L^p boundedness of Marcinkiewicz multiplier operators on the Heisenberg group. In 2001, Nagel, Ricci and Stein [31] studied a class of operators on nilpotent Lie groups given by convolution with flag kernels and applied this theory to study the $\Box_b$-complex on certain CR submanifolds of $\mathbb{C}^n$. More recently, Nagel, Ricci and Stein [31] and Nagel, Ricci, Stein and Wainger [32, 33] developed the theory of singular integrals with flag kernels in the more general setting of homogeneous groups. See also the recent works in [20, 22, 37].

On the other hand, at the extreme values of $p = \infty$, it is natural to hope that the BMO or Lipchitz spaces boundedness of multi-parameter singular integral operators are available. As is well-known, the classical Lipschitz spaces have had a profound influence in harmonic analysis and partial differential equations. Han et al. [17] constructed flag Lipschitz spaces on Heisenberg groups and prove that Marcinkiewicz multipliers are bounded on them. In addition, multi-parameter Lipschitz spaces associated with mixed homogeneities have been studied in Han and Han [16]. Very recently, the first author and his collaborators Zheng, Chen and others in [38] establish a necessary and sufficient condition for the boundedness of Journé's product singular integral operators on the product Lipschitz spaces. Namely, suppose that T is a singular integral operator in Journé's class with regularity exponent $\varepsilon \in (0, 1]$. Then T is bounded on $\tilde{\Lambda}_\alpha$ with $\alpha = (\alpha_1, \alpha_2)$ if and only if $T_1 1 = T_2 1 = 0$ for $0 < \max\{\alpha_1, \alpha_2\} < \varepsilon$, where $\tilde{\Lambda}_\alpha$ denotes the product homogeneous Lipschitz spaces on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ introduced in [38]. $T_1(1) = T_2(1) = 0$ will be explained below. For more about the Lipschitz spaces, see also [4, 21, 23–26, 39]. A natural question then arises: How about the multi-parameter Lipschitz spaces in the setting of inhomogeneous case and whether we can obtain some certain necessary and sufficient condition for the boundedness of Journé's type singular integral operators on these spaces?

The main goal of this paper is to settle this question. More precisely, we will investigate inhomogeneous product Lipschitz spaces via Littlewood-Paley theory and establish a boundedness criterion of inhomogeneous Journé's product singular integral operators on the inhomogeneous product Lipschitz spaces. The consideration of such inhomogeneous Journé's product singular integral operators is motivated by the study of the multi-parameter pseudo-differential operators. Actually, all results in this paper can be extended to arbitrary number of parameters.

In order to describe the main results in this paper, we first introduce the inhomogeneous product Lipschitz spaces on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. For $x_1, y_1 \in \mathbb{R}^{n_1}$ and $x_2, y_2 \in \mathbb{R}^{n_2}$, we denote that

$$\Delta_{y_1} f(x_1, x_2) = f(x_1 - y_1, x_2) - f(x_1, x_2),$$
$$\Delta^2_{y_1} f(x_1, x_2) = f(x_1 - y_1, x_2) - 2f(x_1, x_2) + f(x_1 + y_1, x_2),$$

and

$$\Delta_{y_2} f(x_1, x_2) = f(x_1, x_2 - y_2) - f(x_1, x_2),$$
$$\Delta^2_{y_2} f(x_1, x_2) = f(x_1, x_2 - y_2) - 2f(x_1, x_2) + f(x_1, x_2 + y_2).$$

Definition 1. Let $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 > 0$. The inhomogeneous product Lipschitz space is defined to be the space of all bounded continuous f defined on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that

(1) when $0 < \alpha_1, \alpha_2 < 1$,

$$\|f\|_{\Lambda_\alpha} := \|f\|_{L^\infty} + \sup_{y_1 \neq 0} \frac{|\Delta_{y_1} f|}{|y_1|^{\alpha_1}} + \sup_{y_2 \neq 0} \frac{|\Delta_{y_2} f|}{|y_2|^{\alpha_2}} + \sup_{y_1, y_2 \neq 0} \frac{|\Delta_{y_2}\Delta_{y_1} f|}{|y_1|^{\alpha_1}|y_2|^{\alpha_2}} < \infty;$$

(2) when $\alpha_1 = 1$, $0 < \alpha_2 < 1$,

$$\|f\|_{\Lambda_\alpha} := \|f\|_{L^\infty} + \sup_{y_1 \neq 0} \frac{|\Delta^2_{y_1} f|}{|y_1|} + \sup_{y_2 \neq 0} \frac{|\Delta_{y_2} f|}{|y_2|^{\alpha_2}} + \sup_{y_1, y_2 \neq 0} \frac{|\Delta_{y_2}\Delta^2_{y_1} f|}{|y_1||y_2|^{\alpha_2}} < \infty;$$

(3) when $0 < \alpha_1 < 1$, $\alpha_2 = 1$,

$$\|f\|_{\Lambda_\alpha} := \|f\|_{L^\infty} + \sup_{y_1 \neq 0} \frac{|\Delta_{y_1} f|}{|y_1|^{\alpha_1}} + \sup_{y_2 \neq 0} \frac{|\Delta^2_{y_2} f|}{|y_2|} + \sup_{y_1, y_2 \neq 0} \frac{|\Delta^2_{y_2}\Delta_{y_1} f|}{|y_1|^{\alpha_1}|v|} < \infty;$$

(4) when $\alpha_1 = \alpha_2 = 1$,

$$\|f\|_{\Lambda_\alpha} := \|f\|_{L^\infty} + \sup_{y_1 \neq 0} \frac{|\Delta^2_{y_1} f|}{|y_1|} + \sup_{y_2 \neq 0} \frac{|\Delta^2_{y_2} f|}{|y_2|} + \sup_{y_1, y_2 \neq 0} \frac{|\Delta^2_{y_2}\Delta^2_{y_1} f|}{|y_1||y_2|} < \infty.$$

When $\alpha_1, \alpha_2 > 1$, we write $\alpha_1 = m_1 + r_1$ and $\alpha_2 = m_2 + r_2$, where m_1, m_2 are integers and $0 < r_1, r_2 \leq 1$. Then $f \in \Lambda_\alpha$ means that $f \in C^{m_1+m_2}(\mathbb{R}^{n_1+n_2})$

such that all partial derivatives $\partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}f$ with $|\beta_1| = m_1$, $|\beta_2| = m_2$ belong to Λ_r for $r = (r_1, r_2)$ and

$$\|f\|_{\Lambda_\alpha} := \sum_{|\beta_1|=m_1, |\beta_2|=m_2} \|\partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}f\|_{\Lambda_r}.$$

We will characterize the inhomogeneous Lipschitz spaces via the Littlewood-Paley theory. For this purpose, we first adapt some notations. Given a function φ on $\mathbb{R}^n$, we denote

$$M_\varphi = \max\{N \in \mathbb{N} : \int_{\mathbb{R}^n} \varphi(x)x^\alpha dx = 0,\ |\alpha| \le N\},$$

where $\mathbb{N}$ denotes the class of all natural numbers, that is, $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let φ_0^1 be a function defined on $\mathbb{R}^{n_1}$ satisfying

$$\varphi_0^1 \in C_0^\infty(\mathbb{R}^{n_1}) \text{ and } \int_{\mathbb{R}^{n_1}} \varphi_0^1(x_1)dx_1 = 1, \tag{1}$$

and let φ_0^2 be a function defined on $\mathbb{R}^{n_2}$ satisfying

$$\varphi_0^2 \in C_0^\infty(\mathbb{R}^{n_2}) \text{ and } \int_{\mathbb{R}^{n_2}} \varphi_0^2(x_2)dx_2 = 1. \tag{2}$$

In what follows, we use $C_0^\infty(\mathbb{R}^n)$ to denote the set of all smooth functions with compact support on $\mathbb{R}^n$. We also use in this note $\mathcal{S}(\mathbb{R}^n)$ to denote the class of Schwartz functions in $\mathbb{R}^n$ and $\mathcal{S}'$ its dual. Given a Schwartz function f on $\mathbb{R}^n$, $\|f\|_{\mathcal{S}(\mathbb{R}^n)}$ denotes its seminorm. Motivated by the one-parameter Calderón reproducing formula in [35], the following multi-parameter local Calderón reproducing formula was proved in [8].

Theorem A ([8]). *Assume that functions φ_0^1 and φ_0^2 satisfy conditions* (1) *and* (2) *respectively, and let*

$$\varphi^1(x_1) = \varphi_0^1(x_1) - 2^{-n_1}\varphi_0^1(\frac{x_1}{2})$$

and

$$\varphi^2(x_2) = \varphi_0^2(x_2) - 2^{-x_2}\varphi_0^2(\frac{x_2}{2}).$$

Then for any given integers $M_i \ge 0$, $i = 1, 2$, there exist $\psi_0^1, \psi^1 \in C_0^\infty(\mathbb{R}^{n_1})$ and $\psi_0^2, \psi^2 \in C_0^\infty(\mathbb{R}^{n_2})$ with $M_{\psi^i} \ge M_i$, $i = 1, 2$, such that

$$f(x_1, x_2) = \sum_{j,k \ge 0} \psi_{j,k} * \varphi_{j,k} * f(x_1, x_2), \tag{3}$$

where the series converges in $L^2(\mathbb{R}^{n_1+n_2})$, $\mathcal{S}(\mathbb{R}^{n_1+n_2})$ and $\mathcal{S}'(\mathbb{R}^{n_1+n_2})$.

In the above local reproducing formula (3), $\psi_{j,k}$ is constructed as follows. For $j, k \ge 1$, let $\psi_j^1(x_1) = 2^{jn}\psi^1(2^jx_1)$, $\psi_k^2(x_2) = 2^{km}\psi^2(2^kx_2)$, and set

$$\psi_{j,k}(x_1, x_2) = \psi_j^1(x_1)\psi_k^2(x_2),\quad \psi_{j,0}(x_1, x_2) = \psi_j^1(x_1)\psi_0^2(x_2),$$
$$\psi_{0,k}(x_1, x_2) = \psi_0^1(x_1)\psi_k^2(x_2),\quad \psi_{0,0}(x_1, x_2) = \psi_0^1(x_1)\psi_0^2(x_2).$$

$\varphi_{j,k}$ can be constructed similarly. Moreover, as pointed out in [8, 35], for any positive integer N_i, φ^i in Theorem A can be chosen such that $M_{\varphi^i} \geq N_i$, $i = 1, 2$. It is noteworthy that the multi-parameter local Hardy spaces $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ were developed in [8, 9] with the local Calderón reproducing formula (3) recently.

