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ON WEIGHTED COMPACTNESS OF COMMUTATORS OF BILINEAR FRACTIONAL MAXIMAL OPERATOR

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Abstract. Let $\mathcal{M}_\alpha$ be a bilinear fractional maximal operator and BM_α be a fractional maximal operator associated with the bilinear Hilbert transform. In this paper, the compactness on weighted Lebesgue spaces are considered for commutators of bilinear fractional maximal operators; these commutators include the fractional maximal linear commutators $\mathcal{M}^j_{\alpha,b}$ and $BM^j_{\alpha,b}$ $(j=1,2)$, the fractional maximal iterated commutator $\mathcal{M}_{\alpha,\vec{b}}$, and $BM_{\alpha,\vec{b}}$, where $b \in \mathrm{BMO}(\mathbb{R}^d)$ and $\vec{b}=(b_1,b_2) \in \mathrm{BMO}(\mathbb{R}^d) \times \mathrm{BMO}(\mathbb{R}^d)$. In particular, we improve the well-known results to a larger scale for $1/2 < q < \infty$ and give positive answers to the questions in [2].

1. Introduction and main theorems

Let L be a linear operator from a Banach space X to another Banach space Y. We call L is a compact operator if the image under L of any bounded subset of X is a relatively compact subset of Y. In functional analysis, an important branch is the theory of compact operators. One of classical examples of compact operators is the compact imbedding of Sobolev spaces, by such imbedding, it can be converted an elliptic boundary value problem into a Fredholm integral equation. We refer the interested reader to [12, 21] and references therein for more background and related results.

In 1978, Uchiyama [29] improved the boundedness to compactness if the symbol is in $\mathrm{CMO}(\mathbb{R}^d)$, where $\mathrm{CMO}(\mathbb{R}^d)$ denotes the closure of $\mathcal{C}_c^\infty(\mathbb{R}^d)$ in the topology of $\mathrm{BMO}(\mathbb{R}^d)$. The interest in the compactness of commutators in complex analysis is from the connection between the commutators and the Hankel-type operators. With the aid of the compactness of $[b,T]$, it is easy to derive a Fredholm alternative for equations with VMO coefficients in all L^p

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spaces for $1 < p < \infty$ (see [19]). In recent years, the compactness of commutators has been extensively studied already, as Beatrous and Li [1] studied the boundedness and compactness of the commutators of Hankel type operators, Krantz and Li [20] applied the compactness characterization of the commutator $[b, T_\Omega]$ to study Hankel type operators on Bergman spaces, Chen and Ding [8] proved that the commutator of singular integrals with variable kernels is compact on $\mathbb{R}^d$ if and only if $b \in \mathrm{CMO}(\mathbb{R}^d)$ and they also established the compactness of Littlewood-Paley square functions in [9]. After that, Liu and Tang [25] studied the compactness for higher order commutators of oscillatory singular integral operators. Li and Peng [24] investigated compact commutators of Riesz transforms associated to Schrödinger operators. The first author and co-author [14] recently studied the weighted compactness of commutators of Schödinger type operators and so on.

Since the multilinear setting is a natural generalization of linear case, compactness results in the multilinear setting have just began to be studied (see [2–7,32]). In [32], Xue proved the following weighted strong type estimates for $\mathcal{M}_{\alpha,\vec{b}}$ with $A_{(\vec{p},q)}$ weights (see Definition 2.1).

Theorem A. *Let $0 < \alpha < 2d$, $1 < p_1, p_2 < \infty$, $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/p - \alpha/d$. For $s > 1$ with $0 < s\alpha < 2d$, if $\vec{w}^s \in A_{(\vec{p}/s, q/s)}$ and $\vec{b} = (b_1, b_2) \in \mathrm{BMO}(\mathbb{R}^d) \times \mathrm{BMO}(\mathbb{R}^d) =: \mathrm{BMO}^2(\mathbb{R}^d)$, where $v_{\vec{w}} = w_1 w_2$, there exists a constant $C > 0$ such that*

$$\|\mathcal{M}_{\alpha,\vec{b}}(f, \mathrm{g})\|_{L^q(v_{\vec{w}}^q)} \leq C \|\vec{b}\|_{\mathrm{BMO}^2} \|f\|_{L^{p_1}(w_1^{p_1})} \|\mathrm{g}\|_{L^{p_2}(w_2^{p_2})},$$

where $\mathcal{M}_\alpha$ is the bilinear maximal operator and $\mathcal{M}_{\alpha,\vec{b}}$ is the maximal iterated commutator of $\mathcal{M}_\alpha$ defined by

$$\mathcal{M}_\alpha(f, \mathrm{g})(x) = \sup_{B \ni x} \frac{1}{|B|^{2-\frac{\alpha}{d}}} \int_B \int_B |f(y)||\mathrm{g}(z)| dy dz,$$

and

$$\mathcal{M}_{\alpha,\vec{b}}(f, \mathrm{g})(x) = \sup_{B \ni x} \frac{1}{|B|^{2-\frac{\alpha}{d}}} \int_B \int_B |b_1(x) - b_1(y)||b_2(x) - b_2(z)||f(y)||\mathrm{g}(z)| dy dz.$$

