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THE EXCEPTIONAL SET OF ONE PRIME SQUARE AND FIVE PRIME CUBES

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Abstract. For a natural number n, let $R(n)$ denote the number of representations of n as the sum of one square and five cubes of primes. In this paper, it is proved that the anticipated asymptotic formula for $R(n)$ fails for at most $O(N^{\frac{4}{9}+\varepsilon})$ positive integers not exceeding N.

1. Introduction

Let $\widetilde{R}(n)$ denote the number of representations of n in the form

$$n = x^2 + x_1^3 + x_2^3 + \cdots + x_5^3\,, \tag{1.1}$$

where $x \in \mathbb{N}$ and $x_i \in \mathbb{N}$ $(1 \le i \le 5)$. We write $e(\alpha)$ for $e^{2\pi i\alpha}$ and $e_q(\alpha)$ for $e(\alpha/q)$. Vaughan [5] proved that

$$\widetilde{R}(n) \sim \frac{\Gamma(\frac{3}{2})\Gamma^5(\frac{4}{3})}{\Gamma(\frac{13}{6})}\widetilde{\mathfrak{S}}(n)n^{\frac{7}{6}}, \tag{1.2}$$

where

$$\widetilde{\mathfrak{S}}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1\\(a,q)=1}}^{q} \frac{S_2(q,a)S_3(q,a)^5 e_q(-an)}{q^6}, \tag{1.3}$$

with

$$S_k(q,a) = \sum_{r=1}^{q} e_q(ar^k) \ \ (k=2,3). \tag{1.4}$$

In view of Vaughan's result, it is reasonable to propose the conjecture that for every sufficiently large even integer n, the equation

$$n = p^2 + p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^3 \tag{1.5}$$

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is solvable, where and below the letter p, with or without subscripts, denotes a prime number, unless otherwise indicated. However, this conjecture is at present far from being solved.

Let P_r denote an almost-prime with at most r prime factors, counted according to multiplicity. By $R(n)$ we denote the number of representations of n in the form (1.5). The previous best result of $R(n)$ is due to Cai [1] who obtained $R(n) \gg \frac{n^{\frac{19}{18}}}{\log^6 n}$ with p an almost-prime P_{36}.

Another topic in the study of (1.1) and (1.5) concerns the exceptional sets $\widetilde{E}(N)$ and $E(N)$. Let $\widetilde{E}(N)$ denote the number of all the positive integers n not exceeding N which cannot be written as (1.1). A conventional application of Bessel's inequality yields that $\widetilde{E}(N) \ll \psi(N)^2 N^{\frac{1}{3}+\varepsilon}$, where $\psi(N)$ denotes a function of uniform growth with $\psi(N) = O(N^{\delta})$ for some sufficiently small positive number δ.

In 2001, Wooley [7] made an important breakthrough in the study of $\widetilde{E}(N)$, and showed that $\widetilde{E}(N) \ll \psi(N)^2 \exp(c \log N / \log\log N)$, where c is a positive number. Wooley's methods avoid a conventional application of Bessel's inequality in favour of explicit control of an exponential sum over the exceptional set itself.

It is of interest to investigate the exceptional set $E(N)$ of (1.5), which denotes the number of integers $n \le N$ and cannot be represented as the sum of one square and five cubes of primes.

In this paper, we apply the Hardy-Littlewood method to establish the following result.

Theorem 1. *For a natural number n, let $R(n)$ denote the number of representations of n as the sum of one square and five cubes of primes, $\varphi(q)$ denote Euler's function and $\mathcal{E}(N)$ denote the number of integers $n \le N$ such that the asymptotic formula*

$$R(n) = \frac{\Gamma(\frac{3}{2})\Gamma^5(\frac{4}{3})}{\Gamma(\frac{13}{6})}\mathfrak{S}(n)\frac{n^{\frac{7}{6}}}{\log^6 n} + O\left(\frac{n^{\frac{7}{6}}\log\log n}{\log^7 n}\right) \tag{1.6}$$

fails to hold, where

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{S_2^*(q,a) S_3^{*5}(q,a) e_q(-an)}{\varphi^6(q)},$$

$$S_k^*(q,a) = \sum_{\substack{r=1 \\ (r,q)=1}}^{q} e_q(ar^k).$$

Then for any $\varepsilon > 0$, we have $\mathcal{E}(N) \ll N^{\frac{4}{9}+\varepsilon}$.

