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# THE IDEAL CLASS GROUP OF POLYNOMIAL OVERRINGS OF THE RING OF INTEGERS

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Abstract. Let $D$ be an integral domain with quotient field $K$, $Pic(D)$ be the ideal class group of $D$, and $X$ be an indeterminate. A polynomial overring of $D$ means a subring of $K[X]$ containing $D[X]$. In this paper, we study almost Dedekind domains which are polynomial overrings of a principal ideal domain $D$, defined by the intersection of $K[X]$ and rank-one discrete valuation rings with quotient field $K(X)$, and their ideal class groups. Next, let $\mathbb{Z}$ be the ring of integers, $\mathbb{Q}$ be the field of rational numbers, and $\mathfrak{G}_f$ be the set of finitely generated abelian groups (up to isomorphism). As an application, among other things, we show that there exists an overring $R$ of $\mathbb{Z}[X]$ such that (i) $R$ is a Bezout domain, (ii) $R\cap\mathbb{Q}[X]$ is an almost Dedekind domain, (iii) $Pic(R\cap\mathbb{Q}[X]) = \bigoplus_{G\in\mathfrak{G}_f} G$, (iv) for each $G \in \mathfrak{G}_f$, there is a multiplicative subset $S$ of $\mathbb{Z}$ such that $R_S \cap \mathbb{Q}[X]$ is a Dedekind domain with $Pic(R_S \cap \mathbb{Q}[X]) = G$, and (v) every invertible integral ideal $I$ of $R \cap \mathbb{Q}[X]$ can be written uniquely as $I = X^n Q_1^{e_1} \cdots Q_k^{e_k}$ for some integer $n \geq 0$, maximal ideals $Q_i$ of $R\cap\mathbb{Q}[X]$, and integers $e_i \neq 0$. We also completely characterize the almost Dedekind polynomial overrings of $\mathbb{Z}$ containing $\operatorname{Int}(\mathbb{Z})$.

## Introduction

Let $D$ be an integral domain with quotient field $K$, $\operatorname{Max}(D)$ be the set of maximal ideals of $D$, $\dim(D)$ denote the (Krull) dimension of $D$, $Pic(D)$ be the ideal class group of $D$, $X$ be an indeterminate over $D$, and $D[X]$ be the polynomial ring over $D$. An *overring* of $D$ means a subring of $K$ containing $D$ and a subring of $K[X]$ containing $D[X]$ will be called a *polynomial overring of $D$*. For example, let $\operatorname{Int}(D)$ be the ring of integer-valued polynomials, i.e., $\operatorname{Int}(D) = \{f \in K[X] \mid f(a) \in D \text{ for all } a \in D\}$, then $\operatorname{Int}(D)$ is an overring of $D[X]$ and a polynomial overring of $D$.

An integral domain is a *rank-one discrete valuation ring* (DVR) if it is a Noetherian valuation domain which is not a field; so $D$ is a DVR if and only if $D$ is a principal ideal domain (PID) with a unique nonzero maximal ideal.

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We say that $D$ is an almost Dedekind domain if $D_M$ is a DVR for all nonzero maximal ideals $M$ of $D$. Hence, $D$ is a Dedekind domain if and only if $D$ is a Noetherian almost Dedekind domain. The ideal class group of a Dedekind domain measures how far it is from being a unique factorization domain (UFD). For example, if $D$ is a Dedekind domain with $|Pic(D)| = n < \infty$, then the $n$th power of every nonzero nonunit of $D$ can be written as a finite product of primary elements that are irreducible [13, Proposition 6.8]; so $n = 1$ if and only if $D$ is a UFD. It is worthwhile to note that a PID is just a UFD of dimension at most one, whence a Dedekind domain is a UFD if and only if it is a PID.

Let $G$ be an arbitrary abelian group. Claborn proved that there is a Dedekind domain $D$ with $Pic(D) = G$ [8, Theorem 7]. Leeham-Green showed that the Dedekind domain with ideal class group $G$ can be constructed to be a quadratic extension of a PID [23, Theorem 2.1]. Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Q}$ be the field of rational numbers, and $X$ be an indeterminate over $\mathbb{Q}$. In [9, Corollary], Eakin and Heinzer showed that if $G$ is finitely generated, then there is a Dedekind domain $R$ with $\mathbb{Z}[X] \subsetneq R \subsetneq \mathbb{Q}[X]$ and $Pic(R) = G$. It is also known that $\text{Int}(\mathbb{Z})$ is a two-dimensional Prüfer domain such that $\mathbb{Z}[X] \subsetneq \text{Int}(\mathbb{Z}) \subsetneq \mathbb{Q}[X]$ and $Pic(\text{Int}(\mathbb{Z}))$ is a free abelian group on a countably infinite basis [5, Chapters VI and VIII]. Motivated by these results, in this paper, we study the class group of polynomial overrings of a UFD defined by the intersection of DVRs. As an application, we construct almost Dedekind polynomial overrings of $\mathbb{Z}$ with very specific types of ideal class groups.

This paper consists of five sections including introduction. In Section 1, for easy reference of the reader, we review some definitions, notations, and preliminary results about the $t$-operation on integral domains, Krull domains, the class group of integral domains, almost and $t$-almost Dedekind domains, finitely generated abelian groups, and the valuation on a field. A reader who is familiar with these notions can skip Section 1.

Let $D$ be a UFD, $\mathfrak{P} = \{p_i \mid i \in \Lambda\}$ be the set of prime elements of $D$ (up to associates, i.e., if $p \in D$ is a prime element, then there is a unique $p_i \in \mathfrak{P}$ so that $p_iD = pD$), $\{V_{ij}^*\}_{j=1}^{k_i}$ be a finite collection of distinct extensions of $D_{p_iD}$ to $K(X)$ containing $D[X]$ such that $V_{ij}^*$ is a DVR with maximal ideal $M_{ij}$ for all $i \in \Lambda$ and $j = 1, \ldots, k_i$, and $R = (\bigcap_{i\in\Lambda}(\bigcap_{j=1}^{k_i} V_{ij}^*)) \cap K[X]$. For each $i, j$, let $e_{ij}$ be the ramification index of $V_{ij}^*$ over $V_i$, i.e., $p_iV_{ij}^* = (M_{ij})^{e_{ij}}$. Set $m = \sum_i k_i$ ($m = |\Lambda|$ when $|\Lambda| = \infty$), and let $G$ be the free abelian group of rank $m$ on the generators $g_{ij}$ and $H$ be the subgroup of $G$ generated by the elements $r_i = \sum_{j=1}^{k_i} e_{ij}g_{ij}$ for all $i, j$. In Section 2, we first show that $R$ is a $t$-almost Dedekind domain. Furthermore, if each $M_{ij} \cap R$ is a height-one prime ideal, then $Cl(R) = G/H$ (see Sections 1.3 and 1.4 for the definitions of $Cl(R)$ and $t$-almost Dedekind domains), $Cl(R)$ is generated by the classes of $M_{ij} \cap R$, and $\sum_j e_{ij}cl(M_{ij} \cap R) = 0$ in $Cl(R)$ for all $i \in \Lambda$ and $j = 1, \ldots, k_i$. We also prove that if each $M_{ij} \cap R$ is a height-one maximal ideal of $R$, then $R$

is an almost Dedekind domain. We finally use these results to show that if $D$ is a PID and $V_{ij}^*/M_{ij}$ is algebraic over $D/p_iD$ for all $i, j$, then $R$ is an almost Dedekind domain, $Pic(R) = G/H$, and $Pic(R)$ is generated by the classes of $M_{ij} \cap R$ for all $i \in \Lambda$ and $j = 1, \dots, k_i$.

As an application of the results of Section 2, in Section 3, we construct two types of interesting almost Dedekind domains. Firstly, we show that if $\alpha$ is a countable cardinal number, there is an almost Dedekind domain $R$ such that (i) $R$ is a Bezout domain with $\mathrm{Int}(\mathbb{Z}) \subseteq R \subseteq \mathbb{Q}(X)$, (ii) $R \cap \mathbb{Q}[X]$ is an almost Dedekind domain, (iii) $Pic(R \cap \mathbb{Q}[X])$ is a free abelian group of rank $\alpha$, (iv) every maximal ideal of $R \cap \mathbb{Q}[X]$ except $X\mathbb{Q}[X] \cap R$ is invertible, (v) if $I$ is a nonzero finitely generated ideal of $R \cap \mathbb{Q}[X]$, then $I = X^n Q_1^{e_1} \cdots Q_k^{e_k}$ for some integer $n \geq 0$, $Q_i \in \mathrm{Max}(R \cap \mathbb{Q}[X])$, and integers $e_i \neq 0$, and (vi) $Pic(R \cap \mathbb{Q}[X])$ is generated by the classes of maximal ideals $Q$ with $Q \cap \mathbb{Z} \neq (0)$. Let $\mathfrak{G}$ be an infinite set of finitely generated abelian groups (up to isomorphism). Secondly, we construct a Bezout overring $R$ of $\mathbb{Z}[X]$ such that (i) $R \cap \mathbb{Q}[X]$ is an almost Dedekind domain, (ii) $Pic(R \cap \mathbb{Q}[X]) = \bigoplus_{G \in \mathfrak{G}} G$, and (iii) for any $G \in \mathfrak{G}$, there is a multiplicative set $S$ of $\mathbb{Z}$ such that $R_S \cap \mathbb{Q}[X]$ is a Dedekind domain with $Pic(R_S \cap \mathbb{Q}[X]) = G$. We also note that two special cases of $\mathfrak{G}$ are the set of finitely generated abelian groups and the set of finite abelian groups.

Finally, in Section 4, we completely characterize the almost Dedekind polynomial overrings of $\mathbb{Z}$ containing $\mathrm{Int}(\mathbb{Z})$.

## 1. The *t*-operation and Krull domains

For better understanding of the results of this paper, in this section, we review some definitions and results on Krull domains. An integral domain $D$ is called a *Krull domain* if there exists a family $\mathcal{F} = \{V_\lambda\}_{\lambda \in \Lambda}$ of valuation overrings of $D$ such that

(i) $D = \bigcap_\lambda V_\lambda$,

(ii) each $V_\lambda$ is a DVR, and

(iii) $D = \bigcap_\lambda V_\lambda$ has finite character, i.e., each $0 \neq x \in D$ is a nonunit in only finitely many $V_\lambda$'s;

in this case, $\mathcal{F}$ is called a defining family for $D$. Clearly, $\Lambda = \emptyset$ if and only if $D$ is a field; and hence a field can be considered as a Krull domain. It is well known that if $D$ is not a field, then $D$ is a Dedekind domain if and only if $D$ is a Krull domain with $\dim(D) = 1$ [16, Theorem 43.16].

### 1.1. The *t*-operation

Let $D$ be an integral domain with quotient field $K$. A $D$-submodule $A$ of $K$ is called a fractional ideal of $D$ if $dA \subseteq D$ for some $0 \neq d \in D$. Let $F(D)$ be the set of nonzero fractional ideals of $D$. For $A \in F(D)$, let $A^{-1} = \{x \in K \mid xA \subseteq D\}$; then $A^{-1} \in F(D)$. Hence, if we set

- $A_v = (A^{-1})^{-1}$ and
- $A_t = \bigcup\{I_v \mid I \subseteq A, I \in F(D), \text{ and } I \text{ is finitely generated}\}$,

then the $v$- and $t$-operations are well defined.