We continue defining the maximal linear commutators of $\mathcal{M}_\alpha$ as

$$\mathcal{M}^1_{\alpha,b}(f, \mathrm{g})(x) = \sup_{B \ni x} \frac{1}{|B|^{2-\frac{\alpha}{d}}} \int_B \int_B |b(x) - b(y)||f(y)||\mathrm{g}(z)| dy dz$$

and

$$\mathcal{M}^2_{\alpha,b}(f, \mathrm{g})(x) = \sup_{B \ni x} \frac{1}{|B|^{2-\frac{\alpha}{d}}} \int_B \int_B |b(x) - b(z)||f(y)||\mathrm{g}(z)| dy dz.$$

Remark 1.1. Using the method of [32], we can also get the weighted boundedness of $\mathcal{M}^j_{\alpha,b}$, $j = 1, 2$.

$$\|\mathcal{M}^j_{\alpha,b}(f, \mathrm{g})\|_{L^q(v_{\vec{w}}^q)} \leq C \|b\|_{\mathrm{BMO}} \|f\|_{L^{p_1}(w_1^{p_1})} \|\mathrm{g}\|_{L^{p_2}(w_2^{p_2})}$$

with same conditions as in Theorem A.

For $0 \le \alpha < d$, we can define a more singular family of bilinear maximal operators

$$BM_\alpha(f,g)(x) = \sup_{r>0} \frac{1}{|B(0,r)|^{1-\frac{\alpha}{d}}} \int_{B(0,r)} |f(x-y)||g(x+y)|dy.$$

Similarly, the maximal iterated commutator of BM_α is defined by

$$\begin{aligned} &BM_{\alpha,\vec{b}}(f,g)(x) \\ = &\sup_{r>0} \frac{1}{|B(0,r)|^{1-\frac{\alpha}{d}}} \\ &\times \int_{B(0,r)} |b_1(x)-b_1(x-y)||b_2(x)-b_2(x+y)||f(x-y)||g(x+y)|dy. \end{aligned}$$

The maximal linear commutators of BM_α are defined by

$$BM^1_{\alpha,b}(f,g)(x) = \sup_{r>0} \frac{1}{|B(0,r)|^{1-\frac{\alpha}{d}}} \int_{B(0,r)} |b(x)-b(x-y)||f(x-y)||g(x+y)|dy$$

and

$$BM^2_{\alpha,b}(f,g)(x) = \sup_{r>0} \frac{1}{|B(0,r)|^{1-\frac{\alpha}{d}}} \int_{B(0,r)} |b(x)-b(x+y)||f(x-y)||g(x+y)|dy.$$

By the definition of BM_α, it's easy to obtain $BM_\alpha(f,g) \le CB_\alpha(|f|,|g|)$, where

$$B_\alpha(f,g)(x) = \int_{\mathbb{R}^d} \frac{f(x-y)g(x+y)}{|y|^{d-\alpha}} dy.$$

Thus, by the results in [15], we have the following weighted strong type estimates for $BM_{\alpha,\vec{b}}$ with $A_{(\vec{p},q)}$ weights.

Remark 1.2. Let $0 < \alpha < d$, $1 < p_1, p_2 < \infty$, $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/p - \alpha/d$. If $1 < a < \min\{p_1, p_2\}$, then for $\vec{b} = (b_1, b_2) \in \mathrm{BMO}(\mathbb{R}^d) \times \mathrm{BMO}(\mathbb{R}^d) =: \mathrm{BMO}^2(\mathbb{R}^d)$, the following statements holds.

(i) If $1/2 < q < 1$, $v_{\vec{w}} = w_1 w_2 \in A_\infty$ and $\vec{w} \in A_{(\frac{\vec{p}}{a}, \frac{aq}{1-q})}$, then there exists a constant $C > 0$ such that

$$\|BM_{\alpha,\vec{b}}(f,g)\|_{L^q(v^q_{\vec{w}})} \le C\|\vec{b}\|_{\mathrm{BMO}^2}\|f\|_{L^{p_1}(w_1^{p_1})}\|g\|_{L^{p_2}(w_2^{p_2})}.$$

(ii) If $q > 1$, $p_j > s_j$ $(j = 1, 2)$, $v_{\vec{w}} = w_1 w_2 \in A_\infty$ and $\vec{w} \in A_{(\frac{\vec{p}}{a}, aq)}$, where $\frac{1}{s_1} + \frac{1}{s_2} = 1$, then there exists a constant $C > 0$ such that

$$\|BM_{\alpha,\vec{b}}(f,g)\|_{L^q(v^q_{\vec{w}})} \le C\|\vec{b}\|_{\mathrm{BMO}^2}\|f\|_{L^{p_1}(w_1^{p_1})}\|g\|_{L^{p_2}(w_2^{p_2})}.$$

Remark 1.3. Similarly, we can also get the weighted boundedness of $BM^j_{\alpha,b}$ $(j = 1, 2)$ with the fact $BM_\alpha(f,g) \le CB_\alpha(|f|,|g|)$,

$$\|BM^j_{\alpha,b}(f,g)\|_{L^q(v^q_{\vec{w}})} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^{p_1}(w_1^{p_1})}\|g\|_{L^{p_2}(w_2^{p_2})}$$

with same conditions as in Remark 1.2.