The first main result of this paper is the following Littlewood-Paley characterization:

Theorem 1.1. *$f \in \Lambda_\alpha$ with $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 > 0$ if and only if $f \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$ and*

$$\sup_{j,k \geq 0} 2^{j\alpha_1} 2^{k\alpha_2} \|\varphi_{j,k} * f\|_{L^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq C < \infty.$$

Furthermore,

$$\|f\|_{\Lambda_\alpha} \approx \sup_{j,k \geq 0} 2^{j\alpha_1} 2^{k\alpha_2} \|\varphi_{j,k} * f\|_{L^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}.$$

It is known that the theory of one-parameter singular integral operators has been generalized in two ways. First, the convolution singular integral operators were replaced by non-convolution singular integral operators associated with a kernel in the following sense.

Definition 2. A locally integrable function defined from the diagonal $x = y$ in $\mathbb{R}^n \times \mathbb{R}^n$ is called a one-parameter Calderón-Zygmund kernel if there exist constants $C > 0$ and a regularity exponent $\varepsilon \in (0, 1]$ such that

$$|\mathcal{K}(x, y)| \leq C \frac{1}{|x - y|^n}, \tag{4}$$

and

$$|\mathcal{K}(x, y) - \mathcal{K}(x', y)| \leq C \frac{|x - x'|^\varepsilon}{|x - y|^{n+\varepsilon}}, \tag{5}$$

whenever $|x - x'| \leq |x - y|/2$, and

$$|\mathcal{K}(x, y) - \mathcal{K}(x, y')| \leq C \frac{|y - y'|^\varepsilon}{|x - y|^{n+\varepsilon}}, \tag{6}$$

whenever $|y - y'| \leq |x - y|/2$. The smallest such constant C is denoted by $|\mathcal{K}|_{CZ}$.

We call an operator T one-parameter Calderón-Zygmund operator if T is a singular integral operator associated with a one-parameter Calderón-Zygmund kernel $\mathcal{K}(x, y)$ given by

$$\langle Tf, g \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(x) \mathcal{K}(x, y) f(y) dx dy \tag{7}$$

for all $f, g \in C_0^\infty(\mathbb{R}^n)$ with $\text{supp} f \cap \text{supp} g = \emptyset$ and T is bounded on $L^2(\mathbb{R}^n)$. Define $\|T\|_{CZ}$ by

$$\|T\|_{CZ} = \|T\|_{L^2 \mapsto L^2} + |\mathcal{K}|_{CZ}.$$

Then T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and is bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $0 < p \leq 1$ and close to 1, if $\|T\|_{CZ} < \infty$. Nevertheless, to ensure the boundedness of T on local Hardy spaces h^p, $0 < p < 1$, introduced by Goldberg [14], a mild additional size condition

$$|\mathcal{K}(x,y)| \leq C \frac{1}{|x-y|^{n+\delta}}$$

for some $\delta > n(\frac{1}{p} - 1)$ might be included. Hence, this motivates us to define the non-convolution type inhomogeneous Calderón-Zygmund operators on $\mathbb{R}^n$.

Definition 3. A locally integrable function defined away from the diagonal $x = y$ in $\mathbb{R}^n \times \mathbb{R}^n$ is called a one-parameter inhomogeneous Calderón-Zygmund kernel with regularity exponent $\varepsilon \in (0, 1]$ if there exist constants $C > 0$ and $\delta > 0$ such that in addition to (4), (5), (6), $\mathcal{K}$ also satisfies the following

$$(8) \qquad |\mathcal{K}(x,y)| \leq \frac{C}{|x-y|^{n+\delta}}, \text{ if } |x-y| \geq 1.$$

The smallest such constant C in (4), (5), (6) and (8) is still denoted by $|\mathcal{K}|_{CZ}$.

We say that an operator T is a one-parameter inhomogeneous Calderón-Zygmund operator if T is a singular integral operator associated with a one-parameter inhomogeneous Calderón-Zygmund kernel $\mathcal{K}(x,y)$ given by (7) for all $f, g \in C_0^\infty(\mathbb{R}^n)$ with disjoint supports and T is bounded on $L^2(\mathbb{R}^n)$. In addition, the norm of T is still defined by $\|T\|_{CZ} = \|T\|_{L^2 \mapsto L^2} + |\mathcal{K}|_{CZ}$. It is well-known that an inhomogeneous Calderón-Zygmund operator is bounded on $h^p(\mathbb{R}^n)$ if $T^*(1) = 0$ for p near 1. Here $T^*(1) = 0$ means that $\langle 1, T(\psi) \rangle = 0$ for all $\psi \in C_{0,0}^\infty(\mathbb{R}^n)$ defined below.

Secondly, the classical Calderón-Zygmund operators were also extended in another direction to product singular integral operators on the space $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ along with two-parameter family of dilations $\delta : x \to (\delta_1 x_1, \delta_2 x_2)$, $x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where $\delta = (\delta_1, \delta_2)$, $\delta_1, \delta_2 > 0$. After Journé introduced the non-convolution product singular integral operator in [28], there are many significant works involving the boundedness of product singular integral operators on various function spaces. We now introduce the singular integral operator in the inhomogeneous Journé class on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ suitable for our inhomogeneous product Lipschitz spaces.

Definition 4. A singular integral operator T is said to be in inhomogeneous Journé class on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with regularity exponents $\varepsilon \in (0, 1]$ and $\delta > 0$ if

$$T(f)(x_1, x_2) = \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} \mathcal{K}(x_1, x_2, y_1, y_2) f(y_1, y_2) dy_1 dy_2,$$

where (x_1, x_2) is outside the support of the function f, where the kernel $\mathcal{K}$ satisfies the following conditions:

For fixed $x_1, y_1 \in \mathbb{R}^{n_1}$, set $\tilde{\mathcal{K}}^1(x_1, y_1)$ to be the singular integral operator acting on functions on $\mathbb{R}^{n_2}$ with the kernel

$$\tilde{\mathcal{K}}^1(x_1, y_1)(x_2, y_2) = \mathcal{K}(x_1, x_2, y_1, y_2),$$

and similarly,

$$\tilde{\mathcal{K}}^2(x_2, y_2)(x_1, y_1) = \mathcal{K}(x_1, x_2, y_1, y_2),$$

then there exists a constant $C > 0$ such that

(i) T is bounded on $L^2(\mathbb{R}^{n_1+n_2})$.

(ii)

$$\|\tilde{\mathcal{K}}^1(x_1, y_1)\|_{CZ} \leq C \min\{|x_1 - y_1|^{-n_1}, |x_1 - y_1|^{-(n_1+\delta)}\}; \tag{9}$$

$$\begin{aligned} \|\tilde{\mathcal{K}}^1(x_1, y_1) - \tilde{\mathcal{K}}^1(x_1, y_1')\|_{CZ} &\leq C|y_1 - y_1'|^{\varepsilon}|x_1 - y_1|^{-(n_1+\varepsilon)} \\ &\qquad \text{for } |y_1 - y_1'| \leq |x_1 - y_1|/2; \\ \|\tilde{\mathcal{K}}^1(x_1, y_1) - \tilde{\mathcal{K}}^1(x_1', y_1)\|_{CZ} &\leq C|x_1 - x_1'|^{\varepsilon}|x_1 - y_1|^{-(n_1+\varepsilon)} \\ &\qquad \text{for } |x_1 - x_1'| \leq |x_1 - y_1|/2. \end{aligned} \tag{10}$$

(iii)

$$\|\tilde{\mathcal{K}}^2(x_2, y_2)\|_{CZ} \leq C \min\{|x_2 - y_2|^{-n_2}, |x_2 - y_2|^{-(n_2+\delta)}\}; \tag{11}$$

$$\begin{aligned} \|\tilde{\mathcal{K}}^2(x_2, y_2) - \tilde{\mathcal{K}}^2(x_2, y_2')\|_{CZ} &\leq C|y_2 - y_2'|^{\varepsilon}|x_2 - y_2|^{-(n_2+\varepsilon)} \\ &\qquad \text{for } |y_2 - y_2'| \leq |x_2 - y_2|/2; \\ \|\tilde{\mathcal{K}}^2(x_2, y_2) - \tilde{\mathcal{K}}^2(x_2', y_2)\|_{CZ} &\leq C|x_2 - x_2'|^{\varepsilon}|x_2 - y_2|^{-(n_2+\varepsilon)} \\ &\qquad \text{for } |x_2 - x_2'| \leq |x_2 - y_2|/2. \end{aligned} \tag{12}$$

Following Journé in [28], we define the operator T_1 by the following

$$\langle g_2, \langle g_1, T_1 f_1 \rangle f_2 \rangle = \langle g_1 \otimes g_2, T f_1 \otimes f_2 \rangle$$

for $f_1, g_1 \in C_0^\infty(\mathbb{R}^{n_1})$ and $f_2, g_2 \in C_0^\infty(\mathbb{R}^{n_2})$. Observe that when $g_1 \in C_{0,0}^\infty(\mathbb{R}^{n_1}) = \{g \in C_0^\infty(\mathbb{R}^{n_1}) : \int g = 0\}$, $f_1 \in C_b^\infty(\mathbb{R}^{n_1})$, the bounded $C^\infty(\mathbb{R}^{n_1})$ functions, the inner product $\langle g_1, T_1 f_1 \rangle$ is well-defined. Moreover, $\langle g_1, T_1 f_1 \rangle$ is a Calderón-Zygmund singular integral operator on $\mathbb{R}^{n_2}$ with kernel

$$\langle g_1, T_1 f_1 \rangle (x_2, y_2) = \left\langle g_1, \tilde{\mathcal{K}}^2 (x_2, y_2) f_1 \right\rangle.$$

One defines $\langle g_2, T_2 f_2 \rangle$ similarly for $g_2 \in C_{0,0}^\infty(\mathbb{R}^{n_2})$ and $f_2 \in C_b^\infty(\mathbb{R}^{n_1})$. Using these definitions, we can give a meaning of the notation $T_1(1) = 0$. More precisely, $T_1(1) = 0$ is equivalent to

$$\langle g_1, \langle g_2, T_2 f_2 \rangle 1 \rangle = \langle g_1 \otimes g_2, T1 \otimes f_2 \rangle = 0\rangle$$

for all $g_1 \in C_{0,0}^\infty(\mathbb{R}^{n_1})$ and all $f_2, g_2 \in C_0^\infty(\mathbb{R}^{n_2})$, that is, for $g_1 \in C_{0,0}^\infty(\mathbb{R}^{n_1})$, $g_2 \in C_{0,0}^\infty(\mathbb{R}^{n_2})$, and almost everywhere $y_2 \in \mathbb{R}^{n_2}$,

$$\int g(x_1) g(x_2) \mathcal{K}(x_1, x_2, y_1, y_2) dx_1 dx_2 dy_1 = 0.$$

Which $T_1^*(1) = 0$ means $\langle g_2, T_2 f_2\rangle^* 1 = 0$ in the same conditions. Exchanging the role of indices one can obtain the meaning of $T_2(1) = 0$ and $T_2^*(1) = 0$.