Let $* = v$ or $t$. The $*$-operation has the following properties for all $0 \neq a \in K$ and $I, J \in F(D)$: (i) $(aD)_* = aD$ and $(aI)_* = aI_*$, (ii) $I \subseteq I_*$, and $I \subseteq J$ implies $I_* \subseteq J_*$, (iii) $(I_*)_* = I_*$, (iv) $(IJ)_* = (IJ_*)_*$, and (v) $I \subseteq I_t \subseteq I_v$, and $I_t = I_v$ if $I$ is finitely generated. An $I \in F(D)$ is called a *$*$-ideal* if $I_* = I$; so by (iii), $I_*$ is a $*$-ideal. Clearly, each nonzero principal ideal is a $*$-ideal and a $v$-ideal is a $t$-ideal. A $*$-ideal is a *maximal $*$-ideal* if it is maximal among proper integral $*$-ideals. Let $*$-Max$(D)$ be the set of maximal $*$-ideals of $D$. It may happen that $v$-Max$(D) = \emptyset$ even though $D$ is not a field as in the case of a rank-one nondiscrete valuation domain $D$. However, by Zorn's lemma, one can show that each proper integral $t$-ideal of $D$ is contained in a maximal $t$-ideal, and hence $D$ is not a field if and only if $t$-Max$(D) \neq \emptyset$. It is also easy to see that each maximal $t$-ideal is a prime ideal; each prime ideal of $D$ minimal over a $t$-ideal is a $t$-ideal; each height-one prime ideal is a $t$-ideal; and $D = \bigcap_{P \in t\text{-Max}(D)} D_P$. The reader can refer to [16, Sections 32 and 34] for basic properties of $v$- and $t$-operations.

An $I \in F(D)$ is said to be *invertible* (resp., $t$-invertible, $v$-invertible) if $II^{-1} = D$ (resp., $(II^{-1})_t = D$, $(II^{-1})_v = D$). Note that $II^{-1} \subseteq (II^{-1})_t \subseteq (II^{-1})_v \subseteq D$. Note also that if $0 \neq a \in K$, then $(aD)(aD)^{-1} = (aD)(a^{-1}D) = D$. Hence, a nonzero principal fractional ideal is invertible; an invertible ideal is $t$-invertible; and a $t$-invertible ideal is $v$-invertible.

**Lemma 1.1.** *The following statements hold for $I \in F(D)$.*

(1) *$I$ is $t$-invertible if and only if $I_t = J_t$ for some finitely generated $J \in F(D)$ and $ID_P$ is principal for all $P \in t$-Max$(D)$.*
(2) *If $I$ is $t$-invertible, then $ID_S$ is $t$-invertible and $(ID_S)_t = I_tD_S$ for a multiplicative set $S$ of $D$.*
(3) *If $I$ is invertible, then $I$ is a $t$-invertible $t$-ideal.*
(4) *If $I$ is a $t$-invertible $t$-ideal, then $I$ is a $v$-invertible $v$-ideal.*

*Proof.* (1) [21, Proposition 2.6]. (2) [30, Lemma 1.4]. (3) [30, Theorem 1.1].

(4) Clearly $I$ is $v$-invertible. Moreover, by (1), $I = J_t$ for some finitely generated $J \in F(D)$. Thus, $I = J_t = J_v = (J_t)_v = I_v$. □

An integral domain $D$ is a *Prüfer domain* (resp., *Prüfer $v$-multiplication domain* (P$v$MD)) if each nonzero finitely generated ideal of $D$ is invertible (resp., $t$-invertible). Clearly, Prüfer domains are P$v$MDs. It is known that $D$ is a Prüfer domain (resp., P$v$MD) if and only if $D_M$ is a valuation domain for all maximal ideals (resp., maximal $t$-ideals) $M$ of $D$ [16, Theorem 22.1] (resp., [18, Theorem 5]). A *Bezout domain* is an integral domain whose finitely generated ideals are principal. An integral domain $D$ is a GCD-domain if $aD \cap bD$ (equivalently, $(a,b)_v$) is principal for all $0 \neq a, b \in D$. Hence,

$$\text{Bezout domain} = \text{Prüfer domain} + \text{GCD-domain},$$

and GCD-domains are P$v$MDs.

## 1.2. Krull domains

Let $X^1(D)$ be the set of height-one prime ideals of an integral domain $D$. It is known that every nonzero ideal of $D$ is $v$-invertible if and only if $D$ is completely integrally closed [16, Theorem 34.4]. The next theorem, which is a very useful characterization of Krull domains, shows when every nonzero ideal of $D$ is $t$-invertible.

**Theorem 1.2.** *The following statements are equivalent for an integral domain $D$.*

(1) *$D$ is a Krull domain.*
(2) (i) $D = \bigcap_{P \in X^1(D)} D_P$, (ii) *$D_P$ is a DVR for all $P \in X^1(D)$, and* (iii) *$D = \bigcap_{P \in X^1(D)} D_P$ has finite character.*
(3) *Every nonzero ideal of $D$ is $t$-invertible.*
(4) *Every nonzero prime ideal of $D$ is $t$-invertible.*
(5) *For each nonzero ideal $I$ of $D$, there are height-one prime ideals $P_1, \dots,$ $P_k$ and positive integers $n_1, \dots, n_k$ such that $I_t = (P_1^{n_1} \cdots P_k^{n_k})_t$.*
(6) *Every nonzero prime ideal of $D$ contains a $t$-invertible prime ideal.*
(7) *$D[X]$ is a Krull domain.*

*In this case, $t = v$, i.e., $I_v = I_t$ for all $I \in F(D)$, $t$-Max$(D) = X^1(D)$, and $\{D_P \mid P \in X^1(D)\}$ is a unique defining family for $D$.*

*Proof.* (1) $\Leftrightarrow$ (2) [13, Proposition 1.9].

(1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) [22, Theorems 3.6 and 3.9].

(1) $\Leftrightarrow$ (7) [13, Proposition 1.6].

In this case, $I_v = I_t$ for all $I \in F(D)$ [22, Theorems 2.1 and 3.2], $t$-Max$(D) =$ $X^1(D)$ [13, Theorem 3.12], and $\{D_P \mid P \in X^1(D)\}$ is a unique defining family for $D$ [16, Corollary 43.9]. □

It is easy to see that $I \in F(D)$ is $t$-invertible if and only if $II^{-1} \not\subseteq P$ for all $P \in t$-Max$(D)$. Hence, by Theorem 1.2, $D$ is a Dedekind domain if and only if $D$ is a Krull domain with $\dim(D) \leq 1$. Moreover, Krull domains are completely integrally closed because $t$-invertible ideals are $v$-invertible.

**Corollary 1.3** ([13, Proposition 1.8])**.** *Let $S$ be a multiplicative set of a Krull domain $D$. Then $D_S$ is a Krull domain.*

*Proof.* Let $Q$ be a nonzero prime ideal of $D_S$. Then $Q \cap D$ is a nonzero prime ideal of $D$ and $Q = (Q \cap D)D_S$. Hence, $(Q \cap D)D_S$ is $t$-invertible by Lemma 1.1 and Theorem 1.2. Thus, by Theorem 1.2, $D_S$ is a Krull domain. □

Let $D$ be a Dedekind domain that is not a field. Then $D[X]$ is a Krull domain that is not a Dedekind domain. The next result gives another example of Krull domains, which is called the Mori-Nagata theorem.

**Proposition 1.4.** *Let $D$ be a Noetherian domain and $R$ be an overring of $D$.*

(1) $\dim(R) \leq \dim(D)$.

(2) *The integral closure of $D$ is a Krull domain.*
(3) *If* $\dim D \leq 2$, *then every Krull overring of $D$ is a Noetherian domain.*

*Proof.* (1) [16, Corollary 3.10]. (2) [27, Theorem 33.10]. (3) [19, Theorem 9]. □

**Proposition 1.5** ([13, Proposition 1.4]). *Let $\{D_\alpha\}$ be a family of Krull overrings of an integral domain $D$ such that $D = \bigcap_\alpha D_\alpha$. If $D = \bigcap_\alpha D_\alpha$ has finite character, then $D$ is a Krull domain.*

*Proof.* This follows directly from the definition of Krull domains. □

### 1.3. The class group of integral domains

Let $T(D)$ be the set of $t$-invertible fractional $t$-ideals of $D$ and $Prin(D)$ be the set of nonzero principal fractional ideals of $D$. Then $T(D)$ is an abelian group under the $t$-multiplication $I * J = (IJ)_t$ [4, Lemme 1] and $Prin(D)$ is a subgroup of $T(D)$. Let

$$Cl_t(D) = T(D)/Prin(D)$$

be the factor group of $T(D)$ modulo $Prin(D)$. Then $Cl_t(D)$ is an abelian group. For $I \in T(D)$, let $cl(I) \in Cl_t(D)$ denote the equivalence class of $T(D)$ containing $I$. Hence, for all $I, J \in T(D)$, we have

(i) $cl(I) = cl(J)$ if and only if $I = xJ$ for some $0 \neq x \in K$,
(ii) $cl(I) + cl(J) = cl((IJ)_t)$ in $Cl_t(D)$.

The notion of $Cl_t(D)$ was introduced by Bouvier [4] and has been studied by many researchers (only a few of them are, for example, [1, 2, 7, 10, 12, 14, 20]). In particular, Bouvier showed that $D$ is a GCD domain if and only if $D$ is a P$v$MD with $Cl_t(D) = \{0\}$ [4, Proposition 2]. Clearly, if $D$ is a Krull domain, then $t = v$ by Theorem 1.2, and hence $Cl_t(D)$ is the usual divisor class group of $D$ (cf. [16, Section 45], [13] or [28] for the divisor class group of Krull domains). Thus, as in [4], we denote $Cl(D) = Cl_t(D)$ and call $Cl(D)$ the *class group* of $D$ which is also called the $t$-class group of $D$ in the literature.

Let $Inv(D)$ be the set of invertible fractional ideals of $D$. Then $Inv(D)$ is a subgroup of $T(D)$ containing $Prin(D)$ by Lemma 1.1, and thus

$$Pic(D) = Inv(D)/Prin(D)$$

is a subgroup of $Cl(D)$ and called the *Picard group* or the *ideal class group* of $D$. Clearly, if $D$ is a Dedekind domain or a Prüfer domain, then $Cl(D)$ is the ideal class group of $D$, i.e., $Cl(D) = Pic(D)$ [4, Lemma 3]. It is clear that $Cl(D)$ measures how far away a Krull domain (resp., P$v$MD) is from a UFD (resp., GCD-domain). For example, a UFD (resp., GCD-domain) is just a Krull domain (resp., P$v$MD) with $Cl(D) = \{0\}$, and a Krull domain with torsion divisor class group is called an almost factorial domain [13, § 6].

### 1.4. Almost and $t$-almost Dedekind domains

An integral domain $D$ is an almost (resp., $t$-almost) Dedekind domain if $D_M$ is a DVR for all nonzero maximal ideals (resp., maximal $t$-ideals) $M$ of $D$. Hence, if $D$ is not a field, then

(i) A Dedekind domain is exactly a Noetherian almost Dedekind domain, or equivalently, a Krull almost Dedekind domain.

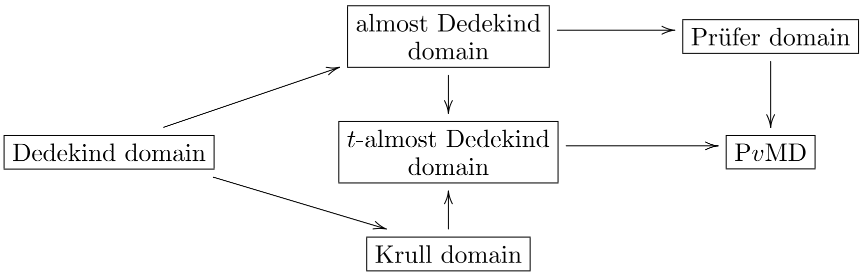
(ii) $D$ is an almost Dedekind domain if and only if $D$ is a $t$-almost Dedekind domain with $\dim(D) = 1$.

(iii) $D$ is a Krull domain if and only if $D$ is a $t$-almost Dedekind domain and the intersection $D = \bigcap_{P \in t\text{-Max}(D)} D_P$ has finite character.

(iv) Almost Dedekind domains are Prüfer domains of Krull dimension one.

Moreover, it is known that if $D$ is a $t$-almost Dedekind domain, then $D$ is a P$v$MD in which each maximal $t$-ideal has height-one [21, Theorems 3.2 and 4.5]. It is also known that if $\text{Int}(D)$ is a Prüfer domain, $D$ is an almost Dedekind domain with finite residue fields [5, Proposition VI.1.5].

A Dedekind domain is an integral domain whose nonzero ideals are invertible. Hence, by Theorem 1.2, we have the following implications:




However, none of the implications is reversible.