Recently, Bényi et al. [2] show that the compactness for commutator of bilinear fractional integrals (include Kenig-Stein type operator B_α) with multiplication by $\mathrm{CMO}(\mathbb{R}^d)$ functions are compact operators from $L^{p_1} \times L^{p_2}$ to L^q for $1 < p_1, p_2 < \infty$ and $1/q = 1/p_1 + 1/p_2 - \alpha/d$. Naturally, it will be a very interesting problem to ask whether we can establish the weighted compactness of commutators of fractional maximal operators with $\mathrm{CMO}(\mathbb{R}^d)$ functions. Furthermore, can we extend the index q to a larger scale? In this paper, we will give positive answers.

Now, we formulate our main results as follows.

Theorem 1.4. *Let $1/2 < q < \infty$, $0 < \alpha < 2d$, $1 < p_1, p_2 < \infty$, $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/p - \alpha/d$. For $s > 1$ with $0 < s\alpha < 2d$, if $\vec{w}^s \in A_{(\vec{p}/s, q/s)}$, $b \in \mathrm{BMO}(\mathbb{R}^d)$ and $\vec{b} = (b_1, b_2) \in \mathrm{BMO}^2(\mathbb{R}^d)$, where $v_{\vec{w}} = w_1 w_2$, then $\mathcal{M}^1_{\alpha,b}$, $\mathcal{M}^2_{\alpha,b}$ and $\mathcal{M}_{\alpha,\vec{b}}$ are compact from $L^{p_1}(w_1^{p_1}) \times L^{p_2}(w_2^{p_2})$ to $L^q(v_{\vec{w}}^q)$.*

We remark that Theorem 1.4 also holds for $\alpha = 0$. In this case, our results not only covers the results in [31] but also improve their results to $1/2 < q \leq 1$.

Theorem 1.5. *Let $q \in (1/2, 1) \cup (1, +\infty)$, $0 < \alpha < d$, $1 < p_1, p_2 < \infty$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/p - \alpha/d$, $1 < a < \min\{p_1, p_2\}$ and $v_{\vec{w}} = w_1 w_2$. If $s > 1$ with $0 < s\alpha < d$, then for $\vec{b} = (b_1, b_2) \in \mathrm{BMO}^2(\mathbb{R}^d)$ and $b \in \mathrm{BMO}(\mathbb{R}^d)$, the following statements holds.*

(i) *If $1/2 < q < 1$ and $\vec{w}^s \in A_{(\frac{\vec{p}}{as}, \frac{aq}{(1-q)s})}$, then $BM^1_{\alpha,b}$, $BM^2_{\alpha,b}$ and $BM_{\alpha,\vec{b}}$ are compact from $L^{p_1}(w_1^{p_1}) \times L^{p_2}(w_2^{p_2})$ to $L^q(v_{\vec{w}}^q)$.*

(ii) *If $q > 1$, $p_j > s_j$ $(j = 1, 2)$ and $\vec{w}^s \in A_{(\frac{\vec{p}}{as}, \frac{aq}{s})}$, where $\frac{1}{s_1} + \frac{1}{s_2} = 1$, then $BM^1_{\alpha,b}$, $BM^2_{\alpha,b}$ and $BM_{\alpha,\vec{b}}$ are compact from $L^{p_1}(w_1^{p_1}) \times L^{p_2}(w_2^{p_2})$ to $L^q(v_{\vec{w}}^q)$.*

We remark that the results are new for fractional maximal operators associated with bilinear Hilbert transform, and we also give the positive answers to Question 3.4 in [2].

The paper is organized as follows: In Section 2, we will give some facts and lemmas. The proof of Theorems 1.4-1.5 are presented in Section 3. A tacit understanding in the present paper is that we use the notation $A \lesssim B$ means that there is an uninteresting constant C such that $A \leq CB$. We do not keep track of dependencies on dimension d.

2. Preliminaries

2.1. Multiple weights class

Given a Lebesgue measurable set $E \subset \mathbb{R}^d$, $|E|$ will denote the Lebesgue measure of E. Let $B = B(x, r)$ be a ball in $\mathbb{R}^d$ centered at x with radius r and $Q(x, r)$ be a cube in $\mathbb{R}^d$ centered at x with the side length $2r$. A weight w is a non-negative measurable and local integrable function on $\mathbb{R}^d$. The measure associated with w is the set function given by $w(E) = \int_E w dx$. For $0 < p < \infty$, we denote by $L^p(w)$ the space of all Lebesgue measurable function $f(x)$ such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^d} |f(x)|^p w(x)dx\right)^{1/p}.$$

Recall that a weight w belongs to the classical Muckenhoupt class A_p $(1 < p < \infty)$, if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{\frac{p}{p'}} < \infty.$$

$w \in A_1$ means if there is a constant C such that

$$\frac{1}{|Q|}\int_Q w \le C \inf_{x\in Q} w \quad \text{for a.e. } x \in \mathbb{R}^d.$$

Now, we defined the following multiple fractional type weights which are somehow different from the weights in [22]. But it is a natural generalization of the classical $A_{p,q}$ weights in [27].