From Theorem 1 and Lemma 2.5, we get the following theorem.

Theorem 2. *For any $\varepsilon > 0$, we have $E(N) \ll N^{\frac{4}{9}+\varepsilon}$.*

2. Notation and some preparatory lemmas

Throughout this paper, by n we denote a sufficiently large integer which satisfies $0.5N \le n \le N$. In addition, let $A = 10^{10}$, $\varepsilon \in (0, 10^{-10})$ be an arbitrarily small positive constant and c be a positive constant not necessarily the same in different formulae. The letter p, with or without subscripts, is reserved for a prime number. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on ε and A. As usual, we use $\varphi(q)$ and $d(n)$ to denote Euler's function and Dirichlet's divisor function. $e(\alpha)$ stands for $e^{2\pi i\alpha}$ and $e_q(\alpha) = e(\alpha/q)$. We denote by $\sum_{r(q)^*}$ sums with r running over a reduced system of residues modulo q. We set

$$N^{\frac{7}{24}} \le X \le N^{\frac{1}{3}}/2,\ \ Q_0 = \log^A N,\ \ Q_1 = N^{\frac{1}{6}},\ \ Q_2 = N^{\frac{5}{6}},$$

$$g_k(\alpha) = \sum_{2\le p\le N^{\frac{1}{k}}} e(\alpha p^k),\ \ g_k(\alpha, X) = \sum_{X\le p\le 2X} e(\alpha p^k),$$

$$S_k^*(q,a) = \sum_{r(q)^*} e_q(ar^k),\ \mathfrak{S}(n) = \sum_{q=1}^{\infty}\sum_{a(q)^*} \frac{S_2^*(q,a)S_3^*(q,a)^5 e_q(-an)}{\varphi(q)^6}.$$

For $(a,q) = 1$, $1 \le a \le q$, put

$$\mathfrak{M}_0 = \bigcup_{1\le q\le Q_0^A} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \left(\frac{a}{q} - \frac{Q_0^A}{N}, \frac{a}{q} + \frac{Q_0^A}{N}\right],$$

$$\mathfrak{J}_0 = \left(\frac{1}{Q_2}, 1 + \frac{1}{Q_2}\right], \qquad \mathfrak{m} = \mathfrak{J}_0 \setminus \mathfrak{M}_0.$$

Then we have the Farey dissection

$$\mathfrak{J}_0 = \mathfrak{M}_0 \bigcup \mathfrak{m}. \tag{2.1}$$

Lemma 2.1. *Define $\omega_3(q) = 3q^{-\frac{1}{2}}$ and let $\mathcal{M}(X)$ be the union of the intervals $\mathcal{M}(q,a,X)$ for $1 \le a \le q \le X^{\frac{3}{4}}$ and $(a,q) = 1$, where $\mathcal{M}(q,a,X) = \{\alpha : |q\alpha - a| \le X^{-\frac{9}{4}}\}$. Let*

$$\mathcal{J}_0(X) = \sup_{\gamma\in[0,1]} \int_{\mathcal{M}(X)} \frac{\omega_3^2(q)|g_3(\alpha+\gamma)|^2}{(1 + X^3|\alpha - \frac{a}{q}|)^2} d\alpha.$$

Suppose that $G(\alpha)$ is an integrable function of period one. Then we have

$$\int_{\mathfrak{m}} h(\alpha)G(\alpha)g_3(\alpha)d\alpha \ll X\mathcal{J}_0^{\frac{1}{4}}(X)\left(\int_{\mathfrak{m}} |G(\alpha)|^2 d\alpha\right)^{\frac{1}{4}} \mathcal{J}^{\frac{1}{2}} + X^{\frac{7}{8}+\varepsilon}\mathcal{J},$$

where

$$\mathcal{J} = \int_{\mathfrak{m}} |G(\alpha)g_3(\alpha)| d\alpha,$$

and

$$\mathcal{J}_0(X) \ll N^{\frac{2}{3}} X^{-3} (\log N)^c. \tag{2.2}$$

Proof. Lemma 2.1 is Lemma 3.1 in Zhao [10] and the estimate (2.2) is Lemma 2.2 in Zhao [10]. □

