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CONSTRUCTION OF A SOLUTION OF SPLIT EQUALITY VARIATIONAL INEQUALITY PROBLEM FOR PSEUDOMONOTONE MAPPINGS IN BANACH SPACES

Getahun Bekele Wega

Abstract. The purpose of this paper is to introduce an iterative algorithm for approximating a solution of split equality variational inequality problem for pseudomonotone mappings in the setting of Banach spaces. Under certain conditions, we prove a strong convergence theorem for the iterative scheme produced by the method in real reflexive Banach spaces. The assumption that the mappings are uniformly continuous and sequentially weakly continuous on bounded subsets of Banach spaces are dispensed with. In addition, we present an application of our main results to find solutions of split equality minimum point problems for convex functions in real reflexive Banach spaces. Finally, we provide a numerical example which supports our main result. Our results improve and generalize many of the results in the literature.

1. Introduction

Let E be a real Banach space with its dual E^*. Let C be a nonempty subset of E. A mapping $G : C \to E^*$ is called monotone if for all $x, y \in C$,

$$\langle Gx - Gy, x - y\rangle \geq 0. \tag{1}$$

It is called maximal monotone if its graph $Gph(A) = \{(x, Ax) \in E \times E^* : x \in C\}$ is not properly contained in the graph of any other monotone mapping. A mapping $G : E \to E^*$ is called α-inverse strongly monotone if there exists a positive real number α such that

$$\langle Gx - Gy, x - y\rangle \geq \alpha||Gx - Gy||^2 \tag{2}$$

for any $x, y \in E$.

We can observe from (2) that an α-inverse strongly monotone mapping is Lipschitz continuous monotone mapping, where a mapping $T : C \to E$ is called

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Lipschitz continuous on C if there exists a constant $L \geq 0$ such that

$$||Tx - Ty|| \leq L||x - y||, \ \forall x, y \in C. \tag{3}$$

A mapping $G : C \to E^*$ is called a pseudomonotone mapping if for all $x, y \in C$,

$$\langle Gx, y - x \rangle \geq 0 \Rightarrow \langle Gy, y - x \rangle \geq 0. \tag{4}$$

It is known that the class of pseudomonotone mappings contains the class of monotone mappings but the converse is not true in general (see, e.g., [32]).

Let C be a nonempty, closed and convex subset of E and let $G : C \to E^*$ be a pseudomonotone mapping. The variational inequality problem is formulated as finding a point u^* in C such that for all $u \in C$

$$\langle Gu^*, u - u^* \rangle \geq 0. \tag{5}$$

We denote the solution set of problem (5) by $VI(C, G)$. This problem was first introduced by Hartman and Stampacchia [15] in 1966. Since then, it has been studied by several authors (see, e.g., [6, 11, 16, 23, 26, 35, 38–40] and references therein).

Several researchers have also studied, different iterative algorithms for approximating a solution of split equality variational inequality problem for pseudomonotone mapping which is the generalization of split variational inequality problem introduced by Censor *et al.* [13] (see, e.g., [7, 12, 13, 18, 21, 22, 33] and references therein).

Let E_1, E_2 and E_3 be real Banach spaces. Let $G : C \to E_1^*$ and $H : D \to E_2^*$ be pseudomonotone mappings, where C and D are nonempty, closed and convex subsets of E_1 and E_2, respectively and let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear mappings. Then, split equality variational inequality problem for pseudomonotone mappings is mathematically formulated as the problem of finding $u^* \in C$ and $v^* \in D$ such that

$$u^* \in VI(C, G), \ v^* \in VI(D, H) \text{ such that } Au^* = Bv^*. \tag{6}$$

The split equality problem was introduced by Moudafi [21] in 2014 and it has been received much attention due to its application in various disciplines such as decomposition methods for partial differential equations, game theory, medical image reconstruction and radiation therapy treatment planning (see, e.g., [1, 2]).

We remark that, if in problem (6), we consider $B = I$ and $E_2 = E_3$, then the split equality problem (6) reduces to split variational inequality problem for pseudomonotone mappings which can be formulated as the problem of finding $(u^*, v^*) \in C \times D$ such that

$$u^* \in VI(C, G) \text{ and } Au^* = v^* \in VI(D, H). \tag{7}$$

Moreover, if in problem (6), we consider $A = I = B$ and $E_1 = E_2 = E_3$, then the split equality problem (6) reduces to a common solution of variational inequality problem for pseudomonotone mapping which is finding $(u^*, v^*) \in C \times D$ such that

$$u^* \in VI(C, G), \ v^* \in VI(D, H) \ \text{ and } \ u^* = v^*. \tag{8}$$

In 2010, Censor *et al.* [13] established the following algorithm for solving problem (7): For any $u_0 \in H_1$,

$$u_{n+1} = S(u_n + \gamma A^*(T - I)Au_n), \ \forall n \geq 0, \tag{9}$$

where $\lambda > 0$, $\gamma \in (0, \frac{1}{||A||^2})$, A^* is the adjoint operator of A, $T = P_D(I - \lambda H)$, $S = P_C(I - \lambda G)$. Under certain conditions they proved weak convergence of the sequence (9) to the solution of problem (7) in the setting of Hilbert spaces provided that G and H are α-inverse strongly monotone mappings.

Recently, Kwelegano *et al.* [18] established an iterative algorithm for approximating a solution of problem (6). Under certain conditions, they proved a strong convergence theorem for the iterative scheme produced by their method in Hilbert spaces provided that the mappings are uniformly continuous pseudomonotone mappings which are sequentially weakly continuous on bounded subsets of C and D.

More recently, Boikanyo and Zegeye [7] extended the results of Kwelegano *et al.* [18] from real Hilbert spaces to uniformly smooth and uniformly convex real Banach spaces.

Question. Can we obtain a strong convergence result for approximating a solution of split equality variational inequality problem for pseudomonotone mapping in spaces more general than uniformly convex and uniformly smooth real Banach spaces?

Motivated and inspired by the above results, it is our purpose in this study to construct an iterative algorithm for approximating a solution of split equality variational inequality for pseudomonotone mappings in the setting of real reflexive Banach spaces. In addition, we apply our main results to find a solution of split equality minimum point problem for convex functions. Our results provide an affirmative answer to our question. Our results improve, and generalize many results in the literature.

2. Preliminaries

In this section, we present some basic notations and results which are needed in the sequel. Throughout this paper let E be a real reflexive Banach space with its dual space E^* and C be a nonempty, closed and convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a proper, convex and lower semi continuous function. We denote the family of those functions by $\mathcal{F}(E)$. The domain, Fenchel conjugate, subdifferential of $f \in \mathcal{F}(E)$ are defined as $dom\, f = \{x \in E : f(x) < \infty\}$, $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}$ for $x^* \in E^*$ and

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y), \ \forall y \in E\},$$

respectively. In addition, for any $x \in int\, dom\, f$ and any $y \in E$, we denote by $f'(x, y)$ the right-hand derivative of f at x in the direction of y, that is,

$$f'(x,y) = \lim_{t \to 0+} \frac{f(x+ty) - f(x)}{t} \tag{10}$$

and the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle \nabla f(x), y \rangle := f'(x, y)$ for all $y \in E$.

Remark 2.1. When the subdifferential of f is single-valued, it coincides with the gradient of f, that is, $\partial f = \nabla f$ (see, e.g., [27]).

Definition. The function f is called:

(i) Gâteaux differentiable at x if limit as $t \to 0$ in (10) exists for any $y \in E$.
(ii) Gâteaux differentiable if it is Gâteaux differentiable at every point $x \in int\ dom f$.
(iii) uniformly Fréchet differentiable on a subset C of E if the limit as $t \to 0$ in (10) is attained uniformly for $x \in C$ and $||y|| = 1$.
(iv) strongly coercive if $\lim_{||x|| \to \infty} \frac{f(x)}{||x||} = \infty$.

Remark 2.2. If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is norm-to-norm uniformly continuous on bounded subsets of E and hence both f and ∇f are bounded on bounded subset of E.

Definition. A function f is said to be Legendre if it satisfies the following conditions:

(i) f is Gâteaux differentiable, $int\, dom\, f \neq \emptyset$ and $dom\, \nabla f = int\, dom\, f$;
(ii) f^* is Gâteaux differentiable, $int\, dom\, f^* \neq \emptyset$ and $dom\, \nabla f^* = int\, dom\, f^*$.

Remark 2.3. If $f : E \to (-\infty, \infty]$ is a Legendre function, then $\nabla f^* = (\nabla f)^{-1}$ (see [8]). It also follows that f is Legendre if and only if f^* is Legendre (see [5]) and that the functions f and f^* are Gâteaux differentiable and strictly convex in the interior of their respective domains.

If E is a real reflexive, smooth and strictly convex Banach space, then the function $f(x) = \frac{1}{p}||x||^p$ $(1 < p < \infty)$ is an example of a proper, lower semi continuous Legendre function with conjugate function $f^*(x^*) = \frac{1}{q}||x^*||^q$ $(1 < q < \infty)$, where $\frac{1}{p} + \frac{1}{q} = 1$ (see, e.g., [4]). In this case, ∇f coincides with the *generalized duality mapping* J_p of E, that is, $\nabla f = J_p$, where $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{y^* \in E^* : \langle x, y^* \rangle = ||x||^p,\ ||y^*|| = ||x||^{p-1}\}. \tag{11}$$

Definition. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable convex function. The function $D_f : dom\, f \times int\ dom\, f \to [0, \infty)$ defined by

$$D_f(y, x) = f(y) - f(x) - \langle \nabla f(x), y - x \rangle,\ x, y \in E \tag{12}$$

is called the Bregman distance with respect to f (see, e.g., [14]).