Definition 2.1. Let $1 \le p_1, p_2 \le \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\vec{p} = (p_1, p_2)$ and $0 < q \le \infty$. One says that a vector of weights $\vec{w} = (w_1, w_2)$ is in the multiple weights class $\vec{w} \in A_{(\vec{p},q)}$ if it satisfies

$$[\vec{w}]_{A_{(\vec{p},q)}} := \sup_Q \left(\frac{1}{|Q|}\int_Q v_{\vec{w}}^q\right)^{\frac{1}{q}} \left(\frac{1}{|Q|}\int_Q w_1^{-p_1'}\right)^{\frac{1}{p_1'}} \left(\frac{1}{|Q|}\int_Q w_2^{-p_2'}\right)^{\frac{1}{p_2'}} < \infty,$$

where $v_{\vec{w}} = w_1 w_2$ and each w_i $(i = 1, 2)$ is a nonnegative function on $\mathbb{R}^d$. When $q = \infty$, $\left(\frac{1}{|Q|}\int_Q v_{\vec{w}}^q\right)^{\frac{1}{q}}$ is understood as $\operatorname{ess\,sup}_{x\in Q} v_{\vec{w}}$. Moreover when $p_i = 1$, $\left(\frac{1}{|Q|}\int_Q w_i^{-p_i'}\right)^{\frac{1}{p_i'}}$ is understood as $(\inf_Q w_i)^{-1}$.

We need the following characterization of multiple weights given by Iida [17].

Lemma 2.2. *Let $1 \le p_1, p_2 \le \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $0 < q \le \infty$. A vector $\vec{w}$ of weights satisfies $\vec{w} \in A_{(\vec{p},q)}$ if and only if*

(i) $v_{\vec{w}}^q \in A_{1+q\left(2-\frac{1}{p}\right)}$;

(ii) $w_i^{-p_i'} \in A_{1+p_i' s_i}$ $(i = 1, 2)$,

where $s_i = \frac{1}{q} + 2 - \frac{1}{p} - \frac{1}{p_i'}$ $(i = 1, 2)$. An analogy is available for $q = \infty$, if we regard the condition $v_{\vec{w}} \in A_{1+q\left(2-\frac{1}{p}\right)}$ as the condition $v_{\vec{w}}^{-\frac{1}{2-\frac{1}{p}}} \in A_1$.

We remark that the $A_{(\vec{p},q)}$ is multiple weights class $A_{\vec{p}}$ for $p = q$, which first introduced in [22]. This weights class $A_{\vec{p}}$ also have similarly characterizations in Lemma 2.2.

For the classical Mukenhoupt weights class A_p, it enjoys the following properties:

Lemma 2.3. *Let $1 < p < \infty$. Then*

(i) *If* $w \in A_p$, *there exists a constant* $\theta \in (0,1)$ *such that* $w^{1+\theta} \in A_p$. *Both* θ *and the* A_p *constant of* $w^{1+\theta}$ *defend only on* p *and the* A_p *constant of* w;

(ii) *If* $w \in A_p(\mathbb{R}^d)$, *we have*

$$\lim_{N\to\infty}\int_{|x|>N}\frac{w(x)}{|x|^{dp}}dx = 0 \quad \textit{and} \quad \lim_{N\to\infty}\int_{|x|>N}\frac{w^{1-p'(x)}}{|x|^{dp'}}dx = 0;$$

(iii) *If* $w \in A_\infty = \bigcup_{1\le p<\infty} A_p$, *then there exists a constant* $\theta \in (0,1)$ *such that for all cubes* Q *and any set* $E \subset Q$,

$$\frac{w(E)}{w(Q)} \le C\left(\frac{|E|}{|Q|}\right)^{\theta}.$$

We remark that (i) of Lemma 2.3 follows from [11] and (ii) of Lemma 2.3 follows from [13].

2.2. Bilinear fractional $L(\log L)$ type maximal operators

We first recall some notations about Orlicz spaces, more details can be found in [28]. Let $\phi = t(1+\log^+ t)$, which is an important Young function. Then for a ball B denote that

$$\|f\|_{L(\log L),\,B} = \inf\left\{\lambda > 0 : \frac{1}{|B|}\int_B \phi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\}.$$

Note that the following inequality holds

$$\frac{1}{|B|}\int_B |f(x)|dx \le C\|f\|_{L(\log L),\,B}.$$

Having the preliminary notations, for $0 \le \alpha < 2d$, we give the definition of bilinear fractional $L(\log L)$ type maximal operator as

$$\mathcal{M}_{L(\log L),\,\alpha}(f,\mathrm{g})(x) = \sup_{B\in x}|B|^{\frac{\alpha}{d}}\|f\|_{L(\log L),B}\|\mathrm{g}\|_{L(\log L),B}.$$

Several weighted norm inequalities for $\mathcal{M}_\alpha$ and $\mathcal{M}_{L(\log L),\,\alpha}$ were established in [26] and [10], respectively, we list the results in the following lemma.

Lemma 2.4. *Let* $0 < \alpha < 2d$, $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, $\vec{w} = (w_1, w_2) \in A_{(\vec{p},q)}$ *and* $v_{\vec{w}} = w_1 w_2$. *Then*

$$\|\mathcal{M}_\alpha(f,\mathrm{g})\|_{L^q(v_{\vec{w}}^q)} \le C\|f\|_{L^{p_1}(w_1^{p_1})}\|\mathrm{g}\|_{L^{p_2}(w_2^{p_2})},$$

and for $s > 1$, *if* $\vec{w}^s \in A_{(\frac{\vec{p}}{s},\frac{q}{s})}$,

$$\|\mathcal{M}_{L(\log L),\,\alpha}(f,\mathrm{g})\|_{L^q(v_{\vec{w}}^q)} \le C\|f\|_{L^{p_1}(w_1^{p_1})}\|\mathrm{g}\|_{L^{p_2}(w_2^{p_2})}$$

holds for every $(f,\mathrm{g}) \in L^{p_1}(w_1^{p_1}) \times L^{p_2}(w_2^{p_2})$.