Let $c(f)$ be the ideal of $D$ generated by the coefficients of $f \in D[X]$, $N = \{f \in D[X] \mid c(f) = D\}$, and $N_v = \{f \in D[X] \mid f \neq 0 \text{ and } c(f)_v = D\}$. Then, by Dedekind-Mertens lemma [16, Theorem 28.1], $N$ and $N_v$ are saturated multiplicative sets of $D[X]$ with $N \subseteq N_v$. The localization of $D[X]$ at $N$, denoted by $D(X)$, is called the *Nagata ring* of $D$ and $\text{Max}(D(X)) = \{M(X) \mid M \in \text{Max}(D)\}$ [16, Section 33]. As the $t$-operation analog, we say that $D[X]_{N_v}$ is the *$t$-Nagata ring* of $D$. It is clear that $D(X) = D[X]_{N_v}$ if and only if each maximal ideal of $D$ is a $t$-ideal; if $D$ is quasi-local with maximal ideal $M$, then $D(X) = D[X]_{M[X]}$; $D$ is a valuation domain (resp., DVR) if and only if $D(X)$ is a valuation domain (resp., DVR); and if $D$ is a field, then $D(X)$ is the quotient field of $D[X]$. Also, by [21, Theorem 2.14],

$$Pic(D(X)) = Cl(D[X]_{N_v}) = Pic(D[X]_{N_v}) = \{0\},$$

and thus $D$ is a Dedekind (resp., Krull) domain if and only if $D(X)$ (resp., $D[X]_{N_v}$) is a PID [16, Proposition 38.7] (resp., [15, Theorem 2.2]); and in this case, $D[X]_{N_v}$ is a Euclidean domain [11, Proposition 5.1 and Theorem 5.3]. Moreover, $D$ is a $t$-almost Dedekind domain if and only if $D[X]$ is a $t$-almost Dedekind domain, if and only if $D[X]_{N_v}$ is an almost Dedekind domain [21, Theorems 4.2 and 4.4]. However, $Cl(D[X]_{N_v}) = \{0\}$, and hence $Cl(D) \neq Cl(D[X]_{N_v})$ in general.

Let $\{X_\alpha\}$ be an infinite set of indeterminates over $D$, $S$ be the saturated multiplicative set of $D[\{X_\alpha\}]$ generated by all nonconstant prime polynomials, and $R = D[\{X_\alpha\}]_S$. Then $Cl(D) = Cl(R)$ if and only if $D$ is integrally closed; and $D$ is a P$v$MD if and only if $R$ is a Prüfer domain [7, Theorem 3.5]. Thus, if $D$ is a $t$-almost Dedekind domain, then $R$ is an almost Dedekind domain with $Cl(R) = Cl(D)$.

### 1.5. Finitely generated abelian groups

For a cardinal number $\alpha$, let $\mathbb{Z}^\alpha$ denote the (additive) free abelian group of rank $\alpha$, and for an integer $n \geq 1$, let $\mathbb{Z}_n$ be the cyclic group of order $n$, i.e., $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$ as additive groups. Let $G$ be a finitely generated abelian group. Then, by the fundamental theorem of finitely generated abelian groups,

$$G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k} \oplus \mathbb{Z}^n$$

for some positive integers $n_1, \ldots, n_k$ with $n_1 | n_2 | \cdots | n_k$, where $n_1 \geq 2$ if $k \geq 2$, and a nonnegative integer $n$. Hence, in this paper, a finitely generated abelian group means a group of the form $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k} \oplus \mathbb{Z}^n$ for some nonnegative integers $n_1, \ldots, n_k$ and $n$.

*Remark* 1.6. (1) Let $e = (d_1, d_2, \ldots, d_{n+1}) \in \mathbb{Z}^{n+1}$, $d = \gcd(d_1, d_2, \ldots, d_{n+1})$, and $\langle e \rangle$ be the subgroup of $\mathbb{Z}^{n+1}$ generated by $e$. It is known that $\mathbb{Z}^{n+1}/\langle e \rangle \cong \mathbb{Z}_d \oplus \mathbb{Z}^n$. In particular, $\mathbb{Z}^{n+1}/\langle (1, \ldots, 1) \rangle \cong \mathbb{Z}^n$.

(2) Let $\mathfrak{G}_f$ be the set of finitely generated abelian groups, $\mathcal{G}_f$ be the set of finite abelian groups, and $\mathbb{Z}^+ = \{n \in \mathbb{Z} \mid n \geq 0\}$. Clearly, $\aleph_0 \leq |\mathcal{G}_f| \leq |\mathfrak{G}_f|$. Moreover, it is clear that the map $\varphi : \mathfrak{G}_f \to \bigcup_{m=2}^\infty (\mathbb{Z}^+)^m$, given by

$$\varphi(\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k} \oplus \mathbb{Z}^n) = (n_1, \ldots, n_k, n),$$

is injective. Note that $\bigcup_{m=2}^\infty (\mathbb{Z}^+)^m$ is countable. Thus, $\aleph_0 = |\mathcal{G}_f| = |\mathfrak{G}_f|$. In particular, an infinite set of finitely generated abelian groups is countable.

### 1.6. Valuations on a field

Let $G$ be a totally ordered (additive) abelian group. Consider a map $v : K \to G \cup \{\infty\}$ with the following three properties:

(i) $\langle v(K \setminus \{0\}) \rangle = G$ and $v(0) = \infty$.

(ii) $v(xy) = v(x) + v(y)$ for all $x, y \in K$.

(iii) $v(x + y) \geq \inf\{v(x), v(y)\}$ for all $x, y \in K$.

Then $v$ is called a valuation on $K$ with value group $G$ and $v(\frac{a}{b}) = v(a) - v(b)$ for all $a, b \in D$ with $b \neq 0$.

Let $v$ be a valuation on $K$, $V = \{x \in K \mid v(x) \geq 0\}$, and $M = \{x \in K \mid v(x) > 0\}$. It is well known that $V$ is a valuation domain with quotient field $K$,

$M$ is the maximal ideal of $V$, and $V$ is a DVR if and only if the value group of $v$ is isomorphic to the additive group of integers. We say that $V^*$ is an extension of $V$ to $K(X)$ if $V^*$ is a valuation domain with quotient field $K(X)$ such that $V^* \cap K = V$. Clearly, $V(X)$ is an extension of $V$ to $K(X)$.

## 2. The class group of polynomial overrings of a UFD

Let $D$ be an integral domain with quotient field $K$, $X^1(D)$ be the set of height-one prime ideals of $D$, $X$ be an indeterminate over $D$, $D[X]$ be the polynomial ring over $D$, and $K(X)$ be the quotient field of $D[X]$. Throughout this section, we use the following notation.

**Notation 2.1.**

(i) $D$ is a UFD that is not a field.

(ii) $\mathfrak{P} = \{p_i \mid i \in \Lambda\}$ is the set of prime elements of $D$ (up to associates), $V_i = D_{p_i D}$, and $P_i = p_i D$ for all $i \in \Lambda$.

(iii) $\{V_{ij}^*\}_{j=1}^{k_i}$ is a finite collection of distinct extensions of $V_i$ to $K(X)$ containing $D[X]$ such that $V_{ij}^*$ is a DVR with maximal ideal $M_{ij}$ for all $i \in \Lambda$ and $j = 1, \dots, k_i$.

(iv) $R_i = (\bigcap_{j=1}^{k_i} V_{ij}^*) \cap K[X]$ for each $i \in \Lambda$.

(v) $R = \bigcap_{i \in \Lambda} R_i$, i.e., $R = (\bigcap_{i \in \Lambda}(\bigcap_{j=1}^{k_i} V_{ij}^*)) \cap K[X]$.

Clearly, for all $i \in \Lambda$, both $R$ and $R_i$ are polynomial overrings of $D$, $R_i \cap K = V_i$, and $R_i$ is a Noetherian Krull domain of (Krull) dimension $\leq 2$ by Propositions 1.4 and 1.5, because $R_i$ is a finite intersection of Krull domains, $V_i[X] \subseteq R_i$, and $V_i[X]$ is a two-dimensional Noetherian domain.

**Proposition 2.2.** *Let the notation be as in Notation 2.1.*

(1) $X^1(D) = \{P_i \mid i \in \Lambda\}$.

(2) *If* $S_k = D \setminus P_k$ *for* $P_k \in X^1(D)$, *then* $R_{S_k} = R_k$.

(3) $R$ *is a* $t$*-almost Dedekind domain.*

(4) *The intersection* $R = \bigcap_i R_i$ *is irredundant, i.e., no* $R_i$ *is superflous.*

(5) *Suppose that* $\{i \in \Lambda \mid fV_{ij}^* \neq V_{ij}^*$ *for some* $j$ *with* $1 \leq j \leq k_i\}$ *is finite for all* $0 \neq f \in D[X]$. *Then* $R$ *is a Krull domain.*

(6) *Suppose that* $R_i = V_i(X) \cap K[X]$ *for all but finitely many* $i$ *in* $\Lambda$ (*e.g.,* $|X^1(D)| < \infty$). *Then* $R$ *is a Krull domain.*

*Proof.* (1) Clear.

(2) Let $f \in R_k$. Then $f \in K[X]$, and since $D$ is a UFD, there is an $s \in S_k$ such that $sf \in D_{P_i}[X]$ for all $P_i \in X^1(D)$ with $P_i \neq P_k$. Note that $D_{P_i}[X] \subseteq R_i$. Hence,

$$sf \in \bigcap_{P_i \in X^1(D)} R_i = R,$$

and thus $f \in R_{S_k}$. The reverse containment is clear. Therefore, $R_{S_k} = R_k$.

(3) Let $Q$ be a maximal $t$-ideal of $R$. If $Q \cap D = (0)$, then $K[X] \subseteq R_Q$, and hence $R_Q$ is a DVR. Now, assume that $Q \cap D \neq (0)$. Then $Q \cap D$ contains

a prime element of $D$, say, $p_k \in \mathfrak{P}$. Note that each $R_i$ is a Krull domain and $R_i = R_{S_i}$ by (2); so there is a set $\Omega$ of height-one prime ideals of $R$ such that $R = \bigcap_{P\in\Omega} R_P$. Note also that each height-one prime ideal in $\Omega$ does not contain two nonassociate prime elements of $D$. Hence, if $Q \cap D$ contains another prime element $q$ of $D$, then

$$R \supseteq (p_k, q)_v \supseteq \bigcap_{P\in\Omega} (p_k, q)R_P = \bigcap_{P\in\Omega} R_P = R$$

(cf. [16, Theorems 32.5 and 34.1] for the second containment), whence $R = (p_k, q)_v \subseteq Q \subsetneq R$, a contradiction. Thus, $QR_{S_k} \subsetneq R_{S_k} = R_k$ by (2), and hence $\text{ht}(Q) = \text{ht}(QR_{S_k}) \leq 2$. Assume that $\text{ht}(Q) = 2$. Since $R_k$ is a Krull domain, $\sqrt{p_kR_k}$ is an intersection of finitely many height-one prime ideals, say, $P_1, \dots, P_l$. Note that $\text{ht}(P_i \cap R) = 1$ for $i = 1, \dots, l$; so we can choose $f \in Q \setminus \bigcup(P_i \cap R)$. Then

$$R \supseteq (p_k, f)_v \supseteq \bigcap_{P\in\Omega} (p_k, f)R_P = \bigcap_{P\in\Omega} R_P = R.$$

Hence, $R = (p_k, f)_v \subseteq Q$, a contradiction. Thus, $\text{ht}(Q) = 1$. Therefore, $R_Q = (R_{S_k})_{QR_{S_k}}$ is a DVR.

(4) Note that if $T_k = \bigcap_{i\neq k} R_i$ for $k \in \Lambda$, then

$$\begin{aligned} R \cap K &= \bigcap_{i\in\Lambda}(R_i \cap K) = \bigcap_{P_i\in X^1(D)} D_{P_i} \\ &\subsetneq \bigcap_{P_i\in X^1(D)\setminus\{P_k\}} D_{P_i} = \bigcap_{i\neq k}(R_i \cap K) = (\bigcap_{i\neq k} R_i) \cap K \\ &= T_k \cap K, \end{aligned}$$

where the proper containment follows because $D$ is a Krull domain, and hence $T_k \neq R$. Thus, the intersection is irredundant.