The Bregman projection with respect to f at $x \in int\, dom f$ onto a nonempty, closed and convex set $C \subset int\ dom f$ is denoted by $P_C^f(x) \in C$ which satisfies

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}. \tag{13}$$

Remark 2.4. We note that the Bregman distance is not a distance in the usual sense because it is not symmetric and does not satisfy the triangle inequality. However, it has the following important properties (see, e.g., [9, 28, 29]):

(i) $D_f(y, x) \geq 0\ \forall x, y \in E$ and $D_f(x, y) = 0$ if and only if $x = y$.
(ii) Three point identity: for any $x \in dom\, f$ and $y, z \in int\, dom\, f$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle. \tag{14}$$

Lemma 2.5 ([10])**.** *Let f be Gâteaux differentiable and totally convex on $int\, dom\, f$. Let $x \in int\, dom\, f$ and let $C \subset int\, dom\, f$ be a nonempty, closed and convex set. The Bregman projection P_C^f from E onto C is a unique vector with the following properties*:

(i) $z = P_C^f(x)$ *if and only if* $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0,\ \forall y \in\ C$;
(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x),\ \forall y \in\ C$.

The function $V_f : E \times E^* \to [0, +\infty)$ associated with a Legendre function $f : E \to \mathbb{R}$ is defined by

$$V_f(x, x^*) = f(x) - \langle x^*, x\rangle + \nabla f^*(x^*),\ \forall x \in E,\ x^* \in E^*. \tag{15}$$

V_f is nonnegative (see, e.g., [31]),

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)),\ \forall x \in E,\ x^* \in E^*, \tag{16}$$

and

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leq V_f(x, x^* + y^*),\ \forall x \in E,\ x^*, y^* \in E^*. \tag{17}$$

Lemma 2.6 ([27])**.** *Let $f : E \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function. Then, $f^* : E^* \to (-\infty, +\infty]$ is a proper, weak* lower semi-continuous and convex function and hence for all $z \in\ E$, we have*

$$D_f\Big(z, \nabla f^*\Big(\sum_{i=1}^{N} t_i \nabla f(x_i)\Big)\Big) \leq \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{18}$$

where $\{x_i\} \subseteq E$ and $\{t_i\} \subseteq (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

Lemma 2.7 ([24])**.** *Suppose $f : E \to (-\infty, \infty]$ is a Legendre function with ∇f^* is bounded on bounded subsets of $int\, dom f^*$. Let $x \in int\, dom f$. If $\{D_f(x, x_n)\}$ is bounded, so is the sequence $\{x_n\}$.*

Lemma 2.8 ([24])**.** *Let $f : E \to (-\infty, \infty]$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. Then,* $\lim_{n\to\infty} D_f(x_n, y_n) = 0$ *if and only if* $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

Definition. A Gâteaux differentiable function f is called:

(i) strongly convex with constant $\alpha > 0$ if for all $x, y \in dom f$

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \alpha ||x - y||^2, \tag{19}$$

or equivalently (see, [25]) for all $x, y \in dom f$

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\alpha}{2} ||x - y||^2.$$

(ii) a uniformly convex function with modulus ϕ if

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) - t(1 - t)\phi(||x - y||)$$

for all $x, y \in dom f$ and $t \in [0, 1]$, where ϕ is a function which is increasing and vanishes only at 0.

(iii) modulus of total convexity of f at $x \in int\, dom f$ if the function $\nu_f(\cdot, \cdot) : E \times \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\nu_f(x, t) = \inf_{\{y \in E: ||x - y|| = t\}} D_f(y, x).$$

If $\nu_f(x, t) > 0, \ \forall x \in E, \ t > 0$, then the function f is said to be totally convex.

Remark 2.9. Note that strongly convex function is uniformly convex function with $\phi(\alpha) = \frac{\mu}{2}\alpha^2$. Thus, the class of uniformly convex functions contains the class of strongly convex functions. Moreover, the function f is totally convex on bounded subsets of E if and only if it is uniformly convex on bounded subsets of E (see, e.g., [10]).

If, in addition, E is smooth, strictly convex, then the function $f(x) = \frac{1}{2}||x||^2$ is strongly convex with a constant parameter $0 < \alpha \leq 1$ and conjugate $f^*(x^*) = \frac{1}{2}||x^*||^2$. Moreover, it is a bounded, strongly coercive and uniformly Fréchet differentiable Legendre function in a bounded subset of E.

Lemma 2.10 ([37]). *If $f : E \to (-\infty, +\infty]$ is a convex function, then the following statements are equivalent*:

(i) *∇f^* is uniformly continuous on bounded subsets of E, $dom f^* = E^*$ and f^* is Frèchet differentiable.*

(ii) *f is uniformly convex and strongly coercive.*

Lemma 2.11 ([20]). *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E. Let A be a continuous pseudo-monotone mapping from C into E^*. Then, $VI(C, A)$ is closed and convex. Furthermore, we have $p \in VI(C, A)$ if and only if $\langle Ax, x - p \rangle \geq 0$ for all $x \in C$.*

Lemma 2.12 ([17]). *Let f be a total convex Gâteaux differentiable such that $dom\, f = E$. Then, for each $x^* \in E^* \backslash \{0\}$, $\tilde{y} \in E$, $x \in H^+$ and $\tilde{x} \in H^-$, it holds that*

$$D_f(\tilde{x}, x) \geq D_f(\tilde{x}, z) + D_f(z, x), \tag{20}$$

where $z = argmin_{y \in H} D_f(y, x)$ and $H^- = \{y \in E : \langle x^*, y - \tilde{y} \rangle \leq 0\}$, $H = \{y \in E : \langle x^*, y - \tilde{y} \rangle = 0\}$ and $H^+ = \{y \in E : \langle x^*, y - \tilde{y} \rangle \geq 0\}$.

Lemma 2.13 ([36]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n, \ n \geq n_0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions:

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \ \textit{and} \ \limsup_{n\to\infty} \delta_n \leq 0.$$

Then

$$\lim_{n\to\infty} a_n = 0.$$

Lemma 2.14 ([19]). *Let $\{s_k\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{k_j}\}$ of $\{s_k\}$ such that $s_{k_j} < s_{k_j+1}$ for all $j \geq 0$. Define an integer sequence $\{m_k\}_{k\geq k_0}$ as*

$$m_k = \max\{k_0 \leq l \leq k : s_l < s_{l+1}\}. \tag{21}$$

Then $m_k \to \infty$ as $k \to \infty$ and for all $k \geq k_0$

$$\max\{s_{m_k}, s_k\} \leq s_{m_k+1}. \tag{22}$$

3. Main results

Hereafter, let E_1, E_2 and E_3 be real reflexive Banach spaces with its dual E_1^*, E_2^* and E_3^*, respectively. Let C and D be nonempty, closed and convex subsets of E_1 and E_2, respectively.

In the sequel, we shall make use of the following assumptions.

Assumption 1.

- (A1) Let $f_1 : E_1 \to (-\infty,+\infty] \in \mathcal{F}(E_1)$, $f_2 : E_2 \to (-\infty,+\infty] \in \mathcal{F}(E_2)$ and $f_3 : E_3 \to (-\infty,+\infty] \in \mathcal{F}(E_3)$ be strongly coercive, bounded, uniformly Fréchet differentiable Legendre functions on bounded subset of E_1, E_2 and E_3, respectively. In addition, let f_1, f_2 and f_3 be strongly convex with constants α_1, α_2 and α_3, respectively. We denote $\alpha = \min\{\alpha_1, \alpha_2, \alpha_3\}$.
- (A2) Let $G : C \to E_1^*$ and $H : D \to E_2^*$ be continuous pseudomonotone mappings.
- (A3) Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators with adjoints $A^* : E_3^* \to E_1^*$ and $B^* : E_3^* \to E_2^*$, respectively.
- (A4) Let $\Omega = \{(u,v) : u \in VI(C,G)$ and $v \in VI(D,H)$ such that $Au = Bv\} \neq \emptyset$.

Assumption 2.

- (B1) Let $l \in (0,1)$, $\eta > 0$ and $\theta \in [\underline{\theta}, \overline{\theta}] \subset (0, \frac{1}{\eta})$.
- (B2) Let $\{\alpha_n\} \subset (0,\epsilon) \subset (0,1)$, for some constant $\epsilon > 0$, be such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(B3) Let $0 < \mu \leq \gamma_n \leq \frac{\alpha^2||Au_n - Bv_n||^2}{2[||A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)||^2 + ||B^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)||^2]}$ for $n \in \Upsilon = \{n \in \mathbb{N} : Au_n - Bv_n \neq 0\}$, otherwise $\gamma_n = \gamma > 0$.