2.3. Weighted Frechet-Kolmogrov theorem

The following weighted Frechet-Kolmogrov theorem was proved by Xue, Yabuta and Yan [33].

Lemma 2.5. *Let w be a weight on $\mathbb{R}^d$. Assume that $w^{-1/(p_0-1)}$ is also a weight on $\mathbb{R}^d$ for some $p_0 > 1$. Let $0 < p < \infty$ and $\mathcal{F}$ be a subset in $L^p(w)$. Then $\mathcal{F}$ is sequentially compact in $L^p(w)$ if the following three conditions are satisfied*:

(i) *$\mathcal{F}$ is bounded, i.e.,* $\sup_{f\in\mathcal{F}} \|f\|_{L^p(w)} < \infty$;

(ii) *$\mathcal{F}$ uniformly vanishes at infinitely, i.e.,*

$$\lim_{N\to\infty} \sup_{f\in\mathcal{F}} \int_{|x|>N} |f(x)|^p w(x)dx = 0;$$

(iii) *$\mathcal{F}$ is uniformly equicontinous, i.e.,*

$$\lim_{|h|\to 0} \sup_{f\in\mathcal{F}} \int_{\mathbb{R}^d} |f(x+h) - f(x)|^p w(x)dx = 0.$$

Note that an operator $\mathcal{T} : V \to Y$ is said to be a compact operator if $\mathcal{T}$ is continuous and maps bounded subsets into sequentially compact subsets.

2.4. Some key lemmas

The following lemma due to Li et al. [23], which essentially was proved in [18].

Lemma 2.6. *Under the condition of Theorem 1.5, we can choose auxiliary indices s_1 and s_2 so that $s_1 \in (1, p_1)$ and $s_2 \in (1, p_2)$. Assume in addition that, for these indices,*

$$a \geq \max\left\{ \frac{p_1}{(s_1(\frac{p_1}{s_1})')'}, \frac{p_2}{(s_2(\frac{p_2}{s_2})')'} \right\} > 1.$$

Then we have

$$s_1\left(\frac{p_1}{s_1}\right)' \leq \left(\frac{p_1}{a}\right)' \quad \text{and} \quad s_2\left(\frac{p_2}{s_2}\right)' \leq \left(\frac{p_2}{a}\right)'.$$

In order to obtain our results, we will utilize new auxiliary lemmas for maximal operators.

Lemma 2.7. *Let $0 \leq \alpha < 2d$, $b \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and $x, t \in \mathbb{R}^d$. For any balls $B_1 := B(x_0, r) \ni x$ and $B_2 := B_2(x_0, r + |t|)$, there exists a constant C such that for any $k \in (1, \infty)$,*

$$\int_{B_2}\int_{B_2} \left| \frac{\chi_{B_1}(y)\chi_{B_1}(z)}{|B_1|^{2-\frac{\alpha}{d}}} - \frac{\chi_{B_2}(y)\chi_{B_2}(z)}{|B_2|^{2-\frac{\alpha}{d}}} \right| |b(x+t) - b(y)||f(y)||\mathrm{g}(z)|dydz$$
$$\leq C(|t|\mathcal{M}_\alpha(f, \mathrm{g})(x) + |t|^{\frac{1}{k}}\mathcal{M}_{L(\log L),\alpha}(f, \mathrm{g})(x)).$$

Proof. It's sufficient to consider two cases as follows: $r \le |t|$ and $r > |t|$.

Case I: $r \le |t|$. For $y \in B_2$, we have

$$|y-x-t| \le |y-x_0| + |x-x_0| + |t| \le r + |t| + r + |t| = 2(r+|t|) \le 4|t|.$$

It implies that

$$\begin{aligned}
&\int_{B_2}\int_{B_2}\left|\frac{\chi_{B_1}(y)\chi_{B_1}(z)}{|B_1|^{2-\frac{\alpha}{d}}} - \frac{\chi_{B_2}(y)\chi_{B_2}(z)}{|B_2|^{2-\frac{\alpha}{d}}}\right| |b(x+t)-b(y)||f(y)||g(z)|dydyz \\
&\le |y-x-t|\|\nabla b\|_{L^\infty(\mathbb{R}^d)}\frac{1}{|B_1|^{2-\frac{\alpha}{d}}}\int_{B_1}\int_{B_1} f(y)||g(z)|dydz \\
&\quad + |y-x-t|\|\nabla b\|_{L^\infty(\mathbb{R}^d)}\frac{1}{|B_2|^{2-\frac{\alpha}{d}}}\int_{B_2}\int_{B_2} |f(y)||g(z)|dydz \\
&\le C\|\nabla b\|_{L^\infty(\mathbb{R}^d)}|t|\mathcal{M}_\alpha(f,g)(x).
\end{aligned}$$