Lemma 2.2. *For $\alpha \in \mathfrak{m}_1$, we have*

$$g_3(\alpha) \ll N^{\frac{11}{36}+\varepsilon}.$$

Proof. See Lemma 2.5 i) in Liu [4]. □

Lemma 2.3. *For $\alpha = \frac{a}{q} + \lambda$, $(a,q) = 1$, $q \le Q \le X$ and $|\lambda| \le \frac{Q}{qN}$, we have*

$$g_k(\alpha, X) \ll Q^{\frac{1}{2}} X^{\frac{11}{20}+\varepsilon} + V_k(\alpha, X), \tag{2.3}$$

where

$$V_k(\alpha, X) = \frac{X(\log N)^c}{q^{\frac{1}{2}-\varepsilon}(1 + X^k|\lambda|)^{\frac{1}{2}}}. \tag{2.4}$$

Proof. See Theorem 2 in Kumchev [3]. □

Lemma 2.4. *Let*

$$u_k(\lambda) = \sum_{2<n\le N} \frac{e(n\lambda)}{n^{1-\frac{1}{k}} \log n}.$$

Then for $\alpha = \frac{a}{q} + \lambda \in \mathfrak{M}_0$, we have

$$g_k(\alpha) = \frac{S_k^*(q,a)}{\varphi(q)} u_k(\lambda) + O\left(N^{\frac{1}{k}} \exp(-\log^{\frac{1}{3}} N)\right).$$

Proof. See Hua [2, Lemma 7.15]. □

Lemma 2.5. *The series $\mathfrak{S}(n)$ is convergent and satisfies*

$$0 < c^* \le \mathfrak{S}(n) \ll d(n),$$

where positive constant c^* is defined in Lemma 4.6 in Xue, Zhang and Li [9].

Proof. See (2.4) in Xue, Zhang and Li [9]. □

Lemma 2.6. *For $(a,q) = 1$, we have*

$$|S_k^*(q,a)| \ll q^{\frac{1}{2}+\varepsilon}.$$

Proof. See Vinogradov [6, Chapter VI]. □

3. Auxiliary estimates

We initiate our proof by recalling the Farey dissection (2.1) that

$$R(n) = \int_{\mathfrak{M}_0} g_2(\alpha) g_3^5(\alpha) e(-\alpha n) d\alpha + \int_{\mathfrak{m}} g_2(\alpha) g_3^5(\alpha) e(-\alpha n) d\alpha.$$

3.1. The evaluation of the integral over $\mathfrak{M}_0$

Proposition. *For $\frac{N}{2} < n \le N$, we have*

$$\int_{\mathfrak{M}_0} g_2(\alpha) g_3^5(\alpha) e(-\alpha n) d\alpha = \frac{\Gamma(\frac{3}{2})\Gamma^5(\frac{4}{3})}{\Gamma(\frac{13}{6})\log^6 n}\mathfrak{S}(n) n^{\frac{7}{6}} + O\left(\frac{n^{\frac{7}{6}}\log\log n}{\log^7 n}\right).$$

Proof. For $\alpha = \frac{a}{q} + \lambda$, let $f_k(\alpha) = \frac{S_k^*(q,a)}{\varphi(q)} u_k(\lambda)$ and $\mathfrak{S}(n) = \sum_{q=1}^{\infty} A(q,n)$. Then it follows from Lemma 2.4 that

$$\begin{aligned}
&\int_{\mathfrak{M}_0} g_2(\alpha) g_3^5(\alpha) e(-\alpha n) d\alpha \\
&= \int_{\mathfrak{M}_0} S_2^*(q,a)(S_3^*(q,a))^5 u_2(\lambda) u_3^5(\lambda)/\varphi(q)^6 e(-\alpha n) d\alpha \\
&\quad + O\left(\int_{\mathfrak{M}_0} N^{\frac{13}{6}} \exp(-6\log^{\frac{1}{3}} N) d\alpha\right) \\
&= \int_{\mathfrak{M}_0} f_2(\alpha) f_3^5(\alpha) e(-\alpha n) d\alpha + O\left(n^{\frac{7}{6}} \exp(-\log^{\frac{1}{4}} n)\right).
\end{aligned} \tag{3.1}$$