Algorithm 1: For arbitrary $(u_0, v_0), (u, v) \in C \times D$, define an iterative algorithm by

Step 1. Compute

$$x_n = P_C^{f_1} \nabla f_1^* [\nabla f_1 u_n - \gamma_n A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)] \text{ and } d(x_n) = x_n - u_n.$$

Compute

$$y_n = P_D^{f_2} \nabla f_2^* [\nabla f_2 v_n - \gamma_n B^*(\nabla f_3 Bv_n - \nabla f_3 Au_n)] \text{ and } d(y_n) = y_n - v_n.$$

Step 2. Compute

$$w_n = \nabla f_1^* [\nabla f_1 x_n - \theta G x_n] \text{ and } d(w_n) = x_n - P_C^{f_1} w_n. \tag{23}$$

Compute

$$z_n = \nabla f_2^* [\nabla f_2 y_n - \theta H y_n] \text{ and } d(z_n) = y_n - P_D^{f_2} z_n. \tag{24}$$

If $d(x_n) = d(y_n) = d(w_n) = d(z_n) = 0$, then stop and $(u_n, v_n) \in \Omega$. Otherwise go to Step 2.

Step 3. Compute $p_n = x_n - \tau_n d(w_n)$ where $\tau_n = l^{j_n}$ and j_n is the smallest non-negative integer j satisfying

$$\langle G x_n - G(x_n - l^j d(w_n)), d(w_n) \rangle \leq \eta D_{f_1}(P_C^{f_1} w_n, x_n)$$

and $q_n = y_n - \rho_n d(z_n)$ where $\rho_n = j_n'$ and j_n' is the smallest non-negative integer j' satisfying

$$\langle H y_n - H(y_n - l^{j'} d(z_n)), d(z_n) \rangle \leq \eta D_{f_2}(P_D^{f_2} z_n, y_n).$$

Step 4. Compute

$$\begin{cases} a_n = P_{P_n}^{f_1} \nabla f_1^* (\nabla f_1 x_n - \theta G p_n), \\ b_n = P_{Q_n}^{f_2} \nabla f_2^* (\nabla f_2 y_n - \theta H q_n), \\ u_{n+1} = P_C^{f_1} \nabla f_1^* (\alpha_n \nabla f_1 u + (1 - \alpha_n) \nabla f_1 s_n), \\ v_{n+1} = P_D^{f_2} \nabla f_2^* (\alpha_n \nabla f_2 v + (1 - \alpha_n) \nabla f_2 t_n), \end{cases} \tag{25}$$

where $P_n = \{p \in C : \langle G p_n, p - p_n \rangle = 0\}$, $Q_n = \{q \in D : \langle H q_n, q - q_n \rangle = 0\}$, $s_n = P_C^{f_1} a_n$ and $t_n = P_D^{f_2} b_n$. Set $n := n + 1$ and go to **Step 1**.

We shall need the following lemmas in the sequel.

Lemma 3.1. *Suppose Assumptions* (A1), (A2), *and* (B1) *hold. Let* $\{x_n\}$, $\{y_n\}$, $\{w_n\}$ *and* $\{z_n\}$ *be the sequences generated by Algorithm 1. Then, the inequalities in Step 3 are well defined.*

Proof. Since $l \in (0,1)$ and the fact that G and H are continuous on C and D, respectively, we have

$$\langle Gx_n - G(x_n - l^j d(w_n)), d(w_n)\rangle \to 0$$

as $j \to \infty$ and

$$\langle Hy_n - H(y_n - l^{j'} d(z_n)), d(z_n)\rangle \to 0$$

as $j' \to \infty$.

On the other hand, the fact that $D_{f_1}(P_C^{f_1} w_n, x_n)$ and $D_{f_2}(P_D^{f_2} z_n, y_n) > 0$, there exists non-negative integers j_n and j_n' satisfying inequalities in Step 2 and the claim hold. □

Lemma 3.2. *Suppose Assumptions* (A1), (A2), *and* (B1) *hold. Let* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *and* $\{w_n\}$ *be the sequences generated by Algorithm 1. Then, we have*

$$\langle Gx_n, d(w_n)\rangle \geq \frac{1}{\theta} D_{f_1}(P_C^{f_1} w_n, x_n) \text{ and } \langle Hx_n, d(z_n)\rangle \geq \frac{1}{\theta} D_{f_1}(P_D^{f_2} z_n, y_n).$$

Proof. From (23), we have

$$\nabla f_1 x_n - \nabla f_1 w_n = \theta G x_n. \tag{26}$$

Thus, from (26), (23) and (14), we get

$$\begin{aligned}\langle Gx_n, d(w_n)\rangle &= \frac{1}{\theta}\langle \nabla f_1 x_n - \nabla f_1 w_n, x_n - P_C^{f_1} w_n\rangle \\ &= \frac{1}{\theta}[D_{f_1}(P_C^{f_1} w_n, x_n) + D_{f_1}(x_n, w_n) - D_{f_1}(P_C^{f_1} w_n, w_n)] \\ &\geq \frac{1}{\theta} D_{f_1}(P_C^{f} w_n, x_n). \qquad (27)\end{aligned}$$

Similarly,

$$\langle Hy_n, d(z_n)\rangle \geq \frac{1}{\theta} D_{f_2}(P_D^{f_2} z_n, y_n) \tag{28}$$

and hence the assertion hold. □

Lemma 3.3. *Assume that Assumptions* (A1)-(A3) *and* (B1) *hold. Let* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *and* $\{w_n\}$ *are the sequences generated by Algorithm 1. Then,* $\langle Gp_n, x_n - p_n\rangle \geq \tau_n(\frac{1}{\theta} - \eta) D_{f_1}(P_C^{f_1} y_n, x_n)$ *and* $\langle Hq_n, y_n - q_n\rangle \geq \rho_n(\frac{1}{\theta} - \eta) D_{f_1}(P_D^{f_2} z_n, y_n)$. *In particular if* $d(w_n) \neq 0$ *and* $d(z_n) \neq 0$, *then* $\langle Gp_n, x_n - p_n\rangle > 0$ *and* $\langle Hq_n, y_n - q_n\rangle > 0$, *respectively.*

Proof. Using Step 3 of the algorithm we know that

$$\begin{aligned}\langle Gp_n, x_n - p_n\rangle &= \langle Gp_n, x_n - (x_n - \tau_n d(w_n))\rangle \\ &= \langle Gp_n, \tau_n d(w_n)\rangle = \tau_n \langle Gp_n, d(w_n)\rangle. \qquad (29)\end{aligned}$$

On the other hand,

$$\langle Gx_n - Gp_n, d(w_n)\rangle \leq \eta D_{f_1}(P_C^{f_1} w_n, x_n) \tag{30}$$

which implies

$$\langle Gp_n, d(w_n)\rangle \geq \langle Gx_n, d(w_n)\rangle - \eta D_{f_1}(P_C^{f_1} w_n, x_n). \tag{31}$$

From (31) and Lemma 3.2, we obtain

$$\langle Gp_n, d(w_n)\rangle \geq (\frac{1}{\theta} - \eta) D_{f_1}(P_C^{f_1} w_n, x_n). \tag{32}$$

Thus, by combining (29) and (32), we get

$$\langle Gp_n, x_n - p_n\rangle \geq \tau_n(\frac{1}{\theta} - \eta) D_{f_1}(P_C^{f_1} w_n, x_n). \tag{33}$$

Similarly,

$$\langle Hq_n, y_n - q_n\rangle \geq \rho_n(\frac{1}{\theta} - \eta) D_{f_2}(P_D^{f_2} z_n, y_n) \tag{34}$$

and the proof is complete. □

Theorem 3.4. *Suppose Assumptions* (A1)-(A4) *and* (B1)-(B3) *hold. Then, the sequence* $\{(u_n, v_n)\}$ *generated by Algorithm 1 is bounded in* $C \times D$.

Proof. Let $(u^*, v^*) \in \Omega$. Then, $u^* \in VI(C, G)$, $v^* \in VI(C, H)$ and $Au^* = Bv^*$. Now, for each $n \geq 0$, define the sets: $P_n^- = \{p \in C : \langle Gx_n, p - x_n\rangle \leq 0\}$, $P_n = \{p \in C : \langle Gx_n, p - x_n\rangle = 0\}$, and $P_n^+ = \{p \in C : \langle Gx_n, p - x_n\rangle \geq 0\}$. Hence, from definition of G, we have $\langle Gu^*, x - u^*\rangle \geq 0$ which implies $\langle Gx, x - u^*\rangle \geq 0$ for all $x \in C$ and hence $u^* \in P_n^-$ for all $n \geq 0$. Moreover, from Lemma 3.3, we have $\langle Gp_n, x_n - p_n\rangle > 0$ which implies that $x_n \in P_n^+$ and $x_n \notin P_n^-$ for all $n \geq 0$. Now, from Lemma 2.12, we obtain

$$D_{f_1}(u^*, a_n) + D_{f_1}(a_n, x_n) \leq D_{f_1}(u^*, x_n). \tag{35}$$

Since $s_n = P_C^f a_n$, from Lemma 2.5, we get

$$D_{f_1}(u^*, s_n) + D_{f_1}(s_n, a_n) \leq D_{f_1}(u^*, a_n). \tag{36}$$

By substituting (36) into (35), we get

$$D_{f_1}(u^*, s_n) + D_{f_1}(s_n, a_n) + D_{f_1}(a_n, x_n) \leq D_{f_1}(u^*, x_n), \tag{37}$$

which implies

$$D_{f_1}(u^*, s_n) \leq D_{f_1}(u^*, x_n) - D_{f_1}(s_n, a_n) - D_{f_1}(a_n, x_n). \tag{38}$$