It should be pointed that the singular integral operators in the inhomogeneous Journé's class are consistent with the one-parameter inhomogeneous Calderón-Zygmund operators considered by Goldberg [14] while a mild additional size condition was needed to guarantee their boundedness on the local Hardy spaces $h^p(\mathbb{R}^n)$. Moreover, the boundedness of inhomogeneous Journé's type singular integral operators on the multi-parameter local Hardy spaces from $h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and from $h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ are established in [7, 8]. However, so far, it is not clear that whether the dual of multi-parameter local Hardy spaces $h^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ is the inhomogeneous product Lipschitz spaces Λ_α with $\alpha_1 = n_1(1/p-1)$, $\alpha_2 = n_2(1/p-1)$. Therefore, the boundedness on inhomogeneous product Lipschitz spaces can not be obtained by using the duality argument directly as in the classical setting. This motivates us to study the boundedness of inhomogeneous Journé's type singular integral operators on the inhomogeneous product Lipschitz spaces.

Our last main results are the following boundedness on inhomogeneous product Lipschitz spaces of singular integral operators in inhomogeneous Journé's class.

Theorem 1.2. *Let T be a singular integral operator in inhomogeneous Journé's class with regularity exponent $\varepsilon \in (0,1]$ and $\delta > 0$. If $T_1(1) = T_2(1) = 0$ and the kernel $\mathcal{K}$ of T satisfies the half smoothness conditions* (9)*,* (10)*,* (11)*,* (12)*, then T is bounded on Λ_α with $\alpha = (\alpha_1, \alpha_2)$ for $\max\{\alpha_1, \alpha_2\} < \varepsilon$.*

Note that the hypothesis of Theorem 1.2 may not necessity (see Remark 3.9 in Section 3). However, the inhomogeneous Journé type singular integral operators are contained in the Journé's product singular integral operators. As a consequence of Theorem 1.2 and a necessary and sufficient condition for the boundedness of Journé's product singular integral operators on product Lipschitz spaces obtained in [38], we get that $T_1(1) = T_2(1) = 0$ if and only if T are bounded both on inhomogeneous product Lipschitz spaces Λ_α and homogeneous product Lipschitz spaces $\tilde{\Lambda}_\alpha$ for $\max\{\alpha_1, \alpha_2\} < \varepsilon$ simultaneously, where T is a singular integral operator in inhomogeneous Journé's class with regularity exponent $\varepsilon \in (0,1]$ and $\delta > 0$.

On the other hand, the bi-parameter singular integral operators in the inhomogeneous Journé's class are suitable for the study of bi-parameter pseudo-differential operators. See [5, 27] for one-parameter pseudo-differential operator and multi-parameter pseudo-differential operator. It has been shown by Chen, Ding and Lu in [5] that the inhomogeneous Journé's class of bi-parameter singular integral operators considered this paper concludes a special kind of bi-parameter pseudo-differential operators. Therefore, our main theorem is strictly more applicable for applications to partial differential equations with variable coefficients. Next, although some ideas of our methods are taken from [16, 38], observe that the functions φ_0^i and ψ_0^i, $i = 1, 2$, in the local reproducing formula (3), do not have any vanishing moments unlike those functions in the multi-parameter Calderón reproducing formula. The lack of the vanishing moments makes the boundedness more complicated than the homogeneous case in [38]. Hence, we must obtain some nontrivial estimates in our proof. In addition, the range of the index α_1 and α_2 in this paper are wider than [38]. Moreover, the smoothness conditions for the variable $y = (y_1, y_2)$ of the kernel $\mathcal{K}$ in Theorem 1.2 are not needed, which is weaker than the conditions in [38]. Finally, it is not clear that the dual of $h^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is the inhomogeneous product Lipschitz spaces Λ_α with $\alpha_1 = n_1(1/p-1)$, $\alpha_2 = n_2(1/p-1)$. To avoid using this, we find a special inhomogeneous Besov spaces as the substitute of h^p spaces.

This paper is organized as follows. In Section 2, we will give the proof of Theorem 1.1. Theorem 1.2 will be proved in Section 3.

Throughout this paper, the letter C stands for a positive constant which is independent of the essential variables, but whose value may vary from line to line. We use the notation $A \approx B$ to denote that there exists a positive constant C such that $C^{-1}B \le A \le CB$. Let $j \wedge j'$ be the minimum of j and j'.

2. Proof of Theorem 1.1

Without the loss of generality, we assume that $\varphi_0^i, \varphi^i, \psi_0^i, \psi^i$, $i = 1, 2$, are radial functions. We first prove that if $f \in \Lambda_\alpha$ with $0 < \alpha_1, \alpha_1 < 1$, then $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$. To do this, for each $g \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$, by the local Calderón reproducing formula (3), we have

$$g(x_1, x_2) = \sum_{j,k\ge 0} \psi_{j,k} * \varphi_{j,k} * g(x_1, x_2),$$

where the series converges in $\mathcal{S}(\mathbb{R}^{n_1+n_2})$. Therefore, for $f \in \Lambda_\alpha$ with $0 < \alpha_1, \alpha_1 < 1$, it suffices to show that $\sum_{j,k\ge 0} \langle f,\ \psi_{j,k} * \varphi_{j,k} * g\rangle$ is well-defined for $g \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$. To this end, for all $j, k \ge 0$, we estimate $\langle \varphi_{j,k} * f, \psi_{j,k} * g\rangle$ as follows.

Case 1: $j = k = 0$.

$$\begin{aligned}|\varphi_{0,0} * f(x_1, x_2)| &= \Big| \iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \varphi_0^1(u)\varphi_0^2(v) f(x_1 - u, x_2 - v)dudv\Big| \\ &\le C\|f\|_{L^\infty} \le C\|f\|_{\Lambda_\alpha}.\end{aligned}$$

This implies that

$$|\langle \varphi_{0,0} * f, \psi_{0,0} * g\rangle| \le C\|f\|_{\Lambda_\alpha}\|g\|_{\mathcal{S}}.$$

Case 2: $j \ge 1$, $k = 0$.

By the cancellation condition on φ_j^1, we have

$$\varphi_{j,0} * f(x_1, x_2) = \iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \varphi_j^1(u)\varphi_0^2(v) f(x_1 - u, x_2 - v)dudv$$

$$= \iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \varphi_j^1(u)\varphi_0^2(v)\Delta_u f(x_1, x_2 - v)dudv.$$

The fact $f \in \Lambda_\alpha$ and the size condition of φ_j^1 give us that

$$\begin{aligned}\big|\varphi_{j,0} * f(x_1, x_2)\big| &\le C\|f\|_{\Lambda_\alpha} \iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} |u|^{\alpha_1} \frac{2^{-j}}{(2^{-j}+|u|)^{n+1}} |\varphi_0^2(v)|dudv \\ &\le C2^{-j\alpha_1}\|f\|_{\Lambda_\alpha}.\end{aligned}$$

Therefore, we obtain that

$$|\langle \varphi_{j,0} * f, \psi_{j,0} * g\rangle| \le C2^{-j\alpha_1}\|f\|_{\Lambda_\alpha}\|\psi_{j,0} * g\|_{L^1(\mathbb{R}^{n_1+n_2})} \le C2^{-j\alpha_1}\|f\|_{\Lambda_\alpha}\|g\|_{\mathcal{S}}.$$

Case 3: $j = 0$, $k \ge 1$.

Repeating the similar argument as the Case 3, we get

$$|\langle \varphi_{0,k} * f, \psi_{0,k} * g\rangle| \le C2^{-k\alpha_2}\|f\|_{\Lambda_\alpha}\|g\|_{\mathcal{S}}.$$

Case 4: $j \ge 1$, $k \ge 1$.

Applying the cancellation conditions on both φ_j^1 and φ_k^2, we have

$$\begin{aligned}|\varphi_{j,k} * f(x_1, x_2)| &= \Big|\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \varphi_j^1(u)\varphi_k^2(v) f(x_1 - u, x_2 - v)dudv\Big| \\ &= \Big|\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \varphi_j^1(u)\varphi_k^2(v)(\Delta_v\Delta_u f)(x_1, x_2)dudv\Big| \\ &\le C\|f\|_{\mathrm{Lip}(\alpha_1,\alpha_2)} \iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \Big|\varphi_j^1(u)\varphi_k^2(v)\Big||u|^{\alpha_1}|v|^{\alpha_2}dudv \\ &\le C2^{-j\alpha_1}2^{-k\alpha_2}\|f\|_{\Lambda_\alpha},\end{aligned}$$

which yields

$$|\langle \varphi_{j,k} * f, \psi_{j,k} * g\rangle| \le C2^{-j\alpha_1}2^{-k\alpha_2}\|f\|_{\Lambda_\alpha}\|g\|_{\mathcal{S}}.$$

Combing these four cases, we obtain that

$$|\langle \varphi_{j,k} * f, \psi_{j,k} * g\rangle| \le C2^{-j\alpha_1}2^{-k\alpha_2}\|f\|_{\Lambda_\alpha}\|g\|_{\mathcal{S}}$$

and thus, $\langle f, g\rangle$ is well defined. In addition, we also obtain

$$\sup_{j,k\ge 0} 2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k} * f\|_{L^\infty} \le C\|f\|_{\Lambda_\alpha}$$

for $0 < \alpha_1, \alpha_2 < 1$.