It is easy to see that

$$\begin{aligned}
&\int_{\mathfrak{M}_0} f_2(\alpha) f_3^5(\alpha) e(-\alpha n) d\alpha \\
&= \sum_{q \le Q_0^{100}} A(q,n) \int_{|\lambda| \le \frac{Q_0^{100}}{N}} u_2(\lambda) u_3^5(\lambda) e(-\lambda n) d\lambda.
\end{aligned} \tag{3.2}$$

Moreover, it follows from [2, Lemma 7.16] that

$$\begin{aligned}
&\int_{|\lambda| \le \frac{Q_0^{100}}{N}} u_2(\lambda) u_3^5(\lambda) e(-\lambda n) d\lambda \\
&= \int_0^1 u_2(\lambda) u_3^5(\lambda) e(-\lambda n) d\lambda + O\left(\int_{\frac{Q_0^{100}}{N}}^1 \frac{1}{\lambda^{\frac{13}{6}} \log^6 N} d\lambda\right) \\
&= \int_0^1 u_2(\lambda) u_3^5(\lambda) e(-\lambda n) d\lambda + O(N^{\frac{7}{6}} Q_0^{-100}).
\end{aligned} \tag{3.3}$$

Similar to [2, Lemma 7.19], we have

$$\int_0^1 u_2(\lambda) u_3^5(\lambda) e(-\lambda n) d\lambda = \frac{\Gamma(\frac{3}{2})\Gamma^5(\frac{4}{3})}{\Gamma(\frac{13}{6})\log^6 n} n^{\frac{7}{6}} + O\left(\frac{n^{\frac{7}{6}}\log\log n}{\log^7 n}\right). \tag{3.4}$$

By (3.3) and (3.4), we obtain

$$\int_{|\lambda| \le \frac{Q_0^{100}}{N}} u_2(\lambda) u_3^5(\lambda) e(-\lambda n) d\lambda = \frac{\Gamma(\frac{3}{2})\Gamma^5(\frac{4}{3})}{\Gamma(\frac{13}{6})\log^6 n} n^{\frac{7}{6}} + O\left(\frac{n^{\frac{7}{6}}\log\log n}{\log^7 n}\right). \tag{3.5}$$

From Lemma 2.6 and the inequality $\varphi(q) \gg \frac{q}{\log q}$, we get

$$\begin{aligned}\sum_{q\le Q_0^{100}} A(q,n) &= \mathfrak{S}(n) - \sum_{q>Q_0^{100}} A(q,n)\\ &= \mathfrak{S}(n) + O\left(\sum_{q>Q_0^{100}} q\cdot\frac{(q^{\frac12+\varepsilon})\cdot(q^{\frac12+\varepsilon})^5(\log q)^6}{q^6}\right)\\ &= \mathfrak{S}(n) + O\left(\sum_{q>Q_0^{100}} q^{-2+\varepsilon}\right)\\ &= \mathfrak{S}(n) + O(Q_0^{-100+\varepsilon}). \qquad (3.6)\end{aligned}$$

On combining (3.1), (3.2), (3.5) and (3.6), we get

$$\int_{\mathfrak{M}_0} g_2(\alpha)g_3^5(\alpha)e(-\alpha n)d\alpha = \frac{\Gamma(\frac32)\Gamma^5(\frac43)}{\Gamma(\frac{13}{6})\log^6 n}\mathfrak{S}(n)n^{\frac76} + O\left(\frac{n^{\frac76}\log\log n}{\log^7 n}\right).$$

This completes the proof of the Proposition. □

3.2. The estimation of the integrals over $\mathfrak{m}_j$ $(j=1,2)$

We divide $\mathfrak{m}$ into two parts $\mathfrak{m}_1$ and $\mathfrak{m}_2$, where

$$\mathfrak{M} = \bigcup_{1\le q\le Q_1}\bigcup_{\substack{a=1\\(a,q)=1}}^{q}\left(\frac{a}{q}-\frac{Q_1}{qN}, \frac{a}{q}+\frac{Q_1}{qN}\right],$$

$$\mathfrak{m}_1 = \mathfrak{J}_0\setminus\mathfrak{M}, \quad \mathfrak{m}_2 = \mathfrak{M}\setminus\mathfrak{M}_0.$$