Similarly, we obtain that

$$D_{f_2}(u^*, t_n) \leq D_{f_2}(v^*, y_n) - D_{f_2}(t_n, b_n) - D_{f_2}(b_n, y_n). \tag{39}$$

Denote $e_n = \nabla f_1^*[\nabla f_1 u_n - \gamma_n A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)]$ and $k_n = \nabla f_2^*[\nabla f_2 v_n - \gamma_n B^*(\nabla f_3 Bv_n - \nabla f_3 Au_n)]$. Then, $x_n = P_C^{f_1} e_n$ and $y_n = P_D^{f_2} k_n$. Hence, from Lemma 2.5 and (16), we obtain

$$\begin{aligned} D_{f_1}(u^*, x_n) &\leq D_{f_1}(u^*, \nabla f_1^*[\nabla f_1 u_n - \gamma_n(\nabla f_3 Au_n - \nabla f_3 Bv_n)]) \\ &= V_{f_1}(u^*, \nabla f_1 u_n - \gamma_n A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)) \\ &\leq V_{f_1}(u^*, \nabla f_1 u_n) - \langle e_n - u^*, \gamma_n A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)\rangle \end{aligned}$$

$$= D_{f_1}(u^*, u_n) - \gamma_n \langle e_n - u^*, A^*(\nabla f_3 A u_n - \nabla f_3 B v_n)\rangle$$

$$= D_{f_1}(u^*, u_n) - \gamma_n \langle A e_n - A u^*, \nabla f_3 A u_n - \nabla f_3 B v_n\rangle. \tag{40}$$

Similarly, we get

$$D_{f_2}(v^*, y_n) \le D_{f_2}(v^*, v_n) - \gamma_n \langle B k_n - B v^*, \nabla f_3 B v_n - \nabla f_3 A u_n\rangle. \tag{41}$$

Furthermore, from (25), Lemma 2.5(ii) and Lemma 2.6, we have

$$\begin{aligned} D_{f_1}(u^*, u_{n+1}) &= D_{f_1}(u^*, P_C^{f_1} \nabla f_1^*(\alpha_n \nabla f_1 v + (1-\alpha_n)\nabla f_1 s_n)) \\ &\le D_{f_1}(u^*, \nabla f_1^*(\alpha_n \nabla f_1 v + (1-\alpha_n)\nabla f_1 s_n)) \\ &\le \alpha_n D_{f_1}(u^*, u) + (1-\alpha_n) D_{f_1}(u^*, s_n). \end{aligned} \tag{42}$$

Similarly, we get

$$D_{f_2}(v^*, v_{n+1}) \le \alpha_n D_{f_2}(v^*, v) + (1-\alpha_n) D_{f_1}(v^*, t_n). \tag{43}$$

Now, from inequalities (38), (40) and (42), we get

$$\begin{aligned} &D_{f_1}(u^*, u_{n+1}) \\ &\le \alpha_n D_{f_1}(u^*, u) + (1-\alpha_n)\alpha_n D_{f_1}(u^*, s_n) \\ &\le \alpha_n D_{f_1}(u^*, u) + (1-\alpha_n)[D_{f_1}(u^*, x_n) - D_{f_1}(s_n, a_n) - D_{f_1}(a_n, x_n)] \\ &\le \alpha_n D_{f_1}(u^*, u) + (1-\alpha_n)[D_{f_1}(u^*, u_n) - D_{f_1}(s_n, a_n) - D_{f_1}(a_n, x_n)] \\ &\quad - (1-\alpha_n)\gamma_n \langle A e_n - A u^*, \nabla f_3 A u_n - \nabla f_3 B v_n\rangle. \end{aligned} \tag{44}$$

Similarly, from inequalities (39), (41) and (43), we obtain

$$\begin{aligned} &D_{f_2}(v^*, v_{n+1}) \\ &\le \alpha_n D_{f_2}(v^*, v) + (1-\alpha_n)[D_{f_2}(v^*, v_n) - D_{f_2}(t_n, b_n) - D_{f_2}(b_n, y_n)] \\ &\quad - (1-\alpha_n)\gamma_n \langle B k_n - B v^*, \nabla f_3 A u_n - \nabla f_3 B v_n\rangle. \end{aligned} \tag{45}$$

Denote $\Psi_n = D_{f_1}(u^*, u_n) + D_{f_2}(v^*, v_n)$ and $\Psi = D_{f_1}(u^*, u) + D_{f_2}(v^*, v)$. Then by adding inequalities (44) and (45), we get

$$\begin{aligned} \Psi_{n+1} &\le \alpha_n \Psi + (1-\alpha_n)\Psi_n - (1-\alpha_n)[D_{f_1}(s_n, a_n) + D_{f_1}(a_n, x_n)] \\ &\quad - (1-\alpha_n)[D_{f_2}(t_n, b_n) + D_{f_2}(b_n, y_n)] \\ &\quad - (1-\alpha_n)\gamma_n \langle A e_n - A u^* + B v^* - B k_n, \nabla f_3 A u_n - \nabla f_3 B v_n\rangle \\ &= \alpha_n \Psi + (1-\alpha_n)\Psi_n - (1-\alpha_n)[D_{f_1}(s_n, a_n) + D_{f_1}(a_n, x_n)] \\ &\quad - (1-\alpha_n)[D_{f_2}(t_n, b_n) + D_{f_2}(b_n, y_n)] \\ &\quad - (1-\alpha_n)\gamma_n \langle A e_n - B k_n, \nabla f_3 A u_n - \nabla f_3 B v_n\rangle. \end{aligned} \tag{46}$$

Observe that,

$$\begin{aligned} &- \gamma_n \langle A e_n - B k_n, \nabla f_3 A u_n - \nabla f_3 B v_n\rangle \\ = &- \gamma_n \langle A u_n - B v_n, \nabla f_3 A u_n - \nabla f_3 B v_n\rangle \\ &- \gamma_n \langle e_n - u_n, A^*(\nabla f_3 A u_n - \nabla f_3 B v_n)\rangle \\ &- \gamma_n \langle v_n - k_n, B^*(\nabla f_3 A u_n - \nabla f_3 B v_n)\rangle \end{aligned}$$

$$
\begin{aligned}
&\leq -\alpha_3||Au_n - Bv_n||^2 \\
&\quad + ||e_n - u_n|| ||A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)|| \\
&\quad + ||k_n - v_n|| ||B^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)|| \\
&\leq -\alpha||Au_n - Bv_n||^2 \\
&\quad + ||e_n - u_n|| ||A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)|| \\
&\quad + ||k_n - v_n|| ||B^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)||. \qquad (47)
\end{aligned}
$$

Moreover, from definitions of f_1 and e_n, we get

$$
\begin{aligned}
||u_n - e_n|| &= ||\nabla f_1^*(\nabla f_1 u_n) - \nabla f_1^*[\nabla f_1 u_n - \gamma_n A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)]|| \\
&\leq \frac{\gamma_n}{\alpha_1}||A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)|| \\
&\leq \frac{\gamma_n}{\alpha}||A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)||. \qquad (48)
\end{aligned}
$$

Similarly, from definitions of f_2 and k_n, we get

$$
||v_n - k_n|| \leq \frac{\gamma_n}{\alpha}||B^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)||. \qquad (49)
$$

Thus, from inequalities (47), (48) and (49), we obtain

$$
\begin{aligned}
&-\gamma_n\langle Ae_n - Bk_n, \nabla f_3 Au_n - \nabla f_3 Bv_n\rangle \\
&\leq -\gamma_n\alpha||Au_n - Bv_n||^2 \\
&\quad + \frac{\gamma_n^2}{\alpha}||A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)||^2 \\
&\quad + \frac{\gamma_n^2}{\alpha}||B^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)||^2 \\
&\leq -\frac{\mu\alpha}{2}||Au_n - Bv_n||^2 - \gamma_n\Big[\frac{\alpha}{2}||Au_n - Bv_n||^2 \\
&\quad - \frac{\gamma_n}{\alpha}\Big(||A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)||^2 \\
&\quad + ||B^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)||^2\Big)\Big] \\
&\leq -\frac{\mu\alpha}{2}||Au_n - Bv_n||^2. \qquad (50)
\end{aligned}
$$

Finally, from (46) and (50), we obtain

$$
\begin{aligned}
\Psi_{n+1} &\leq \alpha_n\Psi + (1-\alpha_n)\Psi_n - (1-\alpha_n)[D_{f_1}(s_n, a_n) + D_{f_1}(a_n, x_n)] \\
&\quad - (1-\alpha_n)[D_{f_2}(t_n, b_n) + D_{f_2}(b_n, y_n)] \\
&\quad - (1-\alpha_n)\frac{\mu\alpha}{2}||Au_n - Bv_n||^2 \qquad (51) \\
&\leq \alpha_n\Psi + (1-\alpha_n)\Psi_n \qquad (52)
\end{aligned}
$$

and hence by induction we get

$$
\Psi_n \leq \max\{\Psi_0, \Psi\}. \qquad (53)
$$

Thus, the sequence $\{\Psi_n\}$ is bounded. Therefore, by Lemma 2.7, $\{u_n\}$ and $\{v_n\}$ are bounded and so are $\{a_n\}$, $\{b_n\}$, $\{x_n\}$, $\{y_n\}$, $\{w_n\}$, $\{z_n\}$, $\{Gx_n\}$, $\{Hy_n\}$. □

Theorem 3.5. *Suppose Assumptions* (A1)-(A4) *and* (B1)-(B2) *hold. Then, the sequence* $\{(u_n, v_n)\}$ *generated by Algorithm 1 converges strongly to an element* $(u^*, v^*) = P_\Omega^f(u, v)$.