(5) Let $R'_i = \bigcap_{j=1}^{k_i} V^*_{ij}$ for all $i \in \Lambda$. Then $R'_i$ is a semilocal PID, $K[X]$ is a PID, $R = (\bigcap_{i\in\Lambda} R'_i) \cap K[X]$, and this intersection has finite character by assumption. Thus, $R$ is a Krull domain by Proposition 1.5.

(6) This is an immediate consequence of (5) above, because every nonzero $f \in D[X]$ is contained in $P_i(X)$ for only finitely many $i$ in $\Lambda$. □

**Corollary 2.3.** *Let the notation be as in Notation 2.1, $Q_{ij} = M_{ij} \cap R$, and assume that $htQ_{ij} = 1$ for all $i \in \Lambda$ and $j = 1, \dots, k_i$.*

(1) *$V^*_{ij} = R_{Q_{ij}}$ for all $i, j$. Hence, $Q_{ij}$'s are all distinct.*
(2) *The intersection $R = (\bigcap_{i,j} V^*_{ij}) \cap K[X]$ is irredundant, i.e., no $V^*_{ij}$ is superflous.*
(3) *$fK[X] \cap R = fR$ for all $f \in R \setminus \bigcup_{ij} Q_{ij}$.*
(4) *$t$-Max$(R) = \{Q_{ij} \mid i, j\} \cup \{fK[X] \cap R \mid f \in K[X]$ is irreducible$\}$.*
(5) *$Q_{ij}$ is $t$-invertible for all $i, j$.*

(6) *Let $A$ be an integral $t$-ideal of $R$ with $A \cap D \neq (0)$. Then $A = (\prod_{i,j} Q_{ij}^{c_{ij}})_t$ for some integers $c_{ij} \geq 0$ such that only finitely many $c_{ij}$'s are nonzero.*

(7) *If $Q_{ij}$ is a maximal ideal for all $i, j$, then $R$ is an almost Dedekind domain.*

*Proof.* (1) Clearly, $R_{Q_{ij}} \subseteq (V_{ij}^*)_{M_{ij}} = V_{ij}^*$, and since $R_{Q_{ij}}$ is a DVR by Proposition 2.2(3), $R_{Q_{ij}} = V_{ij}^*$.

(2) For each $m \in \Lambda$ and $j = 1, \dots, k_m$, note that $\bigcap_{j' \neq j} Q_{mj'} \nsubseteq Q_{mj}$ by (1). Hence, if $f \in \bigcap_{j' \neq j} Q_{mj'} \setminus Q_{mj}$, then $\frac{f^n}{p_m} \in (\bigcap_{j' \neq j} V_{mj'}) \cap (\bigcap_{k \neq m} R_k)$ for some integer $n \geq 1$, while $\frac{f^n}{p_m} \notin V_{mj}$. Thus, the intersection is irredundant.

(3) Note that $fV_{ij}^* = V_{ij}^*$ for all $i, j$. Hence,

$$fR = (\bigcap_{i,j} fV_{ij}^*) \cap fK[X] = (\bigcap_{i,j} V_{ij}^*) \cap fK[X] = R \cap fK[X] \supseteq fR.$$

Thus, $fK[X] \cap R = fR$.

(4) By Proposition 2.2(3), every maximal $t$-ideal of $R$ has height-one, and hence $t$-Max$(R) = X^1(R)$. Now, let $Q \in t$-Max$(R)$. There are only two cases that we have to consider. First, if $Q \cap D \neq (0)$, then $p_k R \subseteq Q$ for some $p_k \in \mathfrak{P}$. Note that

$$\sqrt{p_k R} = (\bigcap_{i,j} \sqrt{p_k V_{ij}^*}) \cap \sqrt{p_k K[X]} = (\bigcap_{j} M_{kj}) \cap R = \bigcap_{j} Q_{kj};$$

so $Q_{kj} \subseteq Q$ for some $j$. Thus, $Q = Q_{kj}$. Now, assume $Q \cap D = (0)$. Then $K[X] \subseteq R_Q$, and hence $R_Q = K[X]_{fK[X]}$ for some irreducible $f \in K[X]$. Thus, $Q = fK[X] \cap R$. For the reverse containment, note that each $Q_{ij}$ has height-one, and thus $Q_{ij} \in t$-Max$(R)$. Next, let $f \in K[X]$ be an irreducible polynomial, and put $P = fK[X] \cap R$. Then $P \cap D = (0)$, and hence $P_{D \setminus \{0\}} = fK[X]$. Hence, ht$P = \text{ht}(P_{D \setminus \{0\}}) = 1$. Thus, $P \in t$-Max$(R)$.

(5) Let $p_i \in \mathfrak{P}$ be a prime element such that $p_i \in Q_{ij}$. Note that $R_{Q_{ij}}$ is a DVR and $Q_{ij} \nsubseteq \bigcup_{j' \neq j} Q_{ij'}$ by (1). Hence, there is an $f \in Q_{ij} \setminus \bigcup_{j' \neq j} Q_{ij'}$ such that $Q_{ij} R_{Q_{ij}} = fR_{Q_{ij}}$, whence $Q_{ij} R_Q = (p_i, f)_v R_Q$ for all $Q \in t$-Max$(R)$ by (4). Thus, $Q_{ij} = (p_i, f)_v$ [21, Proposition 2.8], and since $R$ is $t$-almost Dedekind by Proposition 2.2(3), $Q_{ij}$ is $t$-invertible.

(6) Note that $A \cap D$ is contained in only finitely many height-one prime ideals of $D$. Hence, by (4), $A$ is contained in only finitely many maximal $t$-ideals of $R$ which must be $Q_{ij}$. Note also that $AR_{Q_{ij}} = (Q_{ij} R_{Q_{ij}})^{c_{ij}}$ for some integer $c_{ij} \geq 0$ and only finitely many $c_{ij}$'s are nonzero. Hence, if $I = (\prod_{i,j} Q_{ij}^{c_{ij}})_t$, then $AR_Q = IR_Q$ for all $Q \in t$-Max$(R)$ by Lemma 1.1(2). Thus, $A = I$ [21, Proposition 2.8].

(7) Note that a $t$-almost Dedekind domain is an almost Dedekind domain if and only if its Krull dimension is at most one; so it suffices to show that each maximal ideal of $R$ has height-one by Proposition 2.2(3). Let $Q$ be a maximal ideal of $R$ and $S = D \setminus \{0\}$. If $Q \cap S = \emptyset$, then $K[X] = R_S \supsetneq QR_S = QK[X]$,

and since $K[X]$ is a PID, $\text{ht}Q = \dim(QR_S) = 1$. Next, assume that $Q \cap S \neq \emptyset$; so $p \in Q$ for some $0 \neq p \in D$. Since $R$ is a $t$-almost Dedekind domain, there is a height-one prime ideal $P$ of $R$ such that $pR \subseteq P \subseteq Q$. Hence, by (4), $P = Q_{ij}$ for some $i, j$, and hence $Q = P$ by assumption. Thus, $\text{ht}(Q) = 1$. □

Let $P \in X^1(D)$ be $t$-invertible. It is easy to see that $(PD_P)^n \cap D = (P^n)_t$ for any integer $n \geq 1$ and $\{(P^n)_t \mid n \geq 1\}$ is the set of $P$-primary ideals of $D$; hence $(PD_P)^n = (P^n)_t D_P$ (cf. [22, Theorem 2.2]). Furthermore, if $n \geq 1$ is an integer, then $P^{-n}$ means $(P^n)^{-1}$ which is equal to $((P^{-1})^n)_t$. Thus, $(PD_P)^{-n} = P^{-n}D_P = (P^{-n})_t D_P$. We are now ready to prove the main result of this paper.

**Theorem 2.4.** *Let the notation be as in Notation 2.1. For each $i, j$, let $e_{ij}$ be the ramification index of $V_{ij}^*$ over $V_i$. Set $m = \sum_i k_i$ ($m = |X^1(D)|$ when $|X^1(D)| = \infty$), and let $G$ be the free abelian group of rank $m$ on the generators $g_{ij}$ and $H$ be the subgroup of $G$ generated by the elements $r_i = \sum_{j=1}^{k_i} e_{ij} g_{ij}$ for all $i \in \Lambda$ and $j = 1, \ldots, k_i$. If $ht(M_{ij} \cap R) = 1$ for all $i, j$, then*

(1) *$R$ is a $t$-almost Dedekind domain.*
(2) *$Cl(R) = G/H$.*
(3) *$Cl(R)$ is generated by the classes of $M_{ij} \cap R$ and*

$$\sum_{j=1}^{k_i} e_{ij} cl(M_{ij} \cap R) = 0 \text{ in } Cl(R)$$

*for all $i \in \Lambda$ and $j = 1, \ldots, k_i$.*

(4) *$Cl(R)$ is torsion-free if and only if $\gcd(e_{i1}, e_{i2}, \ldots, e_{ik_i}) = 1$ for all $i \in \Lambda$. In this case, $Cl(R)$ is a free abelian group of rank $\sum_{i \in \Lambda}(k_i - 1)$.*

*Proof.* (1) By Proposition 2.2(3), $R$ is a $t$-almost Dedekind domain.

(2) Let $Q_{ij}$ be the center of $V_{ij}^*$ on $R$, i.e., $Q_{ij} = M_{ij} \cap R$ for all $i, j$. Then $Q_{ij}$ is a $t$-invertible maximal $t$-ideal of $R$ by Corollary 2.3(4)-(5). Thus, the subgroup of $T(R)$ generated by $\{Q_{ij}\}$ is isomorphic to $G$ by letting $Q_{ij} \leftrightarrow g_{ij}$. Note that $p_i R = (\prod_{j=1}^{k_i} Q_{ij}^{e_{ij}})_t$ for each prime element $p_i \in \mathfrak{P}$, and hence $H$ is isomorphic to the subgroup of $Prin(R)$ generated by $\{pR \mid p \in \mathfrak{P}\}$.

We first claim that $T(R) = G + Prin(R)$. Clearly, $G + Prin(R) \subseteq T(R)$. For the reverse containment, let $A \in T(R)$ and $S = D \setminus \{0\}$. Clearly, $R_S = K[X]$, and hence $AR_S = uK[X]$ for some $0 \neq u \in K(X)$ because $K[X]$ is a PID. Hence, $u^{-1}AK[X] = K[X]$, and since $u^{-1}A$ is $t$-invertible and $D[X] \subseteq R$, $du^{-1}A \subseteq R$ for some $0 \neq d \in D$. Thus, by Corollary 2.3(6), $du^{-1}A = (\prod_{i,j} Q_{ij}^{c_{ij}})_t$ for some nonnegative integers $c_{ij}$; so $A = ((\prod_{i,j} Q_{ij}^{c_{ij}})_t(d^{-1}uR))_t$. Thus, $T(R) \subseteq G + Prin(R)$, and hence $T(R) = G + Prin(R)$. Therefore,

$$Cl(R) = T(R)/Prin(R) = (G + Prin(R))/Prin(R) = G/G \cap Prin(R).$$

Hence, it suffices to show that $H = G \cap Prin(R)$. Clearly, $H \subseteq G \cap Prin(R)$. For the reverse containment, let $aR = (\prod_{i,j} Q_{ij}^{n_{ij}})_t \in G \cap Prin(R)$, where

$0 \neq a \in K(X)$ and $n_{ij}$ is an integer (note that $n_{ij} = 0$ except finitely many $i$). Then $a \in K$, because $aK[X] = aR_S = (\prod_{i,j} Q_{ij}^{n_{ij}})_t R_S = (\prod_{i,j}(Q_{ij}^{n_{ij}} R_S))_t = K[X]$ by Lemma 1.1(2). So if $aV_i = (P_i)^{\alpha_i}$ for some integer $\alpha_i$, then $aV_{ij}^* = ((M_{ij})^{e_{ij}})^{\alpha_i}$. Note that $V_{ij}^* = R_{Q_{ij}}$ by Corollary 2.3(1), and hence

$$((M_{ij})^{e_{ij}})^{\alpha_i} = (((Q_{ij})^{e_{ij}})^{\alpha_i})_t R_{Q_{ij}}$$

by the remark before Theorem 2.4; thus

$$aR = (\prod_{i,j}(Q_{ij}^{e_{ij}})^{\alpha_i})_t = (\prod_{i\in\Lambda}((\prod_{j=1}^{k_i} Q_{ij}^{e_{ij}})_t)^{\alpha_i})_t = \prod_{i\in\Lambda}(p_i R)^{\alpha_i}.$$

Therefore, $G \cap Prin(R) \subseteq H$.