When $\alpha_1 = 1$, $0 < \alpha_2 < 1$, we only need to consider the cases where $j \ge 1$, $k = 0$ and $j, k \ge 1$ since the other two cases are similar. Indeed, if $j \ge 1$, $k = 0$, applying the cancellation condition on φ_j^1 and noting that φ_j^1 is a radial function, we have

$$\begin{aligned}&|\varphi_{j,0} * f(x_1, x_2)| \\ &= \Big|\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \varphi_j^1(u)\varphi_0^2(v)[f(x_1 - u, x_2 - v) - f(x_1, x_2 - v)]dudv\Big|\end{aligned}$$

$$= \frac{1}{2}\Big|\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \varphi_j^1(u)\varphi_0^2(v)[f(x_1+u,x_2)-2f(x_1,x_2)+f(x_1-u,x_2-v)]dudv\Big|$$
$$\leq C\|f\|_{\Lambda_\alpha}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} |u|\frac{2^{-jL}}{(2^{-j}+|u|)^{n+L}}|\varphi_0^2(v)|dudv$$
$$\leq C2^{-j}\|f\|_{\Lambda_\alpha}.$$

Hence,

$$|\langle\varphi_{j,0}*f,\psi_{j,0}*g\rangle|\leq C2^{-j}\|f\|_{\Lambda_\alpha}\|g\|_{\mathcal{S}}.$$

If $j,k\geq 1$, then

$$\begin{aligned}&\varphi_{j,k}*f(x_1,x_2)\\ &=\frac{1}{2}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\varphi_j^1(u)\varphi_k^2(v)[f(x_1-u,x_2-v)+f(x_1+u,x_2-v)]dudv\\ &=\frac{1}{2}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\varphi_j^1(u)\varphi_k^2(v)(\Delta_v\Delta_u^2f)(x_1,x_2)dudv.\end{aligned}$$

The last equality follows from the cancellation conditions on both φ_j^1 and φ_k^2. Then

$$\begin{aligned}|\varphi_{j,k}*f(x_1,x_2)|&\leq C\|f\|_{\mathrm{Lip}(\alpha_1,\alpha_2)}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}|\varphi_j^1(u)\varphi_k^2(v)||u||v|^{\alpha_2}dudv\\ &\leq C2^{-j}2^{-k\alpha_2}\|f\|_{\Lambda_\alpha},\end{aligned}$$

which implies

$$|\langle\varphi_{j,k}*f,\psi_{j,k}*g\rangle|\leq C2^{-j}2^{-k\alpha_2}\|f\|_{\Lambda_\alpha}\|g\|_{\mathcal{S}}.$$

Thus, $\langle f,g\rangle$ is well defined and

$$\sup_{j,k\geq 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}\leq C\|f\|_{\Lambda_\alpha}$$

for $\alpha_1=1$, $0<\alpha_2<1$.

All other cases where $0<\alpha_1<1$, $\alpha_2=1$ or $\alpha_1=\alpha_2=1$ can be handled similarly. For the case where $\alpha_1,\alpha_2>1$ with $\alpha_1=m_1+r_1$ and $\alpha_2=m_2+r_2$ with $0<r_1,r_2\leq 1$, set $\widehat{\tilde\varphi_j^1}(\xi)=\frac{\widehat{\varphi_j^1}(\xi)}{(-2\pi i\xi)^{\beta_1}}$ and $\widehat{\tilde\varphi_k^2}(\eta)=\frac{\widehat{\varphi_k^2(\xi)}}{(-2\pi i\eta)^{\beta_2}}$, where $|\beta_1|=m_1$, $|\beta_2|=m_2$, then

$$\varphi_{j,k}*f=\partial^{\beta_1}\partial^{\beta_2}\tilde\varphi_{j,k}*f=(-1)^{m_1+m_2}\tilde\varphi_{j,k}*\partial^{\beta_1}\partial^{\beta_2}f,$$

where $\tilde\varphi_{j,k}=\tilde\varphi_j^1\tilde\varphi_k^2$. Note that $2^{jm_1}2^{km_2}\tilde\varphi_{j,k}$ satisfy the similar smoothness, size and cancellation conditions as $\varphi_{j,k}$. Therefore, repeating the similar proof gives that

$$\begin{aligned}\|\varphi_{j,k}*f\|_{L^\infty}&=\|2^{-jm_1}2^{-km_2}(2^{jm_1}2^{km_2}\tilde\varphi_{j,k})*\partial^{\beta_1}\partial^{\beta_2}f\|_{L^\infty}\\ &\leq C2^{-jm_1}2^{-km_2}2^{-jr_1}2^{-kr_2}\|\partial^{\beta_1}\partial^{\beta_2}f\|_{\mathrm{Lip}(r_1,r_2)}\\ &=C2^{-j\alpha_1}2^{-k\alpha_2}\|f\|_{\Lambda_\alpha}.\end{aligned}$$

That is

$$\sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty} \leq C\|f\|_{\Lambda_\alpha}$$

for $\alpha_1, \alpha_2 > 1$.

To prove the converse implication of Theorem 1.1, we first show that for every $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$ satisfying

$$\sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty} \leq C$$

coincides with a bounded continuous function. Recalling the local Calderón reproducing formula (3), $f(x_1,x_2) = \sum_{j,k\geq 0}\psi_{j,k}*\varphi_{j,k}*f(x_1,x_2)$ in $\mathcal{S}'$, we have

$$\begin{aligned}|\psi_{j,k}*\varphi_{j,k}*f(x_1,x_2)| &\leq \|\varphi_{j,k}*f\|_{L^\infty}\|\psi_{j,k}\|_{L^1}\\ &\leq C2^{-j\alpha_1}2^{-k\alpha_2}(\sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}).\end{aligned}$$

Thus, the series $\sum_{j,k\geq 0}\psi_{j,k}*\varphi_{j,k}*f(x_1,x_2)$ converges uniformly in x_1, x_2. Since $\psi_{j,k}*\varphi_{j,k}*f$ is continuous in $\mathbb{R}^{n_1+n_2}$, then the sum function f is also continuous in $\mathbb{R}^{n_1+n_2}$. Moreover,

$$\|f\|_{L^\infty} \leq C\sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}.$$

Now we estimate $\|f\|_{\Lambda_\alpha}$ as follows. We first consider the case where $0 < \alpha_1, \alpha_2 < 1$, and then show that

$$|f(x_1-u,x_2) - f(x_1,x_2)| \leq C|u|^{\alpha_1}\sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}.$$

To do this, write

$$\begin{aligned}&|f(x_1-u,x_2) - f(x_1,x_2)|\\ &= \Big|\sum_{j,k\geq 0}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}[\psi_{j,k}(x_1-u-w,x_2-v) - \psi_{j,k}(x_1-w,x_2-v)]\varphi_{j,k}*f(w,v)dwdv\Big|\\ &\leq \sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}\sum_{j,k\geq 0} 2^{-j\alpha_1}2^{-k\alpha_2}\\ &\quad\times\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}|[\psi_j^1(x_1-u-w) - \psi_j^1(x_1-w)]||\psi_k^2(x_2-v)|dwdv.\end{aligned}$$

Therefore, we only need to consider the case where $|u| < 1$. Let n_1 be the unique nonnegative integer such that $2^{-m_1-1} \leq |u| < 2^{-m_1}$ and set

$$A := \sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}.$$

Then we have

$$|f(x_1-u,x_2) - f(x_1,x_2)|$$

$$\leq A\Big(\sum_{j=0}^{m_1}\sum_{k=0}^{\infty}2^{-j\alpha_1}2^{-k\alpha_2}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}|[\psi_j^1(x_1-u-w)-\psi_j^1(x_1-w)]||\psi_k^2(x_2-v)|dwdv$$
$$+\sum_{j=m_1}^{\infty}\sum_{k=0}^{\infty}2^{-j\alpha_1}2^{-k\alpha_2}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}|[\psi_j^1(x_1-u-w)-\psi_j^1(x_1-w)]||\psi_k^2(x_2-v)|dwdv\Big)$$
$$:= I+II.$$

For I, applying the following smoothness estimates of $\psi_j^{(1)}$ (see Lemma 4 in [6]), i.e., for $x_1,u\in\mathbb{R}^{n_1}$,

$$\begin{aligned}&\left|\psi_j^{(1)}(x_1-u)-\psi_j^{(1)}(x_1)\right|\\ &\leq C\min\left(1,\frac{|u|}{2^{-j}}\right)\left[\frac{2^{-j}}{(2^{-j}+|x_1-u|)^{n+1}}+\frac{2^{-j}}{(2^{-j}+|x_1|)^{n+1}}\right],\end{aligned}\tag{13}$$

we have

$$I\leq CA\sum_{j=0}^{m_1}\sum_{k=0}^{\infty}2^{-j\alpha_1}2^{-k\alpha_2}\frac{|u|}{2^{-j}}\leq CA2^{m_1(1-\alpha_1)}|u|\leq CA|u|^{\alpha_1}.$$

For II, the size conditions on $\psi_j^{(1)}$ and $\psi_k^{(2)}$ imply

$$II\leq CA\sum_{j=m_1}^{\infty}\sum_{k=0}^{\infty}2^{-j\alpha_1}2^{-k\alpha_2}\leq CA2^{-m_1\alpha_1}\leq CA|u|^{\alpha_1}.$$