Proof. Let $(u^*, v^*) \in \Omega$ such that $(u^*, v^*) = P_\Omega^f(u, v)$. From Theorem 3.4, we have that the sequence $\{(u_n, v_n)\}$ is bounded. Denote $r_n = \nabla f_1^*(\alpha_n \nabla f_1 v + (1-\alpha_n)\nabla f_1 s_n)$, $g_n = \nabla f_2^*(\alpha_n \nabla f_2 v + (1-\alpha_n)\nabla f_2 t_n)$. Then, using the same techniques of Theorem 2 of [34] and Theorem 2 of [41], we get

$$\begin{aligned} D_{f_1}(u^*, u_{n+1}) \leq\ & (1-\alpha_n) D_{f_1}(u^*, u_n) + \alpha_n ||u_n - r_n|| ||\nabla f_1 u - \nabla f_1 u^*|| \\ & + \alpha_n \langle \nabla f_1 u - \nabla f_1 u^*, u_n - u^* \rangle \end{aligned} \tag{54}$$

and

$$\begin{aligned} D_{f_2}(v^*, v_{n+1}) \leq\ & (1-\alpha_n) D_{f_2}(v^*, v_n) + \alpha_n ||v_n - g_n|| ||\nabla f_2 v - \nabla f_2 v^*|| \\ & + \alpha_n \langle \nabla f_2 v - \nabla f_2 v^*, v_n - v^* \rangle. \end{aligned} \tag{55}$$

Thus, by adding inequalities (54) and (55), we obtain

$$\begin{aligned} \Psi_{n+1} \leq\ & (1-\alpha_n)\Psi_n + \alpha_n ||u_n - r_n|| ||\nabla f_1 u - \nabla f_1 u^*|| \\ & + \alpha_n ||v_n - g_n|| ||\nabla f_2 v - \nabla f_2 v^*|| \\ & + \alpha_n \langle \nabla f_1 u - \nabla f_1 u^*, u_n - u^* \rangle \\ & + \alpha_n \langle \nabla f_2 v - \nabla f_2 v^*, v_n - v^* \rangle, \end{aligned} \tag{56}$$

where $\Psi_n = D_{f_1}(u^*, u_n) + D_{f_2}(v^*, v_n)$.

Now, we divide the rest of the proof into two parts as follows:

Case 1. Assume that there exists $n_0 \in \mathbb{N}$ such that the sequence of real numbers Ψ_n is decreasing for all $n \geq n_0$. Thus, the sequence Ψ_n is convergence and hence $\Psi_n - \Psi_{n+1} \to 0$ as $n \to \infty$. Hence, from (52) and the conditions on the sequences $\{\alpha_n\}$, we obtain

$$\lim_{n\to\infty} ||Au_n - Bv_n|| = 0 \tag{57}$$

and

$$\begin{aligned} & \lim_{n\to\infty} [D_{f_1}(s_n, a_n) + D_{f_1}(a_n, x_n)] \\ = & \lim_{n\to\infty} [D_{f_2}(t_n, b_n) + D_{f_2}(b_n, y_n)] = 0. \end{aligned} \tag{58}$$

Hence, from (58) and Lemma 2.8, we get

$$\lim_{n\to\infty} ||s_n - a_n|| = \lim_{n\to\infty} ||x_n - a_n|| = 0, \tag{59}$$

and

$$\lim_{n\to\infty} ||t_n - b_n|| = \lim_{n\to\infty} ||y_n - b_n|| = 0. \tag{60}$$

From Step 1 and Lemma 2.5, we get

$$\begin{aligned} D_{f_1}(u_n, x_n) &= D_{f_1}(u_n, P_C^{f_1}\nabla f_1^*[\nabla f_1 u_n - \gamma_n A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)]) \\ &\leq D_{f_1}(u_n, \nabla f_1^*[\nabla f_1 u_n - \gamma_n A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)]) \\ &= V_{f_1}(u_n, \nabla f_1 u_n - \gamma_n A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)) \\ &\leq V_{f_1}(u_n, \nabla f_1 u_n) - \langle \gamma_n A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n), \\ &\quad \nabla f_1^*[\nabla f_1 u_n - \gamma_n A^*(\nabla f_3 Au_n - \nabla f_3 Bv_n)] - u_n\rangle \\ &\leq D_{f_1}(u_n, u_n) + \frac{\gamma_n^2}{\alpha}||A||^2||Au_n - Bv_n||^2. \end{aligned} \tag{61}$$

Thus, from (61) and (57), we get

$$\lim_{n\to\infty} D_{f_1}(u_n, x_n) = 0, \tag{62}$$

which implies

$$\lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{63}$$

Similarly, we get

$$\lim_{n\to\infty} ||v_n - y_n|| = 0. \tag{64}$$

Now, observe that

$$||u_n - a_n|| \leq ||u_n - x_n|| + ||x_n - a_n||. \tag{65}$$

Hence, from (59), (63) and (65), we get

$$\lim_{n\to\infty} ||u_n - a_n|| = 0, \tag{66}$$

which implies

$$\lim_{n\to\infty} D_{f_1}(u_n, a_n) = 0. \tag{67}$$

Moreover, from (25) and Lemma 2.6, we obtain

$$\begin{aligned} D_{f_1}(u_n, r_n) &= D_{f_1}(u_n, \nabla f_1^*[\alpha_n \nabla f_1 u + (1-\alpha_n)\nabla f_1 s_n]) \\ &\leq \alpha_n D_{f_1}(u_n, u) + (1-\alpha_n) D_{f_1}(u_n, s_n) \\ &\leq \alpha_n D_{f_1}(u_n, u) + (1-\alpha_n) D_{f_1}(u_n, a_n). \end{aligned} \tag{68}$$

From (67) and (68), we obtain

$$\lim_{n\to\infty} D_{f_1}(u_n, r_n) = 0, \tag{69}$$

which implies that

$$\lim_{n\to\infty} ||u_n - r_n|| = 0. \tag{70}$$

Similarly, we get

$$\lim_{n\to\infty} ||v_n - g_n|| = 0. \tag{71}$$

Now, since the sequence $\{(u_n, v_n)\}$ is bounded in $C \times D$, there exist $(\hat{u}, \hat{v}) \in C \times D$ and a subsequence $\{(u_{n_k}, v_{n_k})\}$ of (u_n, v_n) such that $\{(u_{n_k}, v_{n_k})\} \rightharpoonup (\hat{u}, \hat{v})$ and

$$\begin{aligned} &\limsup_{n\to\infty}[\langle u_n - u^*, \nabla f_1 u - \nabla f_1 u^*\rangle + \langle v_n - u^*, \nabla f_2 v - \nabla f_2 v^*\rangle] \\ &= \lim_{k\to\infty}[\langle u_{n_k} - u^*, \nabla f_1 u - \nabla f_1 v^*\rangle + \langle v_{n_k} - v^*, \nabla f_2 v - \nabla f_2 v^*\rangle]. \end{aligned} \tag{72}$$

It then follows that $u_{n_k} \rightharpoonup \hat{u}$ in C and $v_{n_k} \rightharpoonup \hat{v}$ in D. From (63) and (67) $x_{n_k} \rightharpoonup \hat{u}$ in C and $y_{n_k} \rightharpoonup \hat{v}$ in D, respectively.

Next, we prove that $(\hat{u}, \hat{v}) \in VI(C, G) \times VI(C, H)$ and $A\hat{u} = B\hat{v}$. Since $a_n \in P_n$, then we obtain

$$\begin{aligned} 0 &= \langle Gp_{n_k}, a_{n_k} - p_{n_k}\rangle \\ &= \langle Gp_{n_k}, a_{n_k} - x_{n_k}\rangle + \langle Gp_{n_k}, x_{n_k} - p_{n_k}\rangle \end{aligned}$$

which implies that

$$\langle Gp_{n_k}, x_{n_k} - p_{n_k}\rangle = \langle Gp_{n_k}, x_{n_k} - a_{n_k}\rangle \leq ||Gp_{n_k}||||x_{n_k} - a_{n_k}||. \tag{73}$$

From (59) and (73), we get

$$\lim_{k\to\infty}\langle Gp_{n_k}, x_{n_k} - p_{n_k}\rangle = 0. \tag{74}$$

Similarly, we obtain

$$\lim_{k\to\infty}\langle Hq_{n_k}, y_{n_k} - q_{n_k}\rangle = 0. \tag{75}$$

Now, we prove

$$\lim_{k\to\infty}||P_C^{f_1} w_{n_k} - x_{n_k}|| = 0. \tag{76}$$

From (74), Lemma 3.3 and Lemma 2.8, we get

$$\lim_{k\to\infty}\tau_{n_k}||P_C^{f_1} w_{n_k} - x_{n_k}|| = 0. \tag{77}$$

First, consider the case when $\liminf_{k\to\infty}\tau_{n_k} > 0$. In this case, there is a constant $\tau > 0$ such that $\tau_{n_k} \geq \tau > 0$ for all $k \in \mathbb{N}$. Thus, we have

$$\begin{aligned} ||P_C^{f_1} w_{n_k} - x_{n_k}|| &= \frac{1}{\tau_{n_k}}\tau_{n_k}||P_C^{f_1} w_{n_k} - x_{n_k}|| \\ &\leq \frac{1}{\tau}\tau_{n_k}||P_C^{f_1} w_{n_k} - x_{n_k}||. \end{aligned} \tag{78}$$