And we simply divide the range of n into dyadic intervals, and denote by $Z_j(N)$ the set of integers with $\frac{N}{2} < n \le N$ for which the inequality

$$\left|\int_{\mathfrak{m}_j} g_2(\alpha)g_3^5(\alpha)e(-\alpha n)d\alpha\right| > \frac{n^{\frac76}}{\log^7 n} \qquad (3.7)$$

holds. For simplicity, we abbreviate the cardinality of $Z_j(N)$ to Z_j. Next, define the complex number η_n by taking $\eta_n = 0$ for $n \notin Z_j(N)$, and for $n \in Z_j(N)$ by means of the equation

$$\left|\int_{\mathfrak{m}_j} g_2(\alpha)g_3^5(\alpha)e(-\alpha n)d\alpha\right| = \eta_n\int_{\mathfrak{m}_j} g_2(\alpha)g_3^5(\alpha)e(-\alpha n)d\alpha. \qquad (3.8)$$

Clearly, one has $|\eta_n| = 1$ whenever η_n is non-zero. Thus, we have

$$\sum_{n\in Z_j(N)}\eta_n\int_{\mathfrak{m}_j} g_2(\alpha)g_3^5(\alpha)e(-\alpha n)d\alpha = \int_{\mathfrak{m}_j} g_2(\alpha)g_3^5(\alpha)K_j(\alpha)d\alpha, \qquad (3.9)$$

where the exponential sum $K_j(\alpha)$ is defined by

$$K_j(\alpha) = \sum_{n\in Z_j(N)} \eta_n e(-\alpha n).$$

Let

$$I_j = \int_{\mathfrak{m}_j} g_2(\alpha) g_3^5(\alpha) K_j(\alpha) d\alpha \ \ (j = 1, 2).$$

By (3.7)-(3.9), we get

$$I_j > \sum_{n\in Z_j(N)} \frac{n^{\frac{7}{6}}}{\log^7 n} \gg \frac{Z_j N^{\frac{7}{6}}}{\log^7 N}. \tag{3.10}$$

3.2.1. *The estimation of* Z_1. We now establish our estimate for Z_1. It is easy to see that

$$\begin{aligned} I_1 &\ll (\log N) \max_{N^{\frac{7}{24}} \le X \le N^{\frac{1}{3}}} \left| \int_{\mathfrak{m}_1} g_3(\alpha, X) g_2(\alpha) g_3^4(\alpha) K_1(\alpha) d\alpha \right| \\ &\quad + N^{\frac{7}{24}} \int_{\mathfrak{m}_1} |g_2(\alpha) g_3^4(\alpha) K_1(\alpha)| d\alpha \\ &= (\log N) \max_{N^{\frac{7}{24}} \le X \le N^{\frac{1}{3}}} |I_{11}| + I_{12}, \ \text{say.} \end{aligned} \tag{3.11}$$

It follows from Cauchy's inequality, Hua's inequality and Wooley [8, Lemma 2.1] with $k = 2$ that

$$\begin{aligned} I_{12} &\ll N^{\frac{7}{24}} \left(\int_0^1 |g_2^2(\alpha) K_j^2(\alpha)| d\alpha \right)^{\frac{1}{2}} \left(\int_0^1 |g_3^8(\alpha)| d\alpha \right)^{\frac{1}{2}} \\ &\ll Z_j N^{\frac{9}{8}+\varepsilon} + Z_j^{\frac{1}{2}} N^{\frac{11}{8}+\varepsilon}. \end{aligned} \tag{3.12}$$

Furthermore, by taking $h(\alpha) = g_3(\alpha, X)$ and $G(\alpha) = g_2(\alpha) g_3^3(\alpha) K_1(\alpha)$ in Lemma 2.1, we have

$$\begin{aligned} I_{11} &= \int_{\mathfrak{m}_1} |g(\alpha) G(\alpha) h(\alpha)| d\alpha \\ &\ll X \mathcal{J}_0^{\frac{1}{4}}(X) \left(\int_{\mathfrak{m}_1} |G(\alpha)|^2 d\alpha \right)^{\frac{1}{4}} \mathcal{J}^{\frac{1}{2}} + X^{\frac{7}{8}+\varepsilon} \mathcal{J}, \end{aligned} \tag{3.13}$$

where $\mathcal{J} = \int_{\mathfrak{m}_1} |g_2(\alpha) g_3^4(\alpha) K_1(\alpha)| d\alpha$.