Thus, we obtain that for any $u\neq 0$, $(x_1,x_2)\in\mathbb{R}^{n_1+n_2}$,

$$\frac{\Delta_u f(x_1,x_2)}{|u|^{\alpha_1}}\leq C\sup_{j,k\geq 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}.$$

Similarly, for any $v\neq 0$, $(x_1,x_2)\in\mathbb{R}^{n_1+n_2}$, there holds

$$\frac{\Delta_v f(x_1,x_2)}{|v|^{\alpha_2}}\leq C\sup_{j,k\geq 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}.$$

Finally, we prove that

$$\begin{aligned}&|\Delta_v\Delta_u f(x_1,x_2)|\\ &=|f(x_1-u,x_2-v)-f(x_1,x_2-v)-f(x_1-u,x_2)+f(x_1,x_2)|\\ &\leq C|u|^{\alpha_1}|v|^{\alpha_2}\sup_{j,k\geq 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^\infty}.\end{aligned}$$

We only consider the case where $|u|<1$ and $|v|<1$ since the other cases are similar and easier. Let m_1, m_2 be the unique nonnegative integer such that $2^{-m_1-1}\leq |u|<2^{-m_1}$ and $2^{-m_2-1}\leq |u|<2^{-m_2}$. In fact,

$$\begin{aligned}&|\Delta_v\Delta_u f(x_1,x_2)|\\ &=\Big|\sum_{j,k\geq 0}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\big[\psi_{j,k}(x_1-u-u',x_2-v-v')-\psi_{j,k}(x_1-u',x_2-v-v')\end{aligned}$$

$$- \psi_{j,k}(x_1 - u - u', x_2 - v') + \psi_{j,k}(x_1 - u', x_2 - v')\big]\varphi_{j,k} * f(u', v')du'dv'\Big|$$

$$\leq A\bigg(\sum_{j=m_1}^{\infty}\sum_{k=m_2}^{\infty} + \sum_{j=0}^{m_1}\sum_{k=m_2}^{\infty} + \sum_{j=m_1}^{\infty}\sum_{k=0}^{m_2} + \sum_{j=0}^{m_1}\sum_{k=0}^{m_2}\bigg)\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} 2^{-j\alpha_1}2^{-k\alpha_2}$$
$$\times \big|\psi_j^1(x_1 - u - u') - \psi_j^1(x_1 - u')\big|\big|\psi_k^2(x_2 - v - v') - \psi_k^2(x_2 - v')\big|du'dv'$$
$$:= A(B_1 + B_2 + B_3 + B_4).$$

To deal with the first term, applying the size conditions on both ψ_j^1 and ψ_k^2 yields that

$$B_1 \leq \sum_{j=m_1}^{\infty}\sum_{k=m_2}^{\infty} 2^{-j\alpha_1}2^{-k\alpha_2} \leq C2^{-m_1\alpha_1}2^{-m_2\alpha_2} \leq C|u|^{\alpha_1}|v|^{\alpha_2}.$$

For the second part, applying the smooth condition on ψ_j^1, i.e., (13), and the size condition on ψ_k^2 implies that

$$B_2 \leq C\sum_{j=0}^{m_1}\sum_{k=m_2}^{\infty} 2^{-j\alpha_1}2^{-k\alpha_2}\frac{|u|}{2^{-j}} \leq C2^{m_1(1-\alpha_1)}2^{-m_2\alpha_2}|u| \leq C|u|^{\alpha_1}|v|^{\alpha_2}.$$

The estimate for third term B_3 is similar to the estimate for B_2. Finally, to handle with the last term, applying the smoothness conditions on both ψ_j^1 and ψ_k^2, we get

$$B_4 \leq C\sum_{j=0}^{m_1}\sum_{k=0}^{m_2} 2^{-j\alpha_1}2^{-k\alpha_2}\frac{|u|}{2^{-j}}\frac{|v|}{2^{-k}} \leq C2^{n_1(1-\alpha_1)}2^{n_2(1-\alpha_2)}|u||v| \leq C|u|^{\alpha_1}|v|^{\alpha_2}.$$

Combing these estimates yields that

$$|\Delta_v\Delta_u f(x_1, x_2)| \leq CA|u|^{\alpha_1}|v|^{\alpha_2}.$$

Repeating a similar calculation, we can handle the other cases where $\alpha_1 = 1$, $0 < \alpha_2 < 1$, $0 < \alpha_1 < 1$, $\alpha_2 = 1$ and $\alpha_1 = \alpha_2 = 1$. Lastly, when $1 < \alpha_1 = m_1 + r_1$, $1 < \alpha_2 = m_2 + r_2$ with $0 < r_1, r_2 \leq 1$, observe that

$$\begin{aligned}
&\Delta_v\Delta_u\partial^{\beta_1}\partial^{\beta_2}f(x_1, x_2)\\
&= \sum_{j,k\geq 0}\iint_{\mathbb{R}^n\times\mathbb{R}^m}[\partial^{\beta_1}\psi_j^{(1)}(x_1 - u') - \partial^{\beta_1}\psi_j^{(1)}(x_1 - u' - u)]\\
&\quad\times[\partial^{\beta_2}\psi_k^{(2)}(x_2 - v') - \partial^{\beta_2}\psi_k^{(2)}(x_2 - v' - v)]\varphi_{j,k} * f(u', v')dudv
\end{aligned}$$

for $|\beta_1| = m_1$ and $|\beta_2| = m_2$. Again observe that the properties of $\partial^{\beta_1}\psi_j^{(1)}$ and $\partial^{\beta_2}\psi_k^{(2)}$ are similar to $2^{jm_1}\psi_j^{(1)}$ and $2^{km_2}\psi_k^{(2)}$, respectively, and hence the estimate for this case is the same as the proof for the case where $0 < \alpha_1, \alpha_2 \leq 1$. Therefore, the proof of Theorem 1.1 is completed.

3. Proof of Theorem 1.2

Before we present the details of the proof of Theorem 1.2, we introduce the definition of inhomogeneous product Besov spaces and some lemmas.

Definition 5. Let $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 \in \mathbb{R}$. Suppose that φ_0^1, φ_0^2 satisfy conditions (1) and (2), respectively, and set $\varphi^1(x_1) = \varphi_0^1(x_1) - 2^{-n_1}\varphi_0^1(\frac{x_1}{2})$ and $\varphi^2(x_2) = \varphi_0^2(x_2) - 2^{-n_2}\varphi_0^2(\frac{x_2}{2})$. Then the inhomogeneous product Besov space $B_1^{\alpha,1}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined by the collection of all $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$ such that

$$\|f\|_{B_1^{\alpha,1}} = \sum_{j,k\geq 0} 2^{j\alpha_1} 2^{k\alpha_2} \|\varphi_{j,k} * f\|_{L^1(\mathbb{R}^{n_1+n_2})} < \infty,$$

where the construction of $\varphi_{j,k}$ is the same as before.

To see that this space is well defined, we need to show that the above definition is independent of the choice of the functions φ_0^1 and φ_0^2. This will directly follow from the following lemma.

Lemma 3.1. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$. Suppose that ψ_0^1, ψ_0^2 satisfy conditions* (1) *and* (2)*, respectively. Set $\psi^1(x_1) = \psi_0^1(x_1) - 2^{-n_1}\psi_0^1(\frac{x_1}{2})$ and $\psi^2(x_2) = \psi_0^2(x_2) - 2^{-n_2}\psi_0^2(\frac{x_2}{2})$. Then*

$$\sum_{j,k\geq 0} 2^{j\alpha_1} 2^{k\alpha_2} \|\varphi_{j,k} * f\|_{L^1(\mathbb{R}^{n_1+n_2})} \approx \sum_{j,k\geq 0} 2^{j\alpha_1} 2^{k\alpha_2} \|\psi_{j,k} * f\|_{L^1(\mathbb{R}^{n_1+n_2})}$$

for every $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$.

Proof. For $\psi_0^1 \in C_0^\infty(\mathbb{R}^{n_1})$, $\psi_0^2 \in C_0^\infty(\mathbb{R}^{n_2})$ with $\int \psi_0^i = 1$, $i = 1, 2$, by Theorem A, we can take ϕ_0^i, ϕ^i with large M_{ϕ^i}, $i = 1, 2$, such that

$$f(x_1, x_2) = \sum_{j,k\geq 0} \phi_{j,k} * \psi_{j,k} * f(x_1, x_2).$$

Hence

$$\begin{aligned} &|\varphi_{j,k} * f(x_1, x_2)| \\ &\leq \sum_{j'=0}^{\infty} \sum_{k'=0}^{\infty} \iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} |\varphi_{j,k} * \phi_{j',k'}(x_1 - u, x_2 - v)||\psi_{j',k'} * f(u, v)| du dv. \end{aligned}$$

By the well-known almost orthogonality estimate,

$$\begin{aligned} &|\phi_{j,k} * \varphi_{j'k'}(x_1, x_2)| \\ &\leq C 2^{-|j-j'|L_1} 2^{-|k-k'|L_2} \frac{2^{-(j\wedge j')M_1}}{(2^{-(j\wedge j')} + |x_1|)^{n_1+M_1}} \frac{2^{-(k\wedge k')M_2}}{(2^{-(k\wedge k')} + |x_2|)^{n_2+M_2}} \end{aligned}$$

for any large positive integers L_i and M_i, $i = 1, 2$. It is easy to verify that

$$\sum_{j,k\geq 0} 2^{j\alpha_1} 2^{k\alpha_2} \|\varphi_{j,k} * f\|_{L^1(\mathbb{R}^{n_1+n_2})} \leq C \sum_{j,k\geq 0} 2^{j\alpha_1} 2^{k\alpha_2} \|\psi_{j,k} * f\|_{L^1(\mathbb{R}^{n_1+n_2})} < \infty.$$

The converse inequality follows by symmetry. □

Lemma 3.2. *Under the same assumptions of Definition 5, $\mathcal{S}(\mathbb{R}^{n_1+n_2})$ is dense in $B_1^{\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ with respect to the norm of $B_1^{\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. Consequently, $L^2\cap B_1^{\alpha,1}$ is dense in $B_1^{\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$.*

Proof. Firstly, if $f\in\mathcal{S}(\mathbb{R}^{n_1+n_2})$, by the well-known almost orthogonality estimate,

$$|\varphi_{j,k}*f(x_1,x_2)|\le C2^{-jL_1}2^{-kL_2}\frac{1}{(1+|x_1|)^{n_1+M_1}}\frac{1}{(1+|x_2|)^{n_2+M_2}}$$

for any large positive integers L_i and M_i, $i=1,2$, then

$$\begin{aligned}\|f\|_{B_1^{\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}&\le C\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\frac{1}{(1+|x_1|)^{n_1+M_1}}\frac{1}{(1+|x_2|)^{n_2+M_2}}dx_1dx_2\\&\le C,\end{aligned}$$

if we choose $L_i>\alpha_i$, $i=1,2$. This means $f\in B_1^{\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$.