(3) Let $A \in T(R)$. By the proof of (2), $A = d^{-1}u(\prod_{i,j} Q_{ij}^{c_{ij}})_t$ for some $d, u \in K(X)$ and $Q_{ij}$. Thus, $cl(A) = cl((\prod_{i,j} Q_{ij}^{c_{ij}})_t) = \sum_{i,j} c_{ij} cl(Q_{ij})$. Moreover, note that $p_i R = (\prod_j Q_{ij}^{e_{ij}})_t$. Thus, $\sum_{j=1}^{k_i} e_{ij} cl(M_{ij} \cap R) = 0$ in $Cl(R)$.

(4) It is obvious that $(e_{i1}, e_{i2}, \ldots, e_{ik_i})$ is an element of $\mathbb{Z}^{k_i}$ for all $i \in \Lambda$ and

$$G/H = \bigoplus_{i\in\Lambda} \mathbb{Z}^{k_i}/\langle(e_{i1}, e_{i2}, \ldots, e_{ik_i})\rangle.$$

Hence, if $d_i = \gcd(e_{i1}, e_{i2}, \ldots, e_{ik_i})$ for each $i$, then $G/H = \bigoplus_i(\mathbb{Z}_{d_i} \oplus \mathbb{Z}^{k_i - 1})$ by Remark 1.6. Thus, $Cl(R)$ is torsion-free if and only if $d_i = 1$ for all $i \in \Lambda$. In this case, $Cl(R)$ is a free abelian group of rank $\sum_{i\in\Lambda}(k_i - 1)$. □

*Remark* 2.5. (1) Theorem 2.4 is not true when $D$ is not a UFD. For example, let $D$ be a Krull domain with non-trivial class group. Then

$$D[X] = (\bigcap_{P\in X^1(D)} D_P(X)) \cap K[X]$$

and $Cl(D[X]) = Cl(D) \neq \{0\}$ [13, Theorem 8.1]. However, if Theorem 2.4 is true of Krull domains, then $Cl(D[X]) = \{0\}$ by Theorem 2.4(3), a contradiction.

(2) If $\{V_{ij}\}_{j=1}^{k_i}$ is not finite, then Theorem 2.4 does not hold. For a concrete example, see Example 3.3.

It is known that if $A$ is an intersection of finitely many DVRs with the same quotient field, then $A$ is a semilocal PID [16, Theorem 22.8]. Thus, if $D$ is a semilocal PID, the next result is the Eakin-Heinzer's result [9, Theorem].

**Corollary 2.6.** *Let the notation be as in Notation 2.1. For each $i, j$, let $e_{ij}$ be the ramification index of $V_{ij}^*$ over $V_i$. Set $m = \sum_i k_i$ ($m = |X^1(D)|$ when $|X^1(D)| = \infty$), and let $G$ be the free abelian group of rank $m$ on the generators $g_{ij}$ and $H$ be the subgroup of $G$ generated by the elements $r_i = \sum_{j=1}^{k_i} e_{ij} g_{ij}$ for all $i \in \Lambda$ and $j = 1, \ldots, k_i$. Assume further that $D$ is a PID and $V_{ij}^*/M_{ij}$ is algebraic over $D/P_i$ for all $i \in \Lambda$ and $j = 1, \ldots, k_i$. Then*

(1) *$R$ is an almost Dedekind domain with $Cl(R) = G/H$.*

(2) *$Cl(R)$ is generated by $M_{ij} \cap R$ and*

$$\sum_{j=1}^{k_i} e_{ij} cl(M_{ij} \cap R) = 0 \text{ in } Cl(R)$$

*for all $i \in \Lambda$ and $j = 1, \ldots, k_i$.*

(3) *$Cl(R)$ is torsion-free if and only if $\gcd(e_{i1}, e_{i2}, \ldots, e_{ik_i}) = 1$ for all $i \in \Lambda$. In this case, $Cl(R)$ is a free abelian group of rank $\sum_{i\in\Lambda}(k_i - 1)$.*

(4) *If $R$ is a PID, then $R$ is a non-Euclidean PID.*

*Proof.* By Theorem 2.4 and Corollary 2.3(7), it's enough to show that each $M_{ij} \cap R$ is a height-one maximal ideal. Let $Q = M_{ij} \cap R$. Then $Q \cap D \neq (0)$; so $p \in Q$ for some prime element $p \in D$. Let $P$ be a prime ideal of $R$ such that $P \subseteq Q$ and $P$ is minimal over $pR$. Note that $\text{ht}P = 1$, because $R$ is a $t$-almost Dedekind domain by Proposition 2.2(3), and if $pD = P_k$ for some $k \in \Lambda$, then $P \cap D = P_k$ and

$$\sqrt{pR} = (\bigcap_{i,j} \sqrt{pV_{ij}^*}) \cap \sqrt{pK[X]} = (\bigcap_j M_{kj}) \cap R = \bigcap_j (M_{kj} \cap R).$$

Hence, $P = M_{kj} \cap R$ for the maximal ideal $M_{kj}$ of some $V_{kj}^*$, and thus

$$D/P_k \hookrightarrow R/P \hookrightarrow V_{kj}^*/M_{kj}.$$

Then, by assumption, $R/P$ is algebraic over $D/P_k$, whence $R/P$ is a field because $D/P_k$ is a field. Hence, $P$ is a maximal ideal and $Q = P$. Therefore, $Q$ is a height-one maximal ideal.

Now, assume that $R$ is a Euclidean domain. For $P_k \in X^1(D)$, let $S_k$ be the multiplicative set of $D$ generated by prime elements $p$ of $D$ with $pD \neq P_k$. Then $R_{S_k} = R_k$ by Proposition 2.2(2), and thus $R_k$ is Euclidean [29, Proposition 7]. However, $R_k$ is non-Euclidean [9, Theorem], a contradiction. Thus, $R$ is non-Euclidean. □

Let $\mathbb{Z}$ be the ring of integers. As an application of this section, in the next section, we construct two types of almost Dedekind polynomial overrings of $\mathbb{Z}$ and compute their ideal class groups.

## 3. Almost Dedekind polynomial overrings of $\mathbb{Z}$

Let $\text{Int}(\mathbb{Z})$ be the ring of integer-valued polynomials, i.e.,

$$\text{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}, \text{ i.e., } f(n) \in \mathbb{Z} \text{ for all } n \in \mathbb{Z}\}.$$

Then

$$\mathbb{Z}[X] \subsetneq \text{Int}(\mathbb{Z}) = \bigcap_{p \text{ is a prime}} \text{Int}(\mathbb{Z}_{p\mathbb{Z}}) \subsetneq \mathbb{Z} + X\mathbb{Q}[X] \subsetneq \mathbb{Q}[X].$$

It is known that $\text{Int}(\mathbb{Z})$ is a two-dimensional Prüfer domain [5, Remarks VI.1.8] and $\text{Pic}(\text{Int}(\mathbb{Z}))$ is a free abelian group on a countably infinite basis [17, Corollary 7].

For a prime number $p$, let $\widehat{\mathbb{Z}_p}$ be the ring of $p$-adic integers and $\mathbb{Q}_p$ be the $p$-adic completion of $\mathbb{Q}$. Then $\widehat{\mathbb{Z}_p}$ is a DVR with maximal ideal $p\widehat{\mathbb{Z}_p}$, $\mathbb{Q}_p$ is the quotient field of $\widehat{\mathbb{Z}_p}$, $\widehat{\mathbb{Z}_p} \cap \mathbb{Q} = \mathbb{Z}_{p\mathbb{Z}}$, and $\widehat{\mathbb{Z}_p}/p\widehat{\mathbb{Z}_p} = \mathbb{Z}/p\mathbb{Z}$. For $\alpha \in \widehat{\mathbb{Z}_p}$, let $\mathcal{N}_{p,\alpha} = \{f \in \text{Int}(\mathbb{Z}) \mid f(\alpha) \in p\widehat{\mathbb{Z}_p}\}$, $V_{p,\alpha} = \text{Int}(\mathbb{Z})_{\mathcal{N}_{p,\alpha}}$, and $M_{p,\alpha} = (\mathcal{N}_{p,\alpha})_{\mathcal{N}_{p,\alpha}}$. Then $V_{p,\alpha}$ is a valuation overring of $\text{Int}(\mathbb{Z})$ with maximal ideal $M_{p,\alpha}$ such that

(1) $\text{Int}(\mathbb{Z})/\mathcal{N}_{p,\alpha} = V_{p,\alpha}/M_{p,\alpha} = \mathbb{Z}/p\mathbb{Z}$,
(2) $M_{p,\alpha} = pV_{p,\alpha}$, and hence $V_{p,\alpha}$ is discrete,
(3) $\dim(V_{p,\alpha}) = 2$ if and only if $\alpha$ is algebraic over $\mathbb{Q}$,
(4) $\alpha \neq \beta \Leftrightarrow \mathcal{N}_{p,\alpha} \neq \mathcal{N}_{p,\beta}$ for all $\alpha, \beta \in \widehat{\mathbb{Z}_p}$.

Moreover, if $V$ is a valuation overring of $\mathbb{Z}[X]$ with maximal ideal $M$ such that $V/M = \mathbb{Z}/p\mathbb{Z}$ and $M = pV$, then $V$ is an overring of $\text{Int}(\mathbb{Z})$ [26, Proposition 2.2]. For more on properties of $\mathcal{N}_{p,\alpha}$, $V_{p,\alpha}$, and $M_{p,\alpha}$, see [5, Chapters V and VI].

Let $R$ be a polynomial overring of $\mathbb{Z}$ containing $\text{Int}(\mathbb{Z})$. As in [5], we say that an integral ideal $I$ of $R$ is unitary if $I \cap \mathbb{Z} \neq (0)$. The next result is the first application of the results of Section 2.

**Theorem 3.1.** *Let $\alpha$ be a countable cardinal number, i.e., $\alpha \leq \aleph_0$. Then there exists an overring $R$ of Int($\mathbb{Z}$) with the following properties.*

(1) *Int($\mathbb{Z}$) $\subsetneq R \cap \mathbb{Q}[X] \subsetneq \mathbb{Q}[X]$.*
(2) *$R \cap \mathbb{Q}[X]$ is a non-Noetherian almost Dedekind domain.*
(3) *$Cl(R \cap \mathbb{Q}[X]) = \mathbb{Z}^\alpha$.*
(4) *Every maximal ideal of $R \cap \mathbb{Q}[X]$ except $X\mathbb{Q}[X] \cap R$ is invertible.*
(5) *$R$ is a Bezout domain.*
(6) *Every invertible integral ideal $I$ of $R \cap \mathbb{Q}[X]$ can be written uniquely as $I = X^n Q_1^{e_1} \cdots Q_k^{e_k}$ for some integer $n \geq 0$, maximal ideals $Q_i$ of $R \cap \mathbb{Q}[X]$, and integers $e_i \neq 0$. In particular, if $I \nsubseteq X\mathbb{Q}[X] \cap R$, then each $e_i$ is positive.*
(7) *$Cl(R \cap \mathbb{Q}[X])$ is generated by the classes of unitary maximal ideals.*

*Proof.* Let $\Lambda$ be a set of prime numbers such that $|\Lambda| = \alpha$ and $\Delta = \{p \mid p \notin \Lambda \text{ is a prime number}\}$ is an infinite set. For each $p \in \Lambda$, we can choose two distinct elements $\alpha_{p,1}, \alpha_{p,2} \in p\widehat{\mathbb{Z}_p}$ that are transcendental over $\mathbb{Q}$. Let $V_{p,i} = \text{Int}(\mathbb{Z})_{\mathcal{N}_{p,\alpha_{p,i}}}$ for $i = 1, 2$. Then $V_{p,i}$ is a DVR with maximal ideal $pV_{p,i}$ and $V_{p,i}/pV_{p,i} = \mathbb{Z}/p\mathbb{Z}$. Similarly, for each $q \in \Delta$, choose $\alpha_q \in q\widehat{\mathbb{Z}_q}$ that is transcendental over $\mathbb{Q}$, and let $V_q = \text{Int}(\mathbb{Z})_{\mathcal{N}_{q,\alpha_q}}$. Then $V_q$ is a DVR with maximal ideal $qV_q$ and $V_q/qV_q = \mathbb{Z}/q\mathbb{Z}$. Note that, since $\alpha_{p,1}, \alpha_{p,2} \in p\widehat{\mathbb{Z}_p}$ and $\alpha_q \in q\widehat{\mathbb{Z}_q}$ for all $p \in \Lambda$ and $q \in \Delta$, we have

$$X \in \Big(\bigcap_{p \in \Lambda} (pV_{p,1} \cap pV_{p,2})\Big) \cap \Big(\bigcap_{q \in \Delta} qV_q\Big).$$

Let

$$R = (\bigcap_{p\in\Lambda}(V_{p,1} \cap V_{p,2})) \cap (\bigcap_{q\in\Delta} V_q).$$

Then $\mathrm{Int}(\mathbb{Z}) \subseteq R \cap \mathbb{Q}[X] \subseteq \mathbb{Q}[X]$.