Similar to (3.12), we have

$$\mathcal{J} \ll Z_j N^{\frac{5}{6}+\varepsilon} + Z_j^{\frac{1}{2}} N^{\frac{13}{12}+\varepsilon}. \tag{3.14}$$

We deduce from Lemma 2.2 and Hua's inequality that

$$\int_{\mathfrak{m}_1} |G(\alpha)|^2 d\alpha \ll Z_1^2 \max_{\alpha\in\mathfrak{m}_1} |g_3^2(\alpha)| \times \left(\int_0^1 |g_2^4(\alpha)| d\alpha \right)^{\frac{1}{2}} \left(\int_0^1 |g_3^8(\alpha)| d\alpha \right)^{\frac{1}{2}}$$

$$\ll Z_1^2 N^{\frac{35}{18}+\varepsilon}. \tag{3.15}$$

On combining Lemma 2.1, (3.13)-(3.15), we have

$$\begin{aligned} I_{11} \ll\ & Z_1 X^{\frac{1}{4}+\varepsilon} N^{\frac{77}{72}+\varepsilon} + Z_1 X^{\frac{7}{8}+\varepsilon} N^{\frac{5}{6}+\varepsilon} \\ & + Z_1^{\frac{3}{4}} X^{\frac{1}{4}+\varepsilon} N^{\frac{43}{36}+\varepsilon} + Z_1^{\frac{1}{2}} X^{\frac{7}{8}+\varepsilon} N^{\frac{13}{12}+\varepsilon}. \end{aligned} \tag{3.16}$$

Therefore, we obtain from (3.11)-(3.12) and (3.16) that

$$I_1 \ll Z_1 N^{\frac{83}{72}+\varepsilon} + Z_1^{\frac{3}{4}} N^{\frac{23}{18}+\varepsilon} + Z_1^{\frac{1}{2}} N^{\frac{11}{8}+\varepsilon}. \tag{3.17}$$

It follows from (3.10) and (3.17) that

$$Z_1 \ll N^{\frac{4}{9}+\varepsilon}. \tag{3.18}$$

3.2.2. *The estimation of Z_2.* Define $V_k(\alpha) = \frac{N^{\frac{1}{k}}(\log N)^c}{q^{\frac{1}{2}-\varepsilon}(1+N|\lambda|)^{\frac{1}{2}}}$. On making use of Lemma 2.3, it is readily seen that

$$\begin{aligned} g_2(\alpha) &\ll N^{\frac{43}{120}+\varepsilon} + (\log N) \max_{N^{\frac{43}{120}} \le X \le N^{\frac{1}{2}}} |g_2(\alpha, X)| \\ &\ll N^{\frac{43}{120}+\varepsilon} + V_2(\alpha). \end{aligned} \tag{3.19}$$

Similarly, we have

$$\begin{aligned} g_3(\alpha) &\ll N^{\frac{4}{15}+\varepsilon} + (\log N) \max_{N^{\frac{4}{15}} \le X \le N^{\frac{1}{3}}} |g_3(\alpha, X)| \\ &\ll N^{\frac{4}{15}+\varepsilon} + V_3(\alpha). \end{aligned} \tag{3.20}$$

It follows from (3.19)-(3.20) that

$$\begin{aligned} I_2 &\ll N^{\frac{43}{120}+\varepsilon} \int_{\mathfrak{m}_2} |g_3^5(\alpha) K_2(\alpha)| d\alpha + \int_{\mathfrak{m}_2} |V_2(\alpha) g_3^5(\alpha) K_2(\alpha)| d\alpha \\ &\ll N^{\frac{139}{120}+\varepsilon} \int_{\mathfrak{m}_2} |g_3^2(\alpha) K_2(\alpha)| d\alpha + N^{\frac{43}{120}+\varepsilon} \int_{\mathfrak{m}_2} |g_3^2(\alpha) V_3^3(\alpha) K_2(\alpha)| d\alpha \\ &\quad + N^{\frac{4}{5}+\varepsilon} \int_{\mathfrak{m}_2} |V_2(\alpha) g_3^2(\alpha) K_2(\alpha)| d\alpha + \int_{\mathfrak{m}_2} |V_2(\alpha) V_3^3(\alpha) g_3^2(\alpha) K_2(\alpha)| d\alpha \\ &\ll I_{21} + I_{22} + I_{23} + I_{24}. \end{aligned} \tag{3.21}$$