Let $f\in B_1^{\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. For any fixed $N>0$, set

$$E=\{(j,k):0\le j\le N,\ 0\le k\le N\}$$

and

$$f_N(x_1,x_2)=\sum_{(j,k)\in E}\psi_{j,k}*\varphi_{j,k}*f(x_1,x_2),$$

where $\psi_{j,k}$ and $\varphi_{j,k}$ are the same as in Theorem A. It is easy to see that $f_N\in\mathcal{S}(\mathbb{R}^{n_1+n_2})$. Repeating the similar proof of Lemma 3.1, we can conclude that

$$\|f_N\|_{B_1^{\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\le C\|f\|_{B_1^{\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.$$

On the other hand,

$$\begin{aligned}&\big|\varphi_{j',k'}*\big(f-f_N\big)\big(x_1,x_2\big)\big|\\&\le\sum_{(j,k)\in E^c}C\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}|\varphi_{j',k'}*\psi_{j,k}(x-u,y-v)||\varphi_{j,k}*f(u,v)|dudv,\end{aligned}$$

then one can repeat the similar proof of Lemma 3.1 again to get

$$\begin{aligned}&\|f-f_N\|_{B_1^{\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\\&\le C\sum_{(j,k)\in E^c}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*f\|_{L^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\to 0\text{ as }N\to\infty.\end{aligned}$$

Hence the proof is finished. □

Lemma 3.3. *If $f\in\Lambda_\alpha$, $g\in B_1^{-\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, then*

$$\big|\langle f,g\rangle\big|\le C\|f\|_{\Lambda_\alpha}\ \|g\|_{B_1^{-\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.$$

Proof. By a dense argument in Lemma 3.2, we only need to prove the lemma for $g \in \mathcal{S}$. Using the local Calderón reproducing formula (3), we have

$$\begin{aligned}
&\left|\langle f, g\rangle\right| \\
\leq &\sum_{j,k\geq 0} \iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} |\psi_{j,k} * f(x_1,x_2)||\varphi_{j,k} * g(x_1,x_2)|dx_1dx_2 \\
\leq &\sum_{j,k\geq 0} 2^{-j\alpha_1}2^{-k\alpha_2} \iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} 2^{j\alpha_1}2^{k\alpha_2}|\psi_{j,k} * f(x_1,x_2)||\varphi_{j,k} * g(x_1,x_2)|dx_1dx_2 \\
\leq &C\|f\|_{\Lambda_\alpha}\ \|g\|_{B_1^{-\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.
\end{aligned}$$

□

Now we show the following so-called weak density argument for Λ_α which will play a crucial role in the proof of Theorem 1.2.

Lemma 3.4. *For any $f \in \Lambda_\alpha$, there exists a sequence $\{f_N\} \subset L^2(\mathbb{R}^{n_1+n_2}) \cap \Lambda_\alpha$ satisfying*

$$\|f_N\|_{\Lambda_\alpha} \leq C\|f\|_{\Lambda_\alpha},$$

and

$$\lim_{N\to\infty} \langle f_N, g\rangle = \langle f, g\rangle \text{ for any } g \in B_1^{-\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}).$$

Proof. Suppose $f \in \Lambda_\alpha$. Note that the local Calderón reproducing formula

$$f(x_1,x_2) = \sum_{j,k\geq 0} \psi_{j,k} * \varphi_{j,k} * f(x_1,x_2) \tag{14}$$

holds in the sense of distributions. For any fixed $N > 0$, denote

$$E = \{(j,k) : 0 \leq j \leq N,\ 0 \leq k \leq N\}$$

and

$$f_N(x_1,x_2) = \sum_{(j,k)\in E} \psi_{j,k} * \varphi_{j,k} * f(x_1,x_2). \tag{15}$$

Obviously, $f_N \in L^2(\mathbb{R}^{n_1+n_2})$. Repeating the same proof as the one in Lemma 3.1 yields

$$|\varphi_{j,k} * f_N(x,y)| \leq C2^{-j\alpha_1}2^{-k\alpha_2}\|f\|_{\Lambda_\alpha}.$$

Therefore, by Theorem 1.1, it follows that

$$\|f_N\|_{\Lambda_\alpha} \approx \sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k} * f_N\|_{L^\infty(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \leq C\|f\|_{\Lambda_\alpha}.$$

For any $g \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$, the local Calderón reproducing formula (14) yields

$$\langle f - f_N, g\rangle = \langle f, g - g_N\rangle.$$

By Lemma 3.2, the function

$$\sum_{(j,k)\in E^c} \psi_{j,k} * \varphi_{j,k} * g(x_1,x_2)$$

belongs to $B_1^{-\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and its $B_1^{-\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ norm tends to zero as $N\to\infty$. Thus, Lemma 3.3 implies that

$$|\langle f-f_N,g\rangle|\le C\|f\|_{\Lambda_\alpha}\|g-g_N\|_{\dot{B}_1^{-\alpha,1}}\to 0$$

as N tends to infinity. Since $\mathcal{S}(\mathbb{R}^{n_1+n_2})$ is dense in $B_1^{-\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, a standard limiting argument concludes the proof of Lemma 3.4. □

Next, we need to estimate the kernel of the operator $\varphi_{j,k}T\psi_{j',k'}$, which will denoted by $\varphi_{j,k}*T\big(\psi_{j',k'}(\cdot-y_1,\cdot-y_2)\big)(x_1,x_2)$. For this goal, we need to recall some basic definitions and notations. For $0<\eta<1$, let $C_0^\eta(\mathbb{R}^n)$ denote the space of continuous functions f with compact support such that

$$\|f\|_{\mathrm{Lip}_\eta(\mathbb{R}^n)}=\sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\eta}<\infty.$$

Notice that if $\psi\in\mathcal{S}(\mathbb{R}^n)$ with supp $\psi\subset B(0,r)$, where $B(0,r)$ denotes the Euclidean ball centered at the origin with radius r, then one can check that

$$\|\psi\|_{\mathrm{Lip}_\eta(\mathbb{R}^n)}<\infty,$$

and for any $j\in\mathbb{Z}$,

$$\|\psi_j\|_{\mathrm{Lip}_\eta(\mathbb{R}^n)}\le C2^{jn+j\eta}.$$

A linear one parameter operator $\mathcal{T}$ from $C_0^\eta(\mathbb{R}^n)$ to $(C_0^\eta(\mathbb{R}^n))'$ is said to satisfy the weak boundedness property if

$$|\langle f,\mathcal{T}(g)\rangle|\le Cr^{n+2\eta}\|f\|_{\mathrm{Lip}_\eta(\mathbb{R}^n)}\|g\|_{\mathrm{Lip}_\eta(\mathbb{R}^n)}$$

for any $f,g\in C_0^\eta(\mathbb{R}^n)$ with supports in $B(0,r)$. It is easy to check that L^2 boundedness of $\mathcal{T}$ implies that $\mathcal{T}$ satisfies the weak boundedness property. This property can be easily generalized to multi-parameter operators.

Lemma 3.5. *Let S be a one-parameter inhomogeneous Calderón-Zygmund operator on $\mathbb{R}^n$ with regularity exponent $\varepsilon\in(0,1]$ and $\delta>0$ associated with a kernel $S(u,v)$. Then for any $\varphi_0,\varphi,\psi_0,\psi\in C_0^\infty(\mathbb{R}^n)$ with zero integral of φ,ψ and nonzero integral of φ_0,ψ_0, the following orthogonal estimate holds*

$$\begin{aligned}|\varphi_jS\psi_{j'}(x_1,x_2)|&=\Big|\iint_{\mathbb{R}^{2n}}\varphi_j(x-u)S(u,v)\psi_{j'}(v-y)dudv\Big|\\ &\le C(1+|j-j'|)(2^{-(j-j')\varepsilon}\wedge 1)\frac{2^{-(j'\wedge j)\varepsilon'}}{(2^{-(j'\wedge j)}+|x-y|)^{n+\varepsilon'}}\end{aligned}\tag{16}$$

provided $S(1)=0$, where $\varepsilon'=\min\{\varepsilon,\delta\}$ and φ_j is φ_0 if $j=0$, otherwise the dilations of φ, and ψ_j' is interpreted similarly. Moreover, the corresponding constant depends only $\|S\|_{CZ}$.