(1) and (2). Note that $X \in \bigcap_{q\in\Delta}(qV_q \cap \mathbb{Q}[X])$, $qV_q \cap \mathbb{Q}[X]$ are distinct from each other for all $q \in \Delta$ by Corollary 2.3(1), and $\Delta$ is an infinite set. Hence, $X$ is contained in infinitely many height-one prime ideals of $R \cap \mathbb{Q}[X]$. Thus, $R \cap \mathbb{Q}[X]$ is non-Noetherian. Moreover, $R \cap \mathbb{Q}[X]$ is an almost Dedekind domain by Corollary 2.6, whence $\dim(R \cap \mathbb{Q}[X]) = 1$. Now, note that $\mathrm{Int}(\mathbb{Z})$ is a 2-dimensional Prüfer domain. Thus, $\mathrm{Int}(\mathbb{Z}) \subsetneq R \cap \mathbb{Q}[X] \subsetneq \mathbb{Q}[X]$.

(3) Let $F$ be a free abelian group of rank $\aleph_0$ on generators $\{g_{p,1}, g_{p,2} \mid p \in \Lambda\} \cup \{g_q \mid q \in \Delta\}$ and $H$ be the subgroup of $F$ generated by $\{g_{p,1} + g_{p,2}, g_q \mid p \in \Lambda \text{ and } q \in \Delta\}$. Note that $\mathbb{Z} \times \mathbb{Z}/\langle(1,1)\rangle \cong \mathbb{Z}$ as groups. Thus, by Corollary 2.6,

$$\begin{aligned} Cl(R \cap \mathbb{Q}[X]) &= F/H \\ &= \bigoplus_{p\in\Lambda}((\mathbb{Z} \times \mathbb{Z})/\langle(1,1)\rangle) \oplus (\bigoplus_{q\in\Delta} \mathbb{Z}/\mathbb{Z}) \\ &= \bigoplus_{p\in\Lambda} \mathbb{Z}. \end{aligned}$$

Thus, $Cl(R \cap \mathbb{Q}[X]) = \mathbb{Z}^{\alpha}$.

(4) and (5). Let $\mathcal{R} = R \cap \mathbb{Q}[X]$, and note that every $t$-invertible ideal of $\mathcal{R}$ is invertible.

**Claim 1.** Every maximal ideal of $\mathcal{R}$ except $X\mathbb{Q}[X] \cap R$ is invertible.

*Proof.* Let $A$ be a maximal ideal of $\mathcal{R}$ distinct from $X\mathbb{Q}[X] \cap R$. Then $A = pV_{p,i} \cap \mathcal{R}$, $A = qV_q \cap \mathcal{R}$, or $A = f\mathbb{Q}[X] \cap R$ for some irreducible $f \in \mathbb{Q}[X]$ with $f\mathbb{Q}[X] \neq X\mathbb{Q}[X]$ by Corollary 2.3(4). Hence, by Corollary 2.3(5), we may assume $A = f\mathbb{Q}[X] \cap R$. Also, we may assume that $f \in \mathbb{Z}[X]$ and $c(f) = \mathbb{Z}$. Then, since $f\mathbb{Q}[X] \neq X\mathbb{Q}[X]$, we have $f(0) \neq 0$. Note that

$$X \in (\bigcap_{p\in\Lambda}(pV_{p,1} \cap pV_{p,2})) \cap (\bigcap_{q\in\Delta} qV_q) \cap \mathbb{Q}[X],$$

so $f \in pV_{p,i}$ (resp., $f \in qV_q$) implies $f(0) \in p\mathbb{Z}$ (resp., $f(0) \in q\mathbb{Z}$). Hence, by Corollary 2.3(4), $f$ is contained in only finitely many maximal ideals of $\mathcal{R}$, and thus there is an element $g \in \mathcal{R}$ such that $A = (f, g)$. Then, since $\mathcal{R}$ is an almost Dedekind domain, $A$ is invertible. □

**Claim 2.** $\mathrm{Max}(R) = \{pV_{p,i} \cap R \mid p \in \Lambda \text{ and } i = 1, 2\} \cup \{qV_q \cap R \mid q \in \Delta\} \cup \{X\mathbb{Q}[X]_{X\mathbb{Q}[X]} \cap R\}$.

*Proof.* Let $M$ be a maximal ideal of $R$ such that $M \cap \mathbb{Q} = (0)$. Then $R_M = \mathbb{Q}[X]_{f\mathbb{Q}[X]}$ for some irreducible $f \in \mathbb{Q}[X]$. Assume $f\mathbb{Q}[X] \neq X\mathbb{Q}[X]$. Then

$M \cap \mathbb{Q}[X]$ is invertible by Claim 1, and since $M = (M \cap \mathbb{Q}[X])R$ [16, Theorem 26.1], $M$ is invertible. Hence,

$$\begin{aligned} M = M_v &\supseteq (\bigcap_{p\in\Lambda}(MV_{p,1} \cap MV_{p,2})) \cap (\bigcap_{q\in\Delta} MV_q) \\ &= (\bigcap_{p\in\Lambda}(V_{p,1} \cap V_{p,2})) \cap (\bigcap_{q\in\Delta} V_q) \\ &= R \end{aligned}$$

(cf. [16, Theorems 32.5 and 34.1] for the first containment), and thus $M = R$, a contradiction. Furthermore, $R$ is not a Dedekind domain. Thus, by Corollary 2.3(4), the proof is completed. □

Hence, by Claim 2, $X$ is contained in all maximal ideals of $R$. Furthermore, if $Q$ is a maximal ideal of $R$ distinct from $X\mathbb{Q}[X]_{X\mathbb{Q}[X]} \cap R$, then $Q$ is invertible (cf. the proof of Claim 2), and thus $Q$ is principal [24, Theorem 2.6]. This implies that every finitely generated ideal of $R$ is principal.

(6) Let $\mathcal{R} = R \cap \mathbb{Q}[X]$ and $I = (f_1, \ldots, f_k)$. Then $f_i \in \mathbb{Q}[X]$, and hence there are unique positive integers $a, b$ such that $(a, b)\mathbb{Z} = \mathbb{Z}$ and $\sum_{i=1}^{k} \frac{a}{b} c(f_i) = \mathbb{Z}$. Also, there is a unique integer $n \geq 0$ such that $X^{-n}I \subseteq \mathbb{Q}[X]$ and $(X^{-n}f_i)(0) \neq 0$ for some $i$. Hence, if we let $J = \frac{a}{b}X^{-n}I$, then $J \subseteq \mathcal{R}$ and $J \nsubseteq X\mathbb{Q}[X] \cap R$. Then by the proof of (4) and (5), every maximal ideal of $\mathcal{R}$ containing $J$ is invertible, whence $J = N_1^{n_1} \cdots N_r^{n_r}$ for some maximal ideals $N_1, \ldots, N_r$ of $\mathcal{R}$ and integers $n_i \geq 1$. Thus, $I = \frac{b}{a}X^n(N_1^{n_1} \cdots N_r^{n_r})$.

Now, it is clear that both $a\mathcal{R}$ and $b\mathcal{R}$ can be written uniquely as the form $A_1^{m_1} \cdots A_{r'}^{m_{r'}}$ for some integers $m_i \geq 1$ and maximal ideals $A_i$ of $\mathcal{R}$ as in the case of $J$ above or by Corollary 2.3(6). Note also that if $a\mathcal{R} = A_1^{m_1} \cdots A_{r'}^{m_{r'}}$, then $\frac{1}{a}\mathcal{R} = (A_1^{m_1} \cdots A_{r'}^{m_{r'}})^{-1} = A_1^{-m_1} \cdots A_{r'}^{-m_{r'}}$. Thus,

$$I = \frac{b}{a}X^n(N_1^{n_1} \cdots N_r^{n_r}) = X^n(Q_1^{e_1} \cdots Q_k^{e_k})$$

for some integer $n \geq 0$, maximal ideals $Q_i$ of $\mathcal{R}$, and integers $e_i \neq 0$. Moreover, this expression is unique by the uniqueness of $n, a, b, N_i, n_i, A_i$ and $m_i$. In particular, if $I \nsubseteq X\mathbb{Q}[X] \cap R$, then $n = 0$, and thus each $e_i$ must be positive.

(7) Let $I$ be an invertible ideal of $R \cap \mathbb{Q}[X]$. Then $I\mathbb{Q}[X] = u\mathbb{Q}[X]$ for some $0 \neq u \in \mathbb{Q}(X)$, because $\mathbb{Q}[X]$ is a PID, so $u^{-1}I$ is invertible and $u^{-1}I \cap \mathbb{Z} \neq (0)$. Hence, by the proof of (6), $u^{-1}I = \frac{b}{a}(N_1^{n_1} \cdots N_r^{n_r})$ for some $0 \neq a, b \in \mathbb{Z}$ and maximal ideals $N_i$ with $N_i \cap \mathbb{Z} \neq (0)$. Thus, $cl(I) = cl(N_1^{n_1} \cdots N_r^{n_r}) = \sum_i n_i cl(N_i)$. □

*Remark* 3.2. (1) Let the notation be as in the proof of Theorem 3.1, and let $\Omega$ be a finite subset of $\Delta$ and put $T = (\bigcap_{p\in\Lambda}(V_{p,1} \cap V_{p,2})) \cap (\bigcap_{q\in\Omega} V_q)$. Then $R \cap \mathbb{Q}[X] \subsetneq T \cap \mathbb{Q}[X]$, $T \cap \mathbb{Q}[X]$ is an almost Dedekind domain, and $T$ satisfies the properties (3)-(7) of Theorem 3.1. Moreover, in the proof of Theorem

3.1(2), $\alpha$ is finite if and only if $\Lambda$ is a finite set, if and only if $T \cap \mathbb{Q}[X]$ is a Krull domain by Proposition 2.2(5), if and only if $T \cap \mathbb{Q}[X]$ is a Dedekind domain.

(2) Let $K$ be a number field and $O_K$ be the ring of algebraic integers in $K$. Then $\mathrm{Int}(O_K) = \{f \in K[X] \mid f(O_K) \subseteq O_K\}$ is a two-dimensional Prüfer domain [5, Remark VI.1.8]. Also, as in the case of Theorem 3.1, we can construct a Bezout domain $R$ such that $\mathrm{Int}(O_K) \subseteq R \cap K[X] \subseteq K[X]$, $R \cap K[X]$ is an almost Dedekind domain, and $Cl(R \cap K[X]) = \mathbb{Z}^\alpha$.

The following example shows that the results of Section 2 are not true if each $\{V_{ij}^*\}$ of Notation 2.1(iii) is not a finite set.