Case II: $r > |t|$. Notice that by adding and subtracting

$$\int_{B_2}\int_{B_2}\frac{\chi_{B_1}(y)\chi_{B_1}(z)}{|B_2|^{2-\frac{\alpha}{d}}}|b(x+t)-b(y)||f(y)||g(z)|dydz$$

and

$$\int_{B_2}\int_{B_2}\frac{\chi_{B_1}(y)\chi_{B_2}(z)}{|B_2|^{2-\frac{\alpha}{d}}}|b(x+t)-b(y)||f(y)||g(z)|dydz,$$

we can compute

$$\begin{aligned}
&\int_{B_2}\int_{B_2}\left|\frac{\chi_{B_1}(y)\chi_{B_1}(z)}{|B_1|^{2-\frac{\alpha}{d}}} - \frac{\chi_{B_2}(y)\chi_{B_2}(z)}{|B_2|^{2-\frac{\alpha}{d}}}\right| |b(x+t)-b(y)||f(y)||g(z)|dydz \\
&\le \int_{B_2}\int_{B_2}\left|\frac{\chi_{B_1}(y)\chi_{B_1}(z)}{|B_1|^{2-\frac{\alpha}{d}}} - \frac{\chi_{B_1}(y)\chi_{B_1}(z)}{|B_2|^{2-\frac{\alpha}{d}}}\right| |b(x+t)-b(y)||f(y)||g(z)|dydz \\
&\quad + \int_{B_2}\int_{B_2}\left|\frac{\chi_{B_1}(y)\chi_{B_1}(z)}{|B_2|^{2-\frac{\alpha}{d}}} - \frac{\chi_{B_1}(y)\chi_{B_2}(z)}{|B_2|^{2-\frac{\alpha}{d}}}\right| |b(x+t)-b(y)||f(y)||g(z)|dydz \\
&\quad + \int_{B_2}\int_{B_2}\left|\frac{\chi_{B_1}(y)\chi_{B_2}(z)}{|B_2|^{2-\frac{\alpha}{d}}} - \frac{\chi_{B_2}(y)\chi_{B_2}(z)}{|B_2|^{2-\frac{\alpha}{d}}}\right| |b(x+t)-b(y)||f(y)||g(z)|dydz \\
&=: A_1 + A_2 + A_3.
\end{aligned}$$

Estimate for A_1. Since $|x+t-y| \le |x-x_0| + |y-x_0| + |t| \le r + r + |t| < 3r$ for $y \in B_1$, it implies that

$$\begin{aligned}
A_1 &= \frac{|B_2|^{2-\frac{\alpha}{d}} - |B_1|^{2-\frac{\alpha}{d}}}{|B_2|^{2-\frac{\alpha}{d}}}\frac{1}{|B_1|^{2-\frac{\alpha}{d}}}\int_{B_1}\int_{B_1}|b(x+t)-b(y)||f(y)||g(z)|dydz \\
&\le Cr\|\nabla b\|_{L^\infty(\mathbb{R}^d)}\frac{|B_2|^{2-\frac{\alpha}{d}} - |B_1|^{2-\frac{\alpha}{d}}}{|B_2|^{2-\frac{\alpha}{d}}}\frac{1}{|B_1|^{2-\frac{\alpha}{d}}}\int_{B_1}\int_{B_1}|f(y)||g(z)|dydz.
\end{aligned}$$

To obtain the desired result, we need to prove that

$$\frac{|B_2|^{2-\frac{\alpha}{d}} - |B_1|^{2-\frac{\alpha}{d}}}{|B_2|^{2-\frac{\alpha}{d}}} \le C\frac{|t|}{r}. \tag{2.1}$$

Since

$$\frac{|B_2|^{2-\frac{\alpha}{d}}-|B_1|^{2-\frac{\alpha}{d}}}{|B_2|^{2-\frac{\alpha}{d}}}=\frac{(r+|t|)^{2n-\alpha}-r^{2n-\alpha}}{(r+|t|)^{2n-\alpha}}\le\left[\left(1+\frac{|t|}{r}\right)^{2-\frac{\alpha}{d}}\right]^d-1,$$

to prove (2.1), we begin with the case $0<2-\frac{\alpha}{d}\le 1$. By the Bernoulli inequality and binomial theorem, for $r>|t|$, we have

$$\left[\left(1+\frac{|t|}{r}\right)^{2-\frac{\alpha}{d}}\right]^d-1\le\left[1+\left(2-\frac{\alpha}{d}\right)\frac{|t|}{r}\right]^d-1\le C\frac{|t|}{r}.$$

Next let us deal with the case $1<2-\frac{\alpha}{d}\le 2$. Through a similar discussion, for $r>|t|$, we have

$$\begin{aligned}\left[\left(1+\frac{|t|}{r}\right)^{2-\frac{\alpha}{d}}\right]^d-1&=\left[\left(1+\frac{|t|}{r}\right)\left(1+\frac{|t|}{r}\right)^{1-\frac{\alpha}{d}}\right]^d-1\\&\le\left(1+\frac{|t|}{r}\right)^d\left[1+\left(1-\frac{\alpha}{d}\right)\frac{|t|}{r}\right]^d-1\\&\le\left(1+C_1\frac{|t|}{r}\right)\left(1+C_2\frac{|t|}{r}\right)-1\\&\le C\frac{|t|}{r}.\end{aligned}$$

This finishes the proof of inequality (2.1).