Proof. For simplicity, we assume that the supports of $\varphi_0,\varphi,\psi_0,\psi$ are all contained in the unit ball. Firstly, we prove the estimate (16) in the case where

$j \geq j'$ and $|x-y| \leq 10\cdot 2^{-j'}$. Note that if $j=0$, then $j'=0$, the L^2-boundedness of S gives that

$$|\varphi_j S\psi_{j'}(x_1,x_2)| = |\langle \varphi_0(x-\cdot), S(\psi_0(\cdot - y))\rangle| \leq C.$$

Therefore, we assume $j>0$. Since $S(1)=0$, we get

$$\begin{aligned}\varphi_j S\psi_{j'}(x_1,x_2) &= \iint_{\mathbb{R}^{2n}} \varphi_j(x-u)S(u,v)\psi_{j'}(v-y)dudv \\ &= \iint_{\mathbb{R}^{2n}} \varphi_j(x-u)S(u,v)[\psi_{j'}(v-y)-\psi_{j'}(x-y)]dudv.\end{aligned}$$

Let ρ_0 be a smooth function on $\mathbb{R}^n$ supported on $\mathrm{B}(0,4)$, identically equal to 1 on $\mathrm{B}(0,2)$. Set $\rho_1 = 1-\rho_0$. Then

$$\begin{aligned}&\varphi_j S\psi_{j'}(x_1,x_2) \\ =\ & \iint_{\mathbb{R}^{2n}} \varphi_j(x-u)S(u,v)[\psi_{j'}(v-y)-\psi_{j'}(x-y)]\rho_0(2^j(v-x))dudv \\ &+ \iint_{\mathbb{R}^{2n}} \varphi_j(x-u)S(u,v)[\psi_{j'}(v-y)-\psi_{j'}(x-y)]\rho_1(2^j(v-x))dudv \\ :=\ & I+II.\end{aligned}$$

Therefore, by the weak boundedness of S on $\mathbb{R}^n$, we have

$$|I| \leq C2^{-j(n+2\eta)}\big\|\varphi_j(x-\cdot)\big\|_{\mathrm{Lip}_\eta(\mathbb{R}^n)}\big\|(\psi_{j'}(\cdot-y)-\psi_{j'}(x-y))\rho_0(2^j(\cdot-x))\big\|_{\mathrm{Lip}_\eta(\mathbb{R}^n)}.$$

The following well known estimates (see the Appendix in [7])

$$\big\|\varphi_j(x-\cdot)\big\|_{\mathrm{Lip}_\eta(\mathbb{R}^n)} \leq C2^{jn+j\eta}$$

and

$$\big\|(\psi_{j'}(\cdot-y)-\psi_{j'}(x-y))\rho_0(2^j(\cdot-x))\big\|_{\mathrm{Lip}_\eta(\mathbb{R}^n)} \leq C2^{-(j-j')}2^{j'n}2^{j\eta}$$

imply

$$|I| \leq C2^{-(j-j')}2^{j'n}.$$

For the term II, using $\int\varphi=0$, one has

$$II = \iint_{\mathbb{R}^{2n}} \varphi_j(x-u)[S(u,v)-S(x,v)][\psi_{j'}(v-y)-\psi_{j'}(x-y)]\rho_0(2^j(v-x))dudv.$$

Using the smoothness of $S(u,v)$ in u, together with

$$|\psi_{j'}(v-y)-\psi_{j'}(x-y)| \leq C\left(\frac{|v-x|}{2^{-j'}+|v-x|}\right)^{\varepsilon}2^{j'n}$$

for any $\varepsilon\in(0,1]$, we obtain

$$\begin{aligned}|II| &\leq C\iint_{|v-x|\geq 2\cdot 2^{-j}} |\varphi_j(x-u)|\frac{|x-u|^{\varepsilon}}{|v-x|^{n+\varepsilon}}\left(\frac{|v-x|}{2^{-j'}+|v-x|}\right)^{\varepsilon}2^{j'n}dudv \\ &\leq C2^{-j\varepsilon}2^{j'n}\iint_{|v-x|\geq 2\cdot 2^{-j'}} \frac{1}{|v-x|^{n+\varepsilon}}dudv\end{aligned}$$

$$+ C2^{-(j-j')\varepsilon}2^{j'n} \iint_{2\cdot 2^{-j}\le|v-x|<2\cdot 2^{-j'}} |v-x|^{-n}dudv$$
$$\le C(1+(j-j'))2^{-(j-j')\varepsilon}2^{j'n}$$
$$\le C(1+(j-j'))2^{-(j-j')\varepsilon}\frac{2^{-j'\varepsilon}}{(2^{-j'}+|x-y|)^{n+\varepsilon}}.$$

Now we deal with the case where $|x-y|>10\cdot 2^{-j'}$. If $j=0$, then $j'=0$, using the size condition of $S(u,v)$, we get

$$\Big|\iint_{\mathbb{R}^{2n}} \varphi_0(x-u)S(u,v)\psi_0(v-y)dudv\Big|$$
$$\le C\iint_{\mathbb{R}^{2n}} |\varphi_j(x-u)|\frac{1}{|u-v|^{n+\delta}}|\psi_{j'}(v-y)|dudv$$
$$\le C\frac{1}{|x-y|^{n+\delta}}.$$

The last inequality follows from $|u-v|\approx|x-y|$. For $j>0$, the cancellation condition of φ yields

$$\Big|\iint_{\mathbb{R}^{2n}} \varphi_j(x-u)S(u,v)\psi_{j'}(v-y)dudv\Big|$$
$$\le \Big|\iint_{\mathbb{R}^{2n}} \varphi_j(x-u)[S(u,v)-S(x,v)]\psi_{j'}(v-y)dudv\Big|$$
$$\le C\iint_{\mathbb{R}^{2n}} |\varphi_j(x-u)|\frac{|x-u|^{\varepsilon}}{|x-v|^{n+\varepsilon}}|\psi_{j'}(v-y)|dudv$$
$$\le C\frac{2^{-j\varepsilon}}{|x-y|^{n+\varepsilon}} = C2^{-(j-j')\varepsilon}\frac{2^{-j'\varepsilon}}{(2^{-j'}+|x-y|)^{n+\varepsilon}}.$$

This proves (16) for this case where $j\ge j'$ and $|x-y|\le 10\cdot 2^{-j'}$, $j\ge j'$ and $|x-y|>10\cdot 2^{-j'}$. The two remaining cases: $j<j'$ and $|x-y|\le 10\cdot 2^{-j}$, $j<j'$ and $|x-y|>10\cdot 2^{-j}$, are similar but easier. □

Lemma 3.6. *Let T be a singular integral operator in inhomogeneous Journé's class on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ with regularity exponent $\varepsilon\in(0,1]$ and $\delta>0$. If $T_1(1)=T_2(1)=0$, then we have the almost orthogonality estimate*

$$\big|\varphi_{j,k} * T\big(\psi_{j',k'}(\cdot-y_1,\cdot-y_2)\big)(x_1,x_2)\big|$$
$$= \Big|\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \varphi_{j,k}(x_1-u_1,x_2-u_2)\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \mathcal{K}(u_1,u_2,v_1,v_2)$$
$$\times\,\psi_{j',k'}(v_1-y_1,v_2-y_2)du_1dv_1du_2dv_2\Big| \tag{17}$$
$$\le C(1+|j-j'|)(1+|k-k'|)(2^{-(j-j')\varepsilon}\wedge 1)(2^{-(k-k')\varepsilon}\wedge 1)$$
$$\times\frac{2^{-(j'\wedge j)\varepsilon'}}{(2^{-(j'\wedge j)}+|x_1-y_1|)^{n_1+\varepsilon'}}\frac{2^{-(k'\wedge k)\varepsilon'}}{(2^{-(k'\wedge k)}+|x_2-y_2|)^{n_2+\varepsilon'}},$$

where $\varepsilon' = \min\{\varepsilon, \delta\}$.

Proof. Set

$$K_2(u_2, v_2) = \iint_{\mathbb{R}^{2n_1}} \varphi_j^1(x_1 - u_1)\tilde{\mathcal{K}}^2(u_2, v_2)(u_1, v_1)\psi_{j'}^1(v_1 - y_1)du_1 dv_1,$$

where $\tilde{\mathcal{K}}^2(u_2, v_2)(u_1, v_1) = \mathcal{K}(u_1, u_2, v_1, v_2)$ is a one-parameter inhomogeneous Calderón-Zygmund kernel on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_1}$ with regularity exponent $\varepsilon \in (0, 1]$ and $\delta > 0$ such that

$$\|\tilde{\mathcal{K}}^2(u_2, v_2)\|_{CZ} \le C \min\{\frac{1}{|u_2 - v_2|^{n_2}}, \frac{1}{|u_2 - v_2|^{n_2+\delta}}\}$$

for fixed u_2, v_2. Since $T_1(1) = 0$, by Lemma 3.5 we get

$$\begin{aligned}
&|K_2(u_2, v_2)| \\
&\le C\|\tilde{\mathcal{K}}^2(u_2, v_2)\|_{CZ}(1 + |j - j'|)(2^{-(j-j')\varepsilon'} \wedge 1)\frac{2^{-(j'\wedge j)\varepsilon'}}{(2^{-(j'\wedge j)} + |x_1 - y_1|)^{n+\varepsilon'}} \\
&\le C \min\left\{\frac{1}{|u_2 - v_2|^{n_2}}, \frac{1}{|u_2 - v_2|^{n_2+\delta}}\right\}(1 + |j - j'|)(2^{-(j-j')\varepsilon'} \wedge 1) \\
&\quad \times \frac{2^{-(j'\wedge j)\varepsilon'}}{(2^{-(j'\wedge j)} + |x_1 - y_1|)^{n_1+\varepsilon'}}.
\end{aligned}$$

Similarly, for $|u_2 - u_2'| \le \frac{|u_1 - v_2|}{2}$,

$$\begin{aligned}
&|K_2(u_2, v_2) - K_2(u_2', v_2)| \\
&= \Big| \iint_{\mathbb{R}^{2n_1}} \varphi_j^1(x_1 - u_1)[\tilde{\mathcal{K}}^2(u_2, v_2)(u_1, v_1) - \tilde{\mathcal{K}}^2(u_2', v_2)(u_1, v_1)] \\
&\qquad \times \psi_{j'}^1(v_1 - y_1, v_2 - y_2)du_1 dv_1\Big| \\
&\le C\|\tilde{\mathcal{K}}^2(u_2, v_2) - \tilde{\mathcal{K}}^2(u_2', v_2)\|_{CZ}(1 + |j - j'|)(2^{-(j-j')\varepsilon'} \wedge 1) \\
&\qquad \times \frac{2^{-(j'\wedge j)\varepsilon'}}{(2^{-(j'\wedge j)} + |x_1 - y_1|)^{n_1+\varepsilon'}} \\
&\le C\frac{|u_2 - u_2'|^{\varepsilon}}{|u_2 - v_2|^{n_2+\varepsilon}}(1 + |j - j'|)(2^{-(j-j')\varepsilon'} \wedge 1)\frac{2^{-(j'\wedge j)\varepsilon'}}{(2^{-(j'\wedge j)} + |x_1 - y_1|)^{n_1+\varepsilon'}},
\end{aligned}$$

and for $|v_2 - v_2'| \le \frac{|u_2 - v_2|}{2}$,

$$\begin{aligned}
&|K_2(u_2, v_2) - K_2(u_2 - v_2')| \\
&\le C\frac{|v_2 - v_2'|^{\varepsilon}}{|u_2 - v_2|^{n_2+\varepsilon}}(1 + |j - j'|)(2^{-(j-j')\varepsilon'} \wedge 1)\frac{2^{-(j'\wedge j)\varepsilon'}}{(2^{-(j'\wedge j)} + |x_1 - y_1|)^{n_1+\varepsilon'}}.
\end{aligned}$$