Thus, from (77) and (78), we obtain

$$\lim_{k\to\infty}||P_C^{f_1} w_{n_k} - x_{n_k}|| = 0. \tag{79}$$

Second, we consider the case when $\liminf_{k\to\infty}\tau_{n_k} = 0$. In this case, we take a subsequence $\{n_{k_j}\}$ of $\{n_k\}$, if necessary, we assume without loss of generality that

$$\lim_{k\to\infty}\tau_{n_k} = 0 \text{ and } \lim_{k\to\infty}||x_{n_k} - P_C^{f_1} w_{n_k}|| = c > 0. \tag{80}$$

Consider $p'_{n_k} = \frac{1}{l}\tau_{n_k} P_C^{f_1} w_{n_k} + \big(1 - \frac{1}{l}\tau_{n_k}\big)x_{n_k}$. Then, from (80), we have

$$\lim_{k\to\infty} ||x_{n_k} - p'_{n_k}|| = \lim_{k\to\infty} \frac{1}{l}\tau_{n_k} ||x_{n_k} - P_C^{f_1} w_{n_k}|| = 0. \tag{81}$$

From Step 3 and the definition of p'_{n_k}, we get

$$\langle Gx_{n_k} - Gp'_{n_k}, x_{n_k} - P_C^{f_1} w_{n_k}\rangle > \eta D_{f_1}(P_C^{f_1} w_{n_k}, x_{n_k}). \tag{82}$$

Using (81), (82), Lemma 2.8 and the fact that G is uniformly continuous on bounded subsets of C, we obtain

$$\lim_{k\to\infty} ||P_C^{f_1} w_{n_k} - x_{n_k}|| = 0,$$

which is a contradiction to (80). Therefore, the equality in (76) holds. Similarly, we get

$$\lim_{k\to\infty} ||P_D^{f_2} z_{n_k} - y_{n_k}|| = 0. \tag{83}$$

By combining Lemmas 2.5 and 23, we get

$$\langle Gx_{n_k}, x - P_C^{f_1} w_{n_k}\rangle \geq \langle \nabla f_1 x_{n_k} - \nabla f_1 P_C^{f_1} w_{n_k}, x - P_C^{f_1} w_{n_k}\rangle,\ \forall x \in C,$$

which implies that

$$\begin{aligned}\langle Gx_{n_k}, x - x_{n_k}\rangle &\geq \langle Gx_{n_k}, P_C^{f_1} w_{n_k} - x_{n_k}\rangle \\ &\quad + \langle \nabla f_1 x_{n_k} - \nabla f_1 P_C^{f_1} w_{n_k}, x - P_C^{f_1} w_{n_k}\rangle,\ \forall x \in C.\end{aligned} \tag{84}$$

Thus, from (84), (83) and the fact that ∇f_1 is uniformly continuous, we obtain

$$\liminf_{k\to\infty} \langle Gx_{n_k}, x - x_{n_k}\rangle \geq 0,\ \forall x \in C. \tag{85}$$

Moreover, let $\{\xi_k\}$ be a sequence of decreasing numbers such that $\{\xi_k\} \to 0$ as $k \to \infty$, w be an arbitrary element of C and z be an arbitrary element of D. Using inequality (85), we can find a large enough N_k such that

$$\langle Gx_{n_k}, w - x_{n_k}\rangle + \xi_k \geq 0,\ \forall k \geq N_k. \tag{86}$$

From (86) and the fact that $Gx_{n_k} \neq 0$, we get

$$\langle Gx_{n_k}, \xi_k d_k + w - x_{n_k}\rangle \geq 0,\ \forall k \geq N_k, \tag{87}$$

for some $d_k \in C$ satisfying $\langle Gx_{n_k}, d_k\rangle = 1$. In addition, from definition of G and inequality (87), we have

$$\langle G(w + \xi_n d_k w), w + \xi_k d_k w - x_{n_k}\rangle \geq 0,\ \forall k \geq N_k, \tag{88}$$

which implies that

$$\begin{aligned}\langle Gw, w - x_{n_k}\rangle &\geq \langle Gw - G(w + \xi_k d_k w), w + \xi_k d_k w - x_{n_k}\rangle \\ &\quad - \xi_k \langle Gw, d_k\rangle,\ \forall k \geq N_k.\end{aligned} \tag{89}$$

Since $\xi_k \to 0$ as $k \to \infty$ and G is continuous, then from inequality (89), we obtain

$$\langle Gw, w - \hat{u}\rangle = \liminf_{k\to\infty} \langle Gw, w - x_{n_k}\rangle \geq 0,\ \forall w \in C. \tag{90}$$

Similarly, we get

$$\langle Hz, z - \hat{v} \rangle = \liminf_{k\to\infty} \langle Hz, z - y_{n_k} \rangle \geq 0, \ \forall z \in D. \tag{91}$$

Hence, by Lemma 2.11, $(\hat{u}, \hat{v}) \in VI(C, G) \times VI(D, H)$. Furthermore, from (19), we get

$$\begin{aligned} ||A\hat{u} - B\hat{v}||^2 &\leq \frac{1}{\alpha} \langle A\hat{u} - B\hat{v}, \nabla f_3 A\hat{u} - \nabla f_3 B\hat{v} \rangle \\ &= \frac{1}{\alpha} \langle Au_{n_k} - Bv_{n_k} + A\hat{u} - Au_{n_k} + Bv_{n_k} - B\hat{v}, \nabla f_3 A\hat{u} - \nabla f_3 B\hat{v} \rangle \\ &\leq \frac{1}{\alpha} ||Au_{n_k} - Bv_{n_k}|| ||\nabla f_3 A\hat{u} - \nabla f_3 B\hat{v}|| \\ &\quad + \frac{1}{\alpha} [\langle A\hat{u} - Au_{n_k} + Bv_{n_k} - B\hat{v}, \nabla f_3 A\hat{u} - \nabla f_3 B\hat{v} \rangle]. \end{aligned} \tag{92}$$

From (92), (57), and the fact that $Au_{n_k} \rightharpoonup \hat{u}$ and $Bv_{n_k} \rightharpoonup \hat{v}$, we conclude that $A\hat{u} = B\hat{v}$. Consequently, $(\hat{u}, \hat{v}) \in \Omega$. It follows from by Lemma 2.5(i), that

$$\begin{aligned} &\limsup_{n\to\infty} [\langle u_n - u^*, \nabla f_1 u - \nabla f_1 u^* \rangle + \langle v_n - u^*, \nabla f_2 v - \nabla f_2 v^* \rangle] \\ &= \lim_{k\to\infty} [\langle u_{n_k} - u^*, \nabla f_1 u - \nabla f_1 v^* \rangle + \langle v_{n_k} - v^*, \nabla f_2 v - \nabla f_2 v^* \rangle] \\ &= \langle \hat{u} - u^*, \nabla f_1 u - \nabla f_1 v^* \rangle + \langle \hat{v} - v^*, \nabla f_2 v - \nabla f_2 v^* \rangle \\ &\leq 0. \end{aligned} \tag{93}$$

Therefore, from (56), (70), (71), (93) and Lemma 2.13, we conclude that $\Psi_n \to 0$ as $n \to \infty$. Hence, by Lemma 2.8, $u_n \to u^*$ and $v_n \to v^*$ as $n \to \infty$.
Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\Psi_{n_i} < \Psi_{n_i+1}, \ \forall i \in \mathbb{N}. \tag{94}$$

Then, by Lemma 2.14, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ as $k \to \infty$ and $\max\{\Psi_{m_k}, \Psi_k\} \leq \Psi_{m_k+1}$ for all $k \in \mathbb{N}$. Thus, from (52) and the conditions on $\{\alpha_n\}$, we get

$$\lim_{k\to\infty} ||Au_{m_k} - Bv_{m_k}|| = 0. \tag{95}$$

From (58) and Lemma 2.8, we get

$$\lim_{k\to\infty} ||s_{m_k} - a_{m_k}|| = \lim_{n\to\infty} ||x_{m_k} - a_{m_k}|| = 0 \tag{96}$$

and

$$\lim_{k\to\infty} ||t_{m_k} - b_{m_k}|| = \lim_{n\to\infty} ||y_{m_k} - b_{m_k}|| = 0. \tag{97}$$

Moreover, following the methods in Case 1, we obtain

$$\lim_{k\to\infty} ||u_{m_k} - r_{m_k}|| = \lim_{k\to\infty} ||v_{m_k} - g_{m_k}|| = 0 \tag{98}$$

and

$$\lim_{k\to\infty} [\langle u_{m_k} - u^*, \nabla f_1 u - \nabla f_1 u^* \rangle + \langle v_{m_k} - u^*, \nabla f_2 v - \nabla f_2 v^* \rangle] \leq 0. \tag{99}$$