**Example 3.3.** Let the notation be as in the remark before Theorem 3.1. For each prime number $p$, let $T_p$ be the set of elements in $\widehat{\mathbb{Z}_p}$ which are transcendental over $\mathbb{Q}$. Then, by [5, Proposition VI.2.2],

$$\mathrm{Int}(\mathbb{Z}) = (\bigcap_{p \text{ is a prime}} (\bigcap_{\alpha \in T_p} V_{p,\alpha})) \cap \mathbb{Q}[X].$$

Hence, $\mathrm{Int}(\mathbb{Z})$ is an intersection of DVRs $V_{p,\alpha}$ and $\mathbb{Q}[X]$. Moreover, for all prime numbers $p$ and $\alpha \in T_p$, $M_{p,\alpha} \cap \mathrm{Int}(\mathbb{Z})$ is a height-one maximal ideal, $V_{p,\alpha} = \mathrm{Int}(\mathbb{Z})_{M_{p,\alpha} \cap \mathrm{Int}(\mathbb{Z})}$, and $V_{p,\alpha}/M_{p,\alpha} \cong \mathbb{Z}/p\mathbb{Z}$. However, $\mathrm{Int}(\mathbb{Z})$ is not an almost Dedekind domain.

For the second application of the results in Section 2 (Theorem 3.5), we first need a lemma which is already known (see, for example, [9, page 68]). But, we give the proof for the completeness of this section.

**Lemma 3.4.** *Let $p$ be a prime number and $n$ be a positive integer. Then there is a DVR $V^*$ with maximal ideal $M$ and quotient field $\mathbb{Q}(X)$ such that $V^*/M \cong \mathbb{Z}/p\mathbb{Z}$ and $pV^* = M^n$. Moreover, $V^* \cap \mathbb{Q}[X]$ is a Dedekind domain with $Cl(V^* \cap \mathbb{Q}[X]) = \mathbb{Z}_n$.*

*Proof.* Choose $\alpha \in p\widehat{\mathbb{Z}_p}$ such that $\alpha\widehat{\mathbb{Z}_p} = p\widehat{\mathbb{Z}_p}$ and $\alpha$ is transcendental over $\mathbb{Q}$, and let $\mathcal{N}_{p,\alpha} = \{f \in \mathrm{Int}(\mathbb{Z}) \mid f(\alpha) \in p\widehat{\mathbb{Z}_p}\}$, $V_{p,\alpha} = \mathrm{Int}(\mathbb{Z})_{\mathcal{N}_{p,\alpha}}$, and $M_{p,\alpha} = (\mathcal{N}_{p,\alpha})_{\mathcal{N}_{p,\alpha}}$. Then $V_{p,\alpha}$ is a DVR with maximal ideal $M_{p,\alpha}$ such that $V_{p,\alpha}/M_{p,\alpha} \cong \mathbb{Z}/p\mathbb{Z}$ and $M_{p,\alpha} = pV_{p,\alpha}$. Note that

$$V_{p,\alpha} \cong \widehat{\mathbb{Z}_p} \cap \mathbb{Q}(\alpha) \text{ and } M_{p,\alpha} \cong p\widehat{\mathbb{Z}_p} \cap \mathbb{Q}(\alpha)$$

by letting $X \mapsto \alpha$. Hence, $pV_{p,\alpha} = XV_{p,\alpha}$.

Now, let $W$ be an extension of $V_{p,\alpha}$ to $\mathbb{Q}(X^{\frac{1}{n}})$, i.e., $W$ is a valuation domain with quotient field $\mathbb{Q}(X^{\frac{1}{n}})$ such that $W \cap \mathbb{Q}(X) = V_{p,\alpha}$. Then $W$ is a DVR [16, Theorem 19.16] because $[\mathbb{Q}(X^{\frac{1}{n}}) : \mathbb{Q}(X)] = n < \infty$. Moreover, if $\mathcal{M}$ is the maximal ideal of $W$, then $\mathcal{M} \cap V_{p,\alpha} = M_{p,\alpha}$. Note that $pW = \mathcal{M}^k$ for some positive integer $k$, so by [3, Lemma 2 on page 417],

$$k[W/\mathcal{M} : V_{p,\alpha}/M_{p,\alpha}] \leq [\mathbb{Q}(X^{\frac{1}{n}}) : \mathbb{Q}(X)] = n.$$

Hence, $pW = \mathcal{M}^n$ and $W/\mathcal{M} \cong \mathbb{Z}/p\mathbb{Z}$ because $pW = XW \subseteq \mathcal{M}^n$. Finally, note that the map $\varphi : \mathbb{Q}(X^{\frac{1}{n}}) \to \mathbb{Q}(X)$ given by $\varphi(f(X^{\frac{1}{n}})/g(X^{\frac{1}{n}})) = f(X)/g(X)$ is an isomorphism. Thus, if we let $V^* = \varphi(W)$, then $V^*$ is the desired valuation domain. Moreover, by Proposition 2.2(6) and Corollary 2.6, $V^* \cap \mathbb{Q}[X]$ is a Dedekind domain and $Cl(V^* \cap \mathbb{Q}[X]) = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$. $\square$

We next give the second application of the results of Section 2.

**Theorem 3.5.** *Let $\mathfrak{G}$ be an infinite set of finitely generated abelian groups* (*up to isomorphism*). *Then there is an overring $R$ of $\mathbb{Z}[X]$ with the following properties.*

(1) *$R \cap \mathbb{Q}[X]$ is an almost Dedekind domain.*
(2) *$Cl(R \cap \mathbb{Q}[X]) = \bigoplus_{G \in \mathfrak{G}} G$.*
(3) *If $G \in \mathfrak{G}$, then there is a multiplicative set $S$ of $\mathbb{Z}$ such that $R_S \cap \mathbb{Q}[X]$ is a Dedekind domain with $Cl(R_S \cap \mathbb{Q}[X]) = G$.*
(4) *$R$ is a Bezout domain.*
(5) *Every invertible integral ideal $I$ of $R \cap \mathbb{Q}[X]$ can be written uniquely as $I = X^n Q_1^{e_1} \cdots Q_k^{e_k}$ for some integer $n \geq 0$, maximal ideals $Q_i$ of $R \cap \mathbb{Q}[X]$, and integers $e_i \neq 0$. In particular, if $I \nsubseteq X\mathbb{Q}[X] \cap R$, then each $e_i$ is positive.*
(6) *Every maximal ideal of $R \cap \mathbb{Q}[X]$ except $X\mathbb{Q}[X] \cap R$ is invertible.*
(7) *$Cl(R \cap \mathbb{Q}[X])$ is generated by the classes of unitary maximal ideals.*

*Proof.* Let $\mathfrak{P} = \{p_i \mid i \in \mathbb{N}\}$ be an infinite set of prime numbers. By Remark 1.6, $\mathfrak{G}$ is countable, whence we write $\mathfrak{G} = \{G_i \mid i \in \mathbb{N}\}$, and for each $k \in \mathbb{N}$, let

$$G_k = \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_{n_k}} \oplus \mathbb{Z}^{m_k}$$

for some positive integers $k_1, \ldots, k_{n_k}$ and a nonnegative integer $m_k$. Now, partition $\mathfrak{P}$ into a family of finite subsets $B_k = \{p_{k_1}, \ldots, p_{k_{n_k}}, p_{m_k}\}$ such that $|B_k| = n_k + 1$, $\mathfrak{P} = \bigcup_{k \in \mathbb{N}} B_k$, and $B_i \cap B_j = \emptyset$ for all $i, j \in \mathbb{N}$ with $i \neq j$.

Let $k \in \mathbb{N}$. By Lemma 3.4 and the proof of Theorem 3.1, there are DVRs $(V_{k_1}, M_{k_1}), \ldots, (V_{k_{n_k}}, M_{k_{n_k}})$, $(W_{k_1}, N_{k_1}), \ldots, (W_{k_{m_k}}, N_{k_{m_k}}), (W_{k_{m_k}+1}, N_{k_{m_k}+1})$, where $k_{m_k} = 0$ when $m_k = 0$, with quotient field $K(X)$ such that

- $X \in (\bigcap_{i=k_1}^{k_{n_k}} M_i) \cap (\bigcap_{j=k_1}^{k_{m_k}+1} N_j)$,
- $[V_i/M_i : \mathbb{Z}/p_i\mathbb{Z}] = [W_j/N_j : \mathbb{Z}/p_{m_k}\mathbb{Z}] = 1$ for $i = k_1, \ldots, k_{n_k}$ and $j = k_1, \ldots, k_{m_k}, k_{m_k} + 1$,
- $p_{k_i} V_{k_i} = (M_{k_i})^{k_i}$ for $i = 1, \ldots, n_k$ and $p_{m_k} W_j = N_j$ for $j = k_1, \ldots, k_{m_k}$ $+1$.

Put

$$R_k = (\bigcap_{i=1}^{n_k} V_{k_i}) \cap (\bigcap_{j=k_1}^{k_{m_k}+1} W_j) \text{ and } R = \bigcap_{k=1}^{\infty} R_k.$$

(1) By Corollary 2.6, $R \cap \mathbb{Q}[X]$ is an almost Dedekind domain.

(2) Let $G$ be the free abelian group of rank $\aleph_0$ on the generators $\{g_{k_i}, h_{j_k} \mid i = 1, \ldots, n_k, j_k = k_1, \ldots, k_{m_k}, k_{m_k}+1$, and $k \in \mathbb{N}\}$ and $H$ be the subgroup of $G$ generated by the elements $\{k_1 g_{k_1}, \ldots, k_{n_k} g_{k_{n_k}}, h_{k_1} + \cdots + h_{k_{m_k}} + h_{k_{m_k}+1} \mid k \in \mathbb{N}\}$. Then $Cl(R \cap \mathbb{Q}[X]) = G/H$ by Corollary 2.6. Moreover,

$$G/H = \bigoplus_k ((\bigoplus_{i=1}^{n_k} \mathbb{Z}/k_i\mathbb{Z}) \oplus (\mathbb{Z}^{m_k+1}/\langle(1, \ldots, 1)\rangle)) = \bigoplus_k G_k.$$

Therefore, $Cl(R \cap \mathbb{Q}[X]) = \bigoplus_k G_k = \bigoplus_{G \in \mathfrak{G}} G$.

(3) If $G \in \mathfrak{G}$, then $G = G_k$ for some $k \in \mathbb{N}$. Let $S$ be the multiplicative set of $D$ generated by $\mathfrak{P} \setminus B_k$. Then, by an argument similar to the proof of Proposition 2.2(2), $(R \cap \mathbb{Q}[X])_S = R_k \cap \mathbb{Q}[X]$. Thus, $(R \cap \mathbb{Q}[X])_S$ is a Dedekind domain with $Cl((R \cap \mathbb{Q}[X])_S) = G_k$ by Corollary 2.6 and Proposition 2.2(6).

(4), (5), (6) and (7). These can be proved by the same argument as the proofs of Theorem 3.1(4)-(7). □

*Remark* 3.6. (1) Two special cases of $\mathfrak{G}$ in Theorem 3.5 are the set of finitely generated abelian groups and the set of finite abelian groups by Remark 1.6.

(2) Recall that $\Lambda := \mathbb{N} \times \mathbb{N}$ is countable, $\Lambda_i := \{(i, n) \mid n \in \mathbb{N}\}$ is countably infinite for each $i \in \mathbb{N}$, $\Lambda_i \cap \Lambda_j \neq \emptyset$ if and only if $i = j$ (i.e., $\Lambda_i$ and $\Lambda_j$ are disjoint), and $\Lambda = \bigcup_{i=1}^{\infty} \Lambda_i$. Hence, if $\mathfrak{P}$ is the set of prime numbers, then $\mathfrak{P} = \bigcup_{i=1}^{\infty} \mathfrak{P}_i$ for countably infinite disjoint sets $\mathfrak{P}_i$ of prime numbers. This implies that there are exactly $2^{\aleph_0}$ distinct countably infinite sets of prime numbers. Thus, by Theorem 3.5, there are at least $2^{\aleph_0}$ distinct Bezout overrings $R$ of $\mathbb{Z}[X]$ with the properties of Theorem 3.5. In fact, there are exactly $2^{\aleph_0}$ overrings $D$ of $\mathbb{Z}$ that are not semilocal, and then there are at least countably infinite Bezout overrings of $D[X]$ with the properties of Theorem 3.5.