The above three estimates imply $K_2(u_2, v_2)$ is a one-parameter inhomogeneous Calderón-Zygmund kernel on $\mathbb{R}^{n_2} \times \mathbb{R}^{n_2}$ with regularity exponent $\varepsilon \in (0, 1]$ and

$\delta > 0$, and

$$\|K_2\|_{CZ} \le C(1+|j-j'|)(2^{-(j-j')\varepsilon'} \wedge 1)\frac{2^{-(j'\wedge j)\varepsilon'}}{(2^{-(j'\wedge j)}+|x_1-y_1|)^{n_1+\varepsilon'}}. \tag{18}$$

Applying Lemma 3.5 again and the condition (18) of the kernel $K_2(u_2, v_2)$ gives the desire estimate (17). □

Repeating the same method of Lemma 3.6, we can obtain the following result.

Lemma 3.7. *Let T be a singular integral operator in inhomogeneous Journé's class on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with regularity exponent $\varepsilon \in (0,1]$ and $\delta > 0$. If $T_1^*(1) = T_2^*(1) = 0$, then we have the almost orthogonality estimate*

$$\begin{aligned} &\big|\varphi_{j,k} * T\big(\psi_{j',k'}(\cdot - y_1, \cdot - y_2)\big)(x_1,x_2)\big| \\ &\le C(1+|j-j'|)(1+|k-k'|)(2^{-(j'-j)\varepsilon} \wedge 1)(2^{-(k'-k)\varepsilon} \wedge 1) \\ &\quad \times \frac{2^{-(j\wedge j')\varepsilon'}}{(2^{-(j\wedge j')}+|x_1-y_1|)^{n_1+\varepsilon'}}\frac{2^{-(k\wedge k')\varepsilon'}}{(2^{-(k\wedge k')}+|x_2-y_2|)^{n_2+\varepsilon'}}, \end{aligned} \tag{19}$$

where $\varepsilon' = \min\{\varepsilon, \delta\}$.

As a consequence of Lemmas 3.6 and 3.7, one can easily get the following boundedness.

Lemma 3.8. *Suppose that T is a singular integral operator in inhomogeneous Journé's class with regularity exponents $\varepsilon \in (0,1]$ and $\delta > 0$. Then*

(1) *T is bounded on $B_1^{\alpha,1}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ if $0 < \max\{\alpha_1,\alpha_2\} < \varepsilon$ and $T_1(1) = T_2(1) = 0$.*
(2) *T is bounded on $B_1^{-\alpha,1}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ if $-\varepsilon < \min\{\alpha_1,\alpha_2\} < 0$ and $T_1^*(1) = T_2^*(1) = 0$.*

Proof. Here we only give the details of the first item, since the second is the same. For $f \in L^2(\mathbb{R}^{n_1+n_2}) \cap B_1^{\alpha,1}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, by the local Calderón reproducing formula (3) and Lemma 3.6, we have

$$\begin{aligned} &\|Tf\|_{B_1^{\alpha,1}(\mathbb{R}^{n_1+n_2})} \\ &= \sum_{j,k\ge 0} 2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k} * Tf\|_{L^1(\mathbb{R}^{n_1+n_2})} \\ &\le \sum_{j,k\ge 0} 2^{j\alpha_1}2^{k\alpha_2}\sum_{j',k'\ge 0}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} |\varphi_{j,k}T\psi_{j',k'}(x_1,x_2,y_1,y_2)| \\ &\quad \times |\varphi_{j',k'} * f(y_1,y_2)|dx_1dx_2dy_1dy_2 \\ &\le C\sum_{j',k'\ge 0}\sum_{j,k\ge 0} 2^{j\alpha_1}2^{k\alpha_2}(1+|j-j'|)(1+|k-k'|)(2^{-(j-j')\varepsilon} \wedge 1) \\ &\quad \times (2^{-(k-k')\varepsilon} \wedge 1)\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \frac{2^{-(j\wedge j')\varepsilon'}}{(2^{-(j\wedge j')}+|x_1-y_1|)^{(n+\varepsilon')}} \end{aligned}$$

$$\times \frac{2^{-(k\wedge k')\varepsilon'}}{(2^{-(k\wedge k')}+|x_2-y_2|)^{(m+\varepsilon')}}|\varphi_{j',k'}*f(y_1,y_2)|dx_1dx_2dy_1dy_2$$
$$\leq C\sum_{j',k'\geq 0}\sum_{j,k\geq 0}2^{(j-j')\alpha_1}2^{(k-k')\alpha_2}(1+|j-j'|)(1+|k-k'|)(2^{-(j-j')\varepsilon}\wedge 1)$$
$$\times(2^{-(k-k')\varepsilon}\wedge 1)\iint_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}2^{j'\alpha_1}2^{k'\alpha_2}|\varphi_{j',k'}*f(y_1,y_2)|dy_1dy_2$$
$$\leq C\|f\|_{B_1^{\alpha,1}(\mathbb{R}^n\times\mathbb{R}^m)}.$$

The last inequality follows from

$$\sum_{j,k\geq 0}2^{(j-j')\alpha_1}2^{(k-k')\alpha_2}(1+|j-j'|)(1+|k-k'|)(2^{-(j-j')\varepsilon'}\wedge 1)(2^{-(k-k')\varepsilon'}\wedge 1)\leq C$$

if $\max\{\alpha_1,\alpha_2\}<\varepsilon$. Then a limiting argument yields Lemma 3.8. □

We now turn to prove Theorem 1.2.

Proof of Theorem 1.2. We first claim that for any $f\in L^2(\mathbb{R}^{n_1+n_2})\cap\Lambda_\alpha$,

$$\|Tf\|_{\Lambda_\alpha}\leq C\|f\|_{\Lambda_\alpha}. \tag{20}$$

To see this, it suffices to show that

$$\sup_{j,k\geq 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*Tf\|_{L^\infty(\mathbb{R}^{n_1+n_2})}\leq C\|f\|_{\Lambda_\alpha}. \tag{21}$$

Repeating the same argument as the proof of Lemma 3.8, we obtain that

$$2^{j\alpha_1}2^{k\alpha_2}|\varphi_{j,k}*Tf(x_1,x_2)|\leq C\|f\|_{\Lambda_\alpha}.$$

Plugging this estimate into (21) yields (20).

Next, we extend T to $\mathrm{Lip}(\alpha_1,\alpha_2)$ as follows. Given $f\in\Lambda_\alpha$, by Lemma 3.4, there is a sequence $\{f_N\}\subset L^2(\mathbb{R}^{n_1+n_2})\cap\Lambda_\alpha$ such that

$$\begin{cases}\{f_N\}\subset L^2(\mathbb{R}^{n_1+n_2})\cap\Lambda_\alpha,\\ \lim\limits_{N\to\infty}\langle f_N,g\rangle=\langle f,g\rangle \text{ for any } g\in B_1^{-\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}).\end{cases}$$

We thus define

$$\langle Tf,g\rangle=\lim_{N\to\infty}\langle Tf_N,g\rangle,\ g\in B_1^{-\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}).$$

To see the existence of the limit, we write $\langle T(f_N-f_{N'}),g\rangle=\langle f_N-f_{N'},T^*g\rangle$. By Lemma 3.8, T^* is bounded on $B_1^{-\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, and thus

$$T^*g\in B_1^{-\alpha,1}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}).$$

Therefore, by Lemma 3.4, $\langle f_N-f_{N'},T^*g\rangle$ tends to zero as $N,N'\to\infty$. It is also easy to check that the definition of Tf is independent of the choice of the sequence f_N satisfying the conditions in Lemma 3.4.

For $f\in\Lambda_\alpha$, by the definition of Tf and the boundedness of T on $L^2(\mathbb{R}^{n_1+n_2})$ $\cap\Lambda_\alpha$,

$$\|Tf\|_{\Lambda_\alpha}\leq C\sup_{j,k\geq 0}2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k}*Tf\|_{L^\infty(\mathbb{R}^{n_1+n_2})}$$

$$
\begin{aligned}
&\leq C \liminf_{N\to\infty} \sup_{j,k\geq 0} 2^{j\alpha_1}2^{k\alpha_2}\|\varphi_{j,k} * Tf_N\|_{L^\infty(\mathbb{R}^{n_1+n_2})} \\
&\leq C \liminf_{N\to\infty} \|Tf_N\|_{\Lambda_\alpha} \\
&\leq C \liminf_{N\to\infty} \|f_N\|_{\Lambda_\alpha} \leq C\|f\|_{\Lambda_\alpha},
\end{aligned}
$$

which concludes the proof of Theorem 1.2. □

Remark 3.9. The sufficient condition $T_1(1) = T_2(1) = 0$ of Theorem 1.2 might not be necessary. Suppose that T is bounded on Λ_α. Take $f_2 \in C_0^\infty(\mathbb{R}^{n_2})$, we note that $\|1 \otimes f_2\|_{\Lambda_\alpha} \neq 0$. The boundedness of T yields that $\|T1 \otimes f_2\|_{\Lambda_\alpha}$ may be not equal to zero in general. That is

$$\langle g_1 \otimes g_2, T1 \otimes f_2\rangle \neq 0$$

for all $g_1 \in C_{0,0}^\infty(\mathbb{R}^{n_1})$ and $g_2, f_2 \in C_0^\infty(\mathbb{R}^{n_2})$. By our definition, $T_1(1) \neq 0$. $T_2(1)$ follows similarly.

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