Hence, using inequality (2.1), we obtain that

$$A_1\le C|t|\mathcal{M}_\alpha(f,\mathrm{g})(x).$$

Estimate for A_2. Note that for any $k>1$

$$\begin{aligned}|b(x+t)-b(y)|&=|b(x+t)-b(y)|^{\frac{1}{k}}|b(x+t)-b(y)|^{1-\frac{1}{k}}\\&\le C\|b\|_{L^\infty(\mathbb{R}^d)}^{1-\frac{1}{k}}\|\nabla b\|_{L^\infty(\mathbb{R}^d)}^{\frac{1}{k}}|x+t-y|^{\frac{1}{k}}\\&\le C\|b\|_{L^\infty(\mathbb{R}^d)}^{1-\frac{1}{k}}\|\nabla b\|_{L^\infty(\mathbb{R}^d)}^{\frac{1}{k}}r^{\frac{1}{k}}.\end{aligned}$$

By the above inequality and Hölder's inequality, we have that for $k>1$

$$\begin{aligned}A_2&\le\frac{1}{|B_2|^{2-\frac{\alpha}{d}}}\int_{B_1}\int_{B_2\setminus B_1}|b(x+t)-b(y)||f(y)||\mathrm{g}(z)|dydyz\\&\le C\frac{r^{\frac{1}{k}}}{|B_2|^{2-\frac{\alpha}{d}}}\int_{B_1}\int_{B_2\setminus B_1}|f(y)||\mathrm{g}(z)|dydz\\&\le Cr^{\frac{1}{k}}\frac{|B_2|-|B_1|}{|B_2|}\mathcal{M}_{L(\log L),\alpha}(f,\mathrm{g})(x).\end{aligned}$$

Since

$$\frac{|B_2|-|B_1|}{|B_2|}=\frac{(r+|t|)^d-r^d}{(r+|t|)^d}\le\frac{r^d+C|t|r^{d-1}-r^d}{r^d}=C\frac{|t|}{r},$$

then we have

$$A_2 \le Ctr^{\frac{1}{k}-1}\mathcal{M}_{L(\log L),\alpha}(f,\mathrm{g})(x) \le Ct^{\frac{1}{k}}\mathcal{M}_{L(\log L),\alpha}(f,\mathrm{g})(x).$$

Estimate for A_3. With the similar argument, we have

$$A_3 \le C(|t|\mathcal{M}_\alpha(f,\mathrm{g})(x) + t^{\frac{1}{k}}\mathcal{M}_{L(\log L),\alpha}(f,\mathrm{g})(x)).$$

Thus, we obtain the desired results. □

The following lemma also plays a key role in the proof of our theorems.

Lemma 2.8. *Let $0 \le \alpha < d$, $b \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and $x, t \in \mathbb{R}^d$. For any balls $B_1 := B(x,r)$ and $B_2 := B_2(x, r+|t|)$, there exists a constant C such that for any $s \in (1,\infty)$ and $k \in (1,\infty)$,*

$$\begin{aligned}&\int_{B_2}\left|\frac{|\mathrm{g}(2x-y)|\chi_{B_1}(y)}{|B_1|^{1-\frac{\alpha}{d}}} - \frac{|\mathrm{g}(2x+2t-y)|\chi_{B_2}(y)}{|B_2|^{1-\frac{\alpha}{d}}}\right||b(x+t)-b(y)||f(y)|dy\\ &\le C|t|(BM_\alpha(f,\mathrm{g})(x) + BM_\alpha(f,\tau_{2t}\mathrm{g})(x))\\ &\quad + C(|t|^{\frac{1}{ks'}}(BM_{\alpha s}(f^s,\mathrm{g}^s)(x))^{\frac{1}{s}} + BM_\alpha(f,\tau_{2t}\mathrm{g}-\mathrm{g})(x)).\end{aligned}$$

Here, τ_a is the shift operator $\tau_a\mathrm{g} = \mathrm{g}(x+a)$ and $s' = s/(s-1)$.

Proof. The same argument as in Lemma 2.7, we also need to consider two cases.

Case I: $r \le |t|$. For $y \in B_2$, we have

$$|x+t-y| \le |y-x| + |t| \le r + 2|t| \le 3|t|,$$

which implies that

$$\begin{aligned}&\int_{B_2}\left|\frac{\mathrm{g}(2x-y)\chi_{B_1}(y)}{|B_1|^{1-\frac{\alpha}{d}}} - \frac{\mathrm{g}(2x+2t-y)\chi_{B_2}(y)}{|B_2|^{1-\frac{\alpha}{d}}}\right||b(x+t)-b(y)||f(y)|dy\\ &\le |x+t-y|\|\nabla b\|_{L^\infty(\mathbb{R}^d)}\frac{1}{|B_1|^{1-\frac{\alpha}{d}}}\int_{B_1}|f(y)||\mathrm{g}(2x-y)|dy\\ &\quad + |x+t-y|\|\nabla b\|_{L^\infty(\mathbb{R}^d)}\frac{1}{|B_2|^{1-\frac{\alpha}{d}}}\int_{B_2}|f(y)||\mathrm{g}(2x+2t-y)|dy\\ &\le C\|\nabla b\|_{L^\infty(\mathbb{R}^d)}(|t|\mathcal{M}_\alpha(f,\mathrm{g})(x) + BM_\alpha(f,\tau_{2t}\mathrm{g})(x)).\end{aligned}$$