In addition, from (54) and inequality (94), we get

$$\begin{aligned}\alpha_{m_k}\Psi_{m_k} \leq\ & \alpha_{m_k}||\nabla f_1 u - \nabla f_1 u^*||||u_{m_k} - r_{m_k}|| \\ & + \alpha_{m_k}||\nabla f_2 v - \nabla f_2 v^*||||v_{m_k} - g_{m_k}|| \\ & + \alpha_{m_k}\langle \nabla f_1 u - \nabla f_1 u^*, u_{m_k} - u^*\rangle \\ & + \alpha_{m_k}\langle \nabla f_2 v - \nabla f_2 v^*, v_{m_k} - v^*\rangle, \end{aligned} \tag{100}$$

which implies

$$\begin{aligned}\Psi_{m_k} \leq\ & ||u_{m_k} - r_{m_k}||||\nabla f_1 u - \nabla f_1 u^*|| + ||v_{m_k} - g_{m_k}||||\nabla f_2 v - \nabla f_2 v^*|| \\ & + \langle \nabla f_1 u - \nabla f_1 u^*, u_{m_k} - u^*\rangle + \langle \nabla f_2 v - \nabla f_2 v^*, v_{m_k} - v^*\rangle, \end{aligned} \tag{101}$$

since $\alpha_{m_k} > 0$. Therefore, from (98), (99) and (101), we get $\lim_{k\to\infty}\Psi_{m_k} = 0$. But from inequality (52), we obtain that $\lim_{k\to\infty}\Psi_{m_k+1} = 0$ and hence the fact that $\Psi_k) \leq \Psi_{m_k+1}$ implies $\lim_{k\to\infty}\Psi_k = 0$. Thus, by Lemma 2.8 $u_k \to u^*$ and $v_k \to v^*$ as $k \to \infty$. □

If in Algorithm 1, we assume $E_2 = E_3$ and $B = I$, then split equality variational inequality problem reduces to split variational inequality problem for pseudomonotone mappings and the method of proof of Theorem 3.5 provides the following corollary for approximating a solution of split variational inequality problem for pseudomonotone mappings in Banach spaces.

Corollary 3.6. *Suppose Assumptions* (A1), (A3), *and* (B1)-(B3) *hold with* $B = I$ *and* $E_2 = E_3$. *Let* $\Omega = \{(u^*, v^*) \in C \times D : u^* \in VI(C, G)$ *and* $v^* \in VI(D, H)$ *such that* $Au^* = v^*\} \neq \emptyset$. *Then, the sequence* $\{(u_n, v_n)\}$ *generated by Algorithm 1 converges strongly to an element* $(u^*, v^*) = P^f_\Omega(u, v)$.

If in Algorithm 1, we assume $C = E_1$ and $D = E_2$, then $P^{f_1}_C = I_1$, $P^{f_2}_D = I_2$ and hence $VI(C, G) = G^{-1}(0)$ and $VI(C, H) = H^{-1}(0)$ where I_1 and I_2 are identity mappings in E_1 and E_2, respectively. Thus, split equality variational inequality problem reduces to split equality zero point problem and the method of proof of Theorem 3.5 provides the following corollary for approximating a solution of split equality zero point problem for pseudomonotone mappings in Banach spaces.

Corollary 3.7. *Suppose Assumptions* (A1), (A3), *and* (B1)-(B3) *hold with* $C = E_1$ *and* $D = E_2$. *Let* $G : E_1 \to E_1^*$ *and* $H : E_2 \to E_2^*$ *be continuous pseudomonotone mappings such that* $\Omega = \{(u^*, v^*) \in E_1 \times E_2 : u^* \in G^{-1}(0)$ *and* $v^* \in H^{-1}(0)$ *such that* $Au^* = Bv^*\} \neq \emptyset$. *Then, the sequence* $\{(u_n, v_n)\}$ *generated by Algorithm 1 converges strongly to an element* $(u^*, v^*) = P^f_\Omega(u, v)$.

If in Algorithm 1, we assume $E_1 = E_2 = E_3$ and $A = I = B$, then split equality variational inequality problem reduces to a common solution of variational inequality problem for pseudomonotone mappings and the method of proof of Theorem 3.5 provides the following corollary for approximating a common solution of variational inequality problem for pseudomonotone mappings in Banach spaces.

Corollary 3.8. *Suppose Assumptions* (A1), (A1) *and* (B1)-(B3) *hold with* $E_1 = E_2 = E_3$ *and* $A = I = B$. *Let* $G : E_1 \to E_1^*$ *and* $H : E_2 \to E_2^*$ *be continuous pseudomonotone mappings such that* $\Omega = \{(u^*, v^*) \in E_1 \times E_2 : u^* \in VI(C, G)$ *and* $v^* \in VI(D, H)$ *such that* $u^* = v^*\} \neq \emptyset$. *Then, the sequence* $\{(u_n, v_n)\}$ *generated by Algorithm 1 converges strongly to an element* $(u^*, v^*) = P_\Omega^f(u, v)$.

The following corollary is for a solution of split equality variational inequality problem for monotone mappings in Banach spaces which follows from Theorem 3.5.

Corollary 3.9. *Suppose Assumptions* (A1), (A3), (A4) *and* (B1)-(B3) *hold. Let* $G : C \to E_1^*$ *and* $H : D \to E_2^*$ *be continuous monotone mappings. Then, the sequence* $\{(u_n, v_n)\}$ *generated by Algorithm 1 converges strongly to an element* $(u^*, v^*) = P_\Omega^f(u, v)$.

4. Application to convex minimization problem

In this section, we apply Theorem 3.5 to find a solution of split equality minimum point problem for convex functions in Banach spaces.

Let $k : E \to \mathbb{R} \in \mathcal{F}(E)$ be a convex lower semicontinuous function and $h : E \to \mathbb{R}$ be a convex smooth function. We consider the problem of approximating $u^* \in E_1$ and $v^* \in E_2$ such that

$$k(u^*) = \min_{u \in E_1}\{k(u)\},\ h(v^*) = \min_{v \in E_2}\{h(v)\} \text{ and } Au^* = Bv^*. \tag{102}$$

This problem is equivalent, by Fermat's rule, to the problem of finding $u^* \in E_1$ and $v^* \in E_2$ such that

$$0 \in \partial k(u^*),\ 0 \in \nabla h(v^*) \text{ and } Au^* = Bv^*, \tag{103}$$

where ∂k is a subdifferential of k and ∇h is a gradient of h. We note that ∇h and ∂k are monotone mappings and hence pseudomonotone mappings (see, for example, [3, 30]).

Now, if in Algorithm 1, we assume $C = E_1$, $D = E_2$, $\partial k = G$ and $\nabla h = H$, then $VI(C, \partial k) = \partial k^{-1}(0)$, $VI(C, \nabla h) = \nabla h^{-1}(0)$. Thus, Algorithm 1 reduces to Algorithm 2 given below. The method of proof of Theorem 3.5 provides the following theorem for approximating a solution of split equality minimum point problem for convex functions in Banach spaces.

Theorem 4.1. *Suppose Assumptions* (A1), (A3), (A4) *and* (B1)-(B3) *hold. Let* $k : E_1 \to \mathbb{R} \in \mathcal{F}(E_1)$ *be a convex lower semicontinuous function with single valued continuous* ∂k *and* $h : E_2 \to \mathbb{R}$ *be a Gâteaux differentiable function with continuous* ∇h *such that* $\Omega = \{(u^*, v^*) \in E_1 \times E_2 : k(u^*) =$

Algorithm 2: For arbitrary $(u_0, v_0), (u, v) \in E_1 \times E_2$, define an iterative algorithm by

Step 1. Compute

$$x_n = \nabla f_1^*[\nabla f_1 u_n - \gamma_n A^*(\nabla f_3 A u_n - \nabla f_3 B v_n)] \text{ and } d(x_n) = x_n - u_n.$$

Compute

$$y_n = \nabla f_2^*[\nabla f_2 v_n - \gamma_n B^*(\nabla f_3 B v_n - \nabla f_3 A u_n)] \text{ and } d(y_n) = y_n - v_n.$$

Step 2. Compute

$$w_n = \nabla f_1^*[\nabla f_1 x_n - \theta \partial k x_n] \text{ and } d(w_n) = x_n - w_n. \tag{104}$$

Compute

$$z_n = \nabla f_2^*[\nabla f_2 y_n - \theta \nabla h y_n] \text{ and } d(z_n) = y_n - z_n. \tag{105}$$

If $d(x_n) = d(y_n) = d(w_n) = d(z_n) = 0$, then stop and $(u_n, v_n) \in \Omega$. Otherwise go to Step 2.

Step 3. Compute $p_n = x_n - \tau_n d(w_n)$ where $\tau_n = l^{j_n}$ and j_n is the smallest non-negative integer j satisfying

$$\langle \partial k x_n - \partial k(x_n - l^j d(w_n)), d(w_n)\rangle \leq \eta D_{f_1}(w_n, x_n)$$

and $q_n = y_n - \rho_n d(z_n)$ where $\rho_n = j_n'$ and j_n' is the smallest non-negative integer j' satisfying

$$\langle \nabla h y_n - \nabla h(y_n - l^{j'} d(z_n)), d(z_n)\rangle \leq \eta D_{f_2}(z_n, y_n).$$

Step 4. Compute

$$\begin{cases} a_n = P_{P_n}^{f_1} \nabla f_1^*(\nabla f_1 x_n - \theta \partial k p_n), \\ b_n = P_{Q_n}^{f_2} \nabla f_2^*(\nabla f_2 y_n - \theta \nabla h q_n), \\ u_{n+1} = \nabla f_1^*(\alpha_n \nabla f_1 u + (1-\alpha_n)\nabla f_1 a_n), \\ v_{n+1} = \nabla f_2^*(\alpha_n \nabla f_2 v + (1-\alpha_n)\nabla f_2 b_n), \end{cases} \tag{106}$$

where $P_n = \{p \in E_1 : \langle G p_n, p - p_n\rangle = 0\}$ and $Q_n = \{q \in E_2 : \langle H q_n, q - q_n\rangle = 0\}$. Set $n := n + 1$ and go to **Step 1**.