(3) The almost Dedekind domains of Theorems 3.1 and 3.5 have exactly one noninvertible maximal ideal. In [24, Theorem 2.5], Loper and Lucas studied the factorization properties of almost Dedekind domains with exactly one noninvertible maximal ideal.

The almost Dedekind domain of Theorem 3.5 is not a Dedekind domain. In fact, every maximal ideal $Q$ of $R \cap \mathbb{Q}[X]$ with $Q \cap \mathbb{Z} \neq (0)$ contains $X$. Hence, we have the following question.

**Question 3.7.** Let $\mathfrak{G}$ be an infinite set of finitely generated abelian groups. Is there a Dedekind domain $R$ with the following three properties?

(i) $\mathbb{Z}[X] \subseteq R \subseteq \mathbb{Q}[X]$.

(ii) $Cl(R) = \bigoplus_{G \in \mathfrak{G}} G$.

(iii) If $G \in \mathfrak{G}$, then there is a multiplicative set $S$ of $R$ such that $R_S$ is a Dedekind domain with $Cl(R_S) = G$.

## 4. Almost Dedekind overrings of Int($\mathbb{Z}$)

Let the notation be as in Section 3 in which we construct two types of almost Dedekind polynomial overrings of $\mathbb{Z}$. Loper and Tartarone [25] studied when the polynomial overrings of $\mathbb{Z}$ defined by the intersection of DVRs are Prüfer domains, Krull domains, or P$v$MDs. In [6], Chabert and Peruginelli studied polynomial overrings $\mathcal{R}$ of $\mathbb{Z}$ containing $\text{Int}(\mathbb{Z})$ with focus on when $\mathcal{R} = \text{Int}(E, \mathbb{Z})$ for some subset $E$ of $\mathbb{Z}$. In this section, we completely characterize the almost Dedekind polynomial overrings of $\mathbb{Z}$ containing $\text{Int}(\mathbb{Z})$.

**Lemma 4.1.** *Let $p$ be a prime number, $T$ be a subset of $\widehat{\mathbb{Z}_p}$, and $\overline{T}$ be the topological closure of $T$ in the $p\widehat{\mathbb{Z}_p}$-adic topology. Then*

$$(\bigcap_{\alpha\in T} V_{p,\alpha}) \cap \mathbb{Q}[X] = (\bigcap_{\alpha\in \overline{T}} V_{p,\alpha}) \cap \mathbb{Q}[X].$$

*Proof.* Clearly, $(\bigcap_{\alpha\in \overline{T}} V_{p,\alpha}) \cap \mathbb{Q}[X] \subseteq (\bigcap_{\alpha\in T} V_{p,\alpha}) \cap \mathbb{Q}[X]$. For the reverse containment, let $f \in (\bigcap_{\alpha\in T} V_{p,\alpha}) \cap \mathbb{Q}[X]$ and $\hat{v}$ be the valuation on $\mathbb{Q}_p$ associated with $\widehat{\mathbb{Z}_p}$. Then $\hat{v}(f(\alpha)) \geq 0$ for all $\alpha \in T$, and since $f \in \mathbb{Q}[X]$ is continuous on $\mathbb{Q}$ [5, Proposition III.2.1], $\hat{v}(f(\beta)) \geq 0$ for all $\beta \in \overline{T}$. Thus, $f \in (\bigcap_{\alpha\in \overline{T}} V_{p,\alpha}) \cap \mathbb{Q}[X]$. □

**Lemma 4.2.** *Let $p$ be a prime number, $A, B$ be subsets of $\widehat{\mathbb{Z}_p}$, and $\overline{A}, \overline{B}$ be the topological closures of $A$ and $B$, respectively, in the $p\widehat{\mathbb{Z}_p}$-adic topology. Then*

$$(\bigcap_{\alpha\in A} V_{p,\alpha}) \cap \mathbb{Q}[X] = (\bigcap_{\alpha\in B} V_{p,\alpha}) \cap \mathbb{Q}[X] \text{ if and only if } \overline{A} = \overline{B}.$$

*Proof.* For convenience, let

$$R_A = (\bigcap_{\alpha\in A} V_{p,\alpha}) \cap \mathbb{Q}[X] \text{ and } R_B = (\bigcap_{\alpha\in B} V_{p,\alpha}) \cap \mathbb{Q}[X].$$

If $\overline{A} = \overline{B}$, then $R_A = R_B$ by Lemma 4.1. Conversely, assume that $R_A = R_B$. For $a \in A$, let $\mathcal{U}$ be an open subset of $\widehat{\mathbb{Z}_p}$ that contains $a$ and $\varphi = \chi_{\mathcal{U}}$ be the characteristic function in $\widehat{\mathbb{Z}_p}$, so $\varphi \in \mathcal{C}(\widehat{\mathbb{Z}_p}, \widehat{\mathbb{Z}_p})$, i.e., $\varphi$ is a continuous function from $\widehat{\mathbb{Z}_p}$ into itself. Then, by Stone-Weierstrass theorem [5, Theorem III.3.4], there is an $f \in \text{Int}(\mathbb{Z}_p)$ such that

$$f \equiv \varphi \pmod{p\widehat{\mathbb{Z}_p}}.$$

Assume that $\mathcal{U} \cap B = \emptyset$. Then $f(b) \notin p\widehat{\mathbb{Z}_p}$ for all $b \in B$, while $f(a) \in p\widehat{\mathbb{Z}_p}$. Note that $R_B$ is a Prüfer domain [16, Theorem 26.1] because $R_B$ is an overring of a Prüfer domain $\text{Int}(\mathbb{Z})$, so $(p, f)R_B$ is invertible. Hence,

$$\begin{aligned}(p, f)R_B = ((p, f)R_B)_v &\supseteq (\bigcap_{\alpha\in B} (p, f)V_{p,\alpha}) \cap (p, f)\mathbb{Q}[X] \\ &= (\bigcap_{\alpha\in B} V_{p,\alpha}) \cap \mathbb{Q}[X] = R_B,\end{aligned}$$

but $(p, f)R_A \subsetneq R_A$ because $(p, f)V_{p,a} \subsetneq V_{p,a}$. Thus, $R_A \neq R_B$, a contradiction. Hence, $\mathcal{U} \cap B \neq \emptyset$. Thus, $a \in \overline{B}$, so $A \subseteq \overline{B}$, and hence $\overline{A} \subseteq \overline{B}$. By symmetry, $\overline{B} \subseteq \overline{A}$. Therefore, $\overline{A} = \overline{B}$. □

**Theorem 4.3.** *Let $p$ be a prime number, $T$ be a subset of $\widehat{\mathbb{Z}_p}$, $\overline{T}$ be the topological closure of $T$ in the $p\widehat{\mathbb{Z}_p}$-adic topology, and*

$$R = (\bigcap_{\alpha \in T} V_{p,\alpha}) \cap \mathbb{Q}[X].$$

*Then the following statements hold.*

(1) *$R$ is an almost Dedekind domain if and only if $\overline{T}$ does not contain an element of $\widehat{\mathbb{Z}_p}$ which is algebraic over $\mathbb{Q}$.*

(2) *$R$ is a Dedekind domain if and only if $T$ is finite. In this case, $Cl(R) = \mathbb{Z}^k$ for $k = |T| - 1$.*

*Proof.* (1) ($\Rightarrow$) Assume that $\overline{T}$ contains an element $\beta \in \widehat{\mathbb{Z}_p}$ that is algebraic over $\mathbb{Q}$. Then $R \subseteq V_{p,\beta}$ by Lemma 4.1, and since $R$ is an almost Dedekind domain, $V_{p,\beta} = R_M$ for some maximal ideal $M$ of $R$. But, in this case, $1 = \dim(R_M) = \dim(V_{p,\beta}) = 2$, a contradiction. ($\Leftarrow$) For the converse, suppose that $R$ is not an almost Dedekind domain. Then there is a maximal ideal $M$ of $R$ such that $R_M$ is not a DVR. Clearly, $M \cap \mathbb{Q} = p\mathbb{Z}_{p\mathbb{Z}}$ and $\mathrm{ht}(M \cap \mathrm{Int}(\mathbb{Z}_{p\mathbb{Z}})) = 2$. Thus, by [5, Proposition V.2.7], $M \cap \mathrm{Int}(\mathbb{Z}_{p\mathbb{Z}}) = \mathcal{M}_{p,\beta}$ for some $\beta \in \widehat{\mathbb{Z}_p}$ which is algebraic over $\mathbb{Q}$, where $\mathcal{M}_{p,\beta} = \{f \in \mathrm{Int}(\mathbb{Z}_{p\mathbb{Z}}) \mid f(\beta) \in p\widehat{\mathbb{Z}_p}\}$. Note that $M = (M \cap \mathrm{Int}(\mathbb{Z}_{p\mathbb{Z}}))R$ [16, Theorem 26.1]. Thus, $R_M = \mathrm{Int}(\mathbb{Z}_{p\mathbb{Z}})_{\mathcal{M}_{p,\beta}} = V_{p,\beta}$. This means that $\beta \in \overline{T}$ by Lemma 4.2, a contradiction.

(2) It is clear that $R$ is a Dedekind domain if and only if $|T| < \infty$. Moreover, by Corollary 2.6, $Cl(R) = \mathbb{Z}^{k+1}/H$, where $H$ is a subgroup of $\mathbb{Z}^{k+1}$ generated by $(1, \ldots, 1)$. Thus, by Remark 1.6(1), $Cl(R) = \mathbb{Z}^k$. □

**Corollary 4.4.** *Let $\Lambda$ be a set of prime numbers, $T_p$ be a subset of $\widehat{\mathbb{Z}_p}$, $\overline{T_p}$ be the topological closure of $T_p$ in the $p\widehat{\mathbb{Z}_p}$-adic topology for all $p \in \Lambda$, and*

$$R = (\bigcap_{p \in \Lambda} (\bigcap_{\alpha \in T_p} V_{p,\alpha})) \cap \mathbb{Q}[X].$$

*Then $R$ is an almost Dedekind domain if and only if $\overline{T_p}$ does not contain an element of $\widehat{\mathbb{Z}_p}$ which is algebraic over $\mathbb{Q}$ for all $p \in \Lambda$.*

*Proof.* For $p \in \Lambda$, let $S_p = \mathbb{Z} \setminus p\mathbb{Z}$, and note that $\mathrm{Int}(\mathbb{Z})_{S_p} = \mathrm{Int}(\mathbb{Z}_{p\mathbb{Z}})$ and $\mathrm{Int}(\mathbb{Z}) \subseteq R$. Hence, it is clear that

$$R_{S_p} = (\bigcap_{\alpha \in T_p} V_{p,\alpha}) \cap \mathbb{Q}[X].$$

Thus, the result follows directly from Theorem 4.3. □

Now, we give the complete characterization of almost Dedekind polynomial overrings of $\mathbb{Z}$ containing $\mathrm{Int}(\mathbb{Z})$.

**Corollary 4.5.** *Let $\mathcal{R}$ be a polynomial overring of $\mathbb{Z}$ containing Int($\mathbb{Z}$). For a prime number $p$, let $\mathrm{Z}_p(\mathcal{R}) = \{\alpha \in \widehat{\mathbb{Z}_p} \mid \mathcal{N}_{p,\alpha}\mathcal{R} \subsetneq \mathcal{R}\}$. Then $\mathcal{R}$ is an almost*

*Dedekind domain if and only if every element of* $\mathrm{Z}_p(\mathcal{R})$ *is transcendental over* $\mathbb{Q}$ *for all prime numbers* $p$.

*Proof.* Let $\Lambda$ be the set of prime numbers $p$ with $\mathrm{Z}_p(\mathcal{R}) \neq \emptyset$. Then

$$\mathcal{R} = \Big(\bigcap_{p\in\Lambda}\Big(\bigcap_{\alpha\in \mathrm{Z}_p(\mathcal{R})} V_{p,\alpha}\Big)\Big) \cap \mathbb{Q}[X]$$

by [6, Proposition 5.1]. Hence, by Lemma 4.2, $\mathrm{Z}_p(\mathcal{R})$ is closed in the $p\widehat{\mathbb{Z}_p}$-adic topology for all $p \in \Lambda$. Thus, the proof is completed by Corollary 4.4. □

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