We have

$$\begin{aligned} I_{21} &\ll Z_2 N^{\frac{139}{120}+\varepsilon} \left(\int_{\mathfrak{m}_2} 1 d\alpha \right)^{\frac{3}{4}} \left(\int_0^1 |g_3(\alpha)|^8 d\alpha \right)^{\frac{1}{4}} \\ &\ll Z_2 N^{\frac{43}{40}+\varepsilon}, \end{aligned} \tag{3.22}$$

where $|K_2(\alpha)| \ll Z_2$, Hölder's inequality and Hua's inequality are employed. Moreover, by the obvious estimate $|V_2(\alpha)| \le N^{\frac{1}{2}} Q_0^{-A}$, $|V_3(\alpha)| \ll N^{\frac{1}{3}}$ and the arguments similar to that leading to (2.2), we get

$$I_{22} \ll Z_2 N^{\frac{43}{120}+\varepsilon}(N^{\frac{1}{3}})^3 \sum_{q\le N^{\frac{1}{3}}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{|\alpha-\frac{a}{q}|\le\frac{1}{qN^{\frac{2}{3}}}} \frac{|\sum_{p\le N^{\frac{1}{3}}} e(p^3\alpha)|^2(\log N)^c}{q(1+N|\alpha-\frac{a}{q}|)}d\alpha$$

$$\ll Z_2 N^{\frac{41}{40}+\varepsilon}, \tag{3.23}$$

$$I_{23} \ll Z_2 N^{\frac{4}{5}+\varepsilon}(N^{\frac{1}{2}}) \sum_{q\le N^{\frac{1}{3}}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{|\alpha-\frac{a}{q}|\le\frac{1}{qN^{\frac{2}{3}}}} \frac{|\sum_{p\le N^{\frac{1}{3}}} e(p^3\alpha)|^2(\log N)^c}{q(1+N|\alpha-\frac{a}{q}|)}d\alpha$$

$$\ll Z_2 N^{\frac{29}{30}+\varepsilon}, \tag{3.24}$$

and

$$I_{24} \ll \max_{\alpha\in\mathfrak{m}_2} |V_2(\alpha)|(N^{\frac{1}{3}})^3 \sum_{q\le N^{\frac{1}{3}}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{|\alpha-\frac{a}{q}|\le\frac{1}{qN^{\frac{2}{3}}}} \frac{|\sum_{p\le N^{\frac{1}{3}}} e(p^3\alpha)|^2(\log N)^c}{q(1+N|\alpha-\frac{a}{q}|)}d\alpha$$

$$\ll Z_2 N^{\frac{7}{6}} Q_0^{-A}. \tag{3.25}$$

A combination of (3.21)-(3.25) then yields

$$I_2 \ll \frac{Z_2 N^{\frac{7}{6}}}{\log^A N}. \tag{3.26}$$

4. Proof of Theorem 1

Let $Z(N)$ denote the number of integers n in the interval $[\frac{N}{2}, N]$ such that the asymptotic formula (1.6) fails to hold. On recalling (3.18) and (3.26), we arrive at the conclusion that

$$Z(N) \le Z_1 + Z_2 \ll N^{\frac{4}{9}+\varepsilon},$$

and

$$\begin{aligned}\mathcal{E}(N) &\le N^{\frac{4}{9}+\varepsilon} + \sum_{0\le j\le J} Z\left(\frac{N}{2^j}\right)\\ &\ll N^{\frac{4}{9}+\varepsilon},\end{aligned}$$

where J is chosen in such a way that $2^{J-1} < N^{\frac{5}{9}} \le 2^J$. Now the proof of Theorem 1 is completed.

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