$\min_{u \in E_1}\{k(u)\}$, $h(v^*) = \min_{v \in E}\{h(v)\}$ *and* $Au^* = Bv^*\} \neq \emptyset$. *Then, the sequence* $\{(u_n, v_n)\}$ *generated by Algorithm 2 converges strongly to an element* $(u^*, v^*) = P_\Omega^f(u, v)$.

If in Algorithm 2, we assume $E_2 = E_3$ and $B = I$, then split equality minimum point problem reduces to split minimum point problem for convex functions and the method of proof of Theorem 3.5 provides the following corollary for approximating a solution of split minimum point problem for convex functions in Banach spaces.

Corollary 4.2. *Suppose Assumptions* (A1), (A3), (A4) *and* (B1)-(B3) *hold with* $B = I$ *and* $E_2 = E_3$. *Let* $k : E_1 \to \mathbb{R} \in \mathcal{F}(E_1)$ *be a convex lower* *semicontinuous function with single valued continuous* ∂k *and* $h : E_2 \to \mathbb{R}$ *be a Gâteaux differentiable function with continuous* ∇h *such that* $\Omega = \{(u^*, v^*) \in E_1 \times E_2 : k(u^*) = \min_{u \in E_1}\{k(u)\},\ h(v^*) = \min_{v \in E}\{h(v)\}$ *and* $Au^* = v^*\} \neq \emptyset$. *Then, the sequence* $\{(u_n, v_n)\}$ *generated by Algorithm 2 converges strongly to an element* $(u^*, v^*) = P^f_\Omega(u, v)$.

If in Algorithm 2, we assume $E_1 = E_2 = E_3$ and $A = I = B$, then split equality minimum point problem reduces to a common minimum point problem for convex functions and the method of proof of Theorem 3.5 provides the following corollary for finding a common minimum point of convex functions in Banach spaces.

Corollary 4.3. *Suppose Assumptions* (A1), (A3) *and* (B1)-(B3) *hold with* $E_1 = E_2 = E_3$ *and* $A = I = B$. *Let* $k : E_1 \to \mathbb{R} \in \mathcal{F}(E_1)$ *be a convex lower semicontinuous function with single valued continuous* ∂k *and* $h : E_2 \to \mathbb{R}$ *be a Gâteaux differentiable function with continuous* ∇h *such that* $\Omega = \{(u^*, v^*) \in E_1 \times E_2 : k(u^*) = \min_{u \in E_1}\{k(u)\},\ h(v^*) = \min_{v \in E}\{h(v)\}$ *and* $u^* = v^*\} \neq \emptyset$. *Then, the sequence* $\{(u_n, v_n)\}$ *generated by Algorithm 2 converges strongly to an element* $(u^*, v^*) = P^f_\Omega(u, v)$.

5. Numerical example

In this section, we provide a numerical example to explain the conclusion of our result. The following numerical example verifies the conclusion of Theorem 3.5.

Example 5.1. Let $E = \mathbb{R}^3$ be with the standard topology. Define $f_1, f_2, f_3 : \mathbb{R}^3 \to \mathbb{R}$, by $f_i(x) = \frac{x^2}{2}$, then $f_i(x^*) = \frac{1}{2}x^{*2}$ and $\nabla f_i(x) = x = \nabla f_i(x^*) = x^*$ for all $i = 1, 2, 3$. Let $C = \{x \in \mathbb{R}^3 : ||x|| \leq 1\}$ and $D = \{x \in \mathbb{R}^3 : ||x|| \leq 2\}$. Let $G : C \to \mathbb{R}^3$ and $H : D \to \mathbb{R}^3$ be defined by $G(x) = (1.5 - ||x||)(x)$ and $H(x) = (x_1 + 1, 2x_2 - 1, 2x_3)$ and let $A, B : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $A(x) = (2x_1, x_2, 3x_3)$ and $B(x) = (0, 0, 2x_3)$, where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, then G is continuous pseudomonotone and H is continuous monotone and hence continuous pseudomonotone with $VI(C, G) = \{(0, 0, 0)\}$, $VI(D, H) = \{(-1, 1, 0)\}$ and $A(0, 0, 0) = (0, 0, 0) = B(-1, 1, 0)$. Thus, $\Omega = \{(u, v) : u \in VI(C, G)$ and $v \in VI(D, H)$ such that $Au = Bv\} \neq \emptyset$. Now, if we assume $(u, v) = ((0.6, 0.2, 0.3), (0.5, 0.0, 0.0))$, $\alpha_n = \frac{1}{n+10^5}$, $l = 0.5$, $\eta = 0.9$ and $\beta = 1$ for all $n \geq 0$, and take different initial points $(u_0, v_0) = ((0.2, 0.8, 0.0), (0.2, 1.3, 0.1))$, $(u_0^{'}, v_0^{'}) = ((0.5, 0.3, 0.0), (0.5, 1.0, 0.5))$ and $(u_0^{''}, v_0^{''}) = ((0.2, 0.5, 0.6), (1.0, 0.5, 0.5))$, then in all cases, the numerical experiment results using MATLAB provide that the sequence $\{(u_n, v_n)\}$ generated by Algorithm 1 converges strongly to $(u^*, v^*) = ((0, 0, 0), (1, 0.5, 0))$ (see Figure 1).

In addition, we have sketched the difference term $||Au_n - Bv_n||$ for each initial point. From the sketch we observe that $||Au_n - Bv_n|| \to 0$ as $n \to \infty$ (see Figure 2).

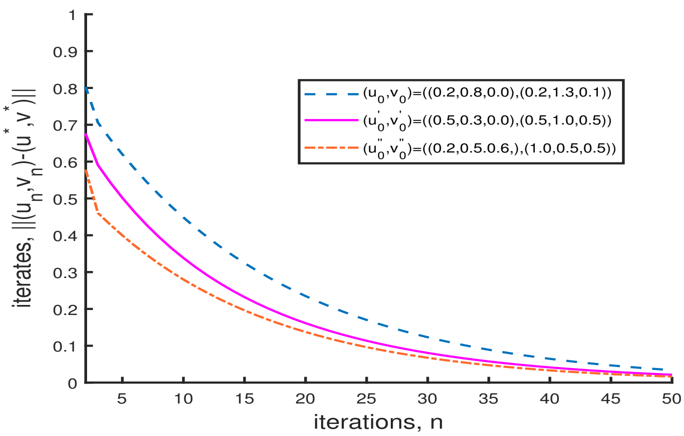

FIGURE 1. The graph of $||(u_n, v_n) - (u^*, v^*)||$ versus number of iterations with different choices of (u_0, v_0)

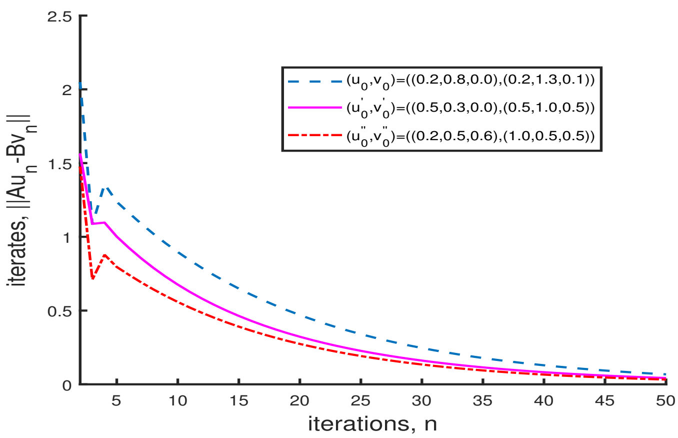

FIGURE 2. The graph of $||Au_n - Bv_n||$ versus number of iterations with different choices of (u_0, v_0)

6. Conclusion

In this paper, we studied the iterative scheme for approximating a solution of split equality variational inequality problem for continuous pseudomonotone mappings in reflexive real Banach spaces. In addition, we give applications of our main results to find a solution of split equality minimum point problem for convex functions in real reflexive Banach spaces. Finally, a numerical example which supports our result is presented. Our results improve and unify most of the results that have been proved for this important class of nonlinear mappings. In particular, Theorem 3.5 extend the results of [12, 13, 18, 21, 22] from real Hilbert spaces to real reflexive Banach spaces. Moreover, Theorem 3.5 extends Theorem 3.7 of Boikanyo and Zegeye [7] from uniformly smooth and uniformly convex real Banach spaces to real reflexive Banach spaces.

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Getahun Bekele Wega
Department of Mathematics
Jimma University
P.O. Box 378
Jimma, Ethiopia
Email address: getahunbekele2012@gmail.com