Case II: $r > |t|$. In this case, we need to add and subtract the following term

$$\int_{B_2}\frac{\chi_{B_1}(y)}{|B_2|^{1-\frac{\alpha}{d}}}|b(x+t)-b(y)||f(y)||\mathrm{g}(2x-y)|dy$$

and

$$\int_{B_2}\frac{\chi_{B_2}(y)}{|B_2|^{1-\frac{\alpha}{d}}}|b(x+t)-b(y)||f(y)||\mathrm{g}(2x-y)|dy.$$

Hence, we obtain that

$$\int_{B_2}\left|\frac{|\mathrm{g}(2x-y)|\chi_{B_1}(y)}{|B_1|^{1-\frac{\alpha}{d}}} - \frac{|\mathrm{g}(2x+2t-y)|\chi_{B_2}(y)}{|B_2|^{1-\frac{\alpha}{d}}}\right||b(x+t)-b(y)||f(y)|dy$$

$$
\begin{aligned}
&\leq \int_{B_2}\left|\frac{\chi_{B_1}(y)}{|B_1|^{1-\frac{\alpha}{d}}}-\frac{\chi_{B_1}(y)}{|B_2|^{1-\frac{\alpha}{d}}}\right||b(x+t)-b(y)||f(y)||\mathrm{g}(2x-y)|dy\\
&\quad+\int_{B_2}\left|\frac{\chi_{B_2}(y)}{|B_2|^{1-\frac{\alpha}{d}}}-\frac{\chi_{B_1}(y)}{|B_2|^{1-\frac{\alpha}{d}}}\right||b(x+t)-b(y)||f(y)||\mathrm{g}(2x-y)|dy\\
&\quad+\int_{B_2}\left|\frac{|\mathrm{g}(2x-y)|\chi_{B_2}(y)}{|B_2|^{1-\frac{\alpha}{d}}}-\frac{|\mathrm{g}(2x+2t-y)|\chi_{B_2}(y)}{|B_2|^{1-\frac{\alpha}{d}}}\right||b(x+t)-b(y)||f(y)|dy\\
&=: E_1+E_2+E_3.
\end{aligned}
$$

Estimate for E_1. Since $|x+t-y|\leq|y-x|+|t|\leq r+|t|\leq 2r$, then use the similar argument of A_1 in Lemma 2.7, we have

$$
\begin{aligned}
E_1&\leq\frac{|B_2|^{1-\frac{\alpha}{d}}-|B_1|^{1-\frac{\alpha}{d}}}{|B_2|^{1-\frac{\alpha}{d}}}\frac{1}{|B_1|^{1-\frac{\alpha}{d}}}\int_{B_1}|b(x+t)-b(y)||f(y)||\mathrm{g}(2x-y)|dy\\
&\leq Ct\|\nabla b\|_{L^\infty(\mathbb{R}^d)}\frac{1}{|B_1|^{1-\frac{\alpha}{d}}}\int_{B_1}|b(x+t)-b(y)||f(y)||\mathrm{g}(2x-y)|dy\\
&\leq CtBM_\alpha(f,\mathrm{g})(x).
\end{aligned}
$$

Estimate for E_2. By Hölder's inequality for $s>1$, we have

$$
E_2\leq\frac{1}{|B_2|^{1-\frac{\alpha}{d}}}\left(\int_{B_2\setminus B_1}|b(x+t)-b(y)|^{s'}dy\right)^{\frac{1}{s'}}\left(\int_{B_2}(|f(y)||\mathrm{g}(2x-y)|)^s dy\right)^{\frac{1}{s}}.
$$

Notice that for $k>1$

$$
\begin{aligned}
|b(x+t)-b(y)|&=|b(x+t)-b(y)|^{\frac{1}{ks'}}|b(x+t)-b(y)|^{1-\frac{1}{ks'}}\\
&\leq C\|b\|_{L^\infty(\mathbb{R}^d)}^{1-\frac{1}{ks'}}\|\nabla b\|_{L^\infty(\mathbb{R}^d)}^{\frac{1}{ks'}}|x+t-y|^{\frac{1}{ks'}}\\
&\leq C\|b\|_{L^\infty(\mathbb{R}^d)}^{1-\frac{1}{ks'}}\|\nabla b\|_{L^\infty(\mathbb{R}^d)}^{\frac{1}{ks'}}r^{\frac{1}{ks'}}.
\end{aligned}
$$

Applying the above inequality, we have

$$
\begin{aligned}
E_2&\leq C\left(\frac{|B_2|-|B_1|}{|B_2|}\right)^{\frac{1}{s'}}r^{\frac{1}{ks'}}(BM_{\alpha s}(f^s,\mathrm{g}^s)(x))^{\frac{1}{s}}\\
&\leq C|t|^{\frac{1}{ks'}}(BM_{\alpha s}(f^s,\mathrm{g}^s)(x))^{\frac{1}{s}}.
\end{aligned}
$$

Estimate for E_3. Using the triangle inequality, we have

$$
\begin{aligned}
E_3&\leq\frac{1}{|B_2|^{1-\frac{\alpha}{d}}}\int_{B_2}|b(x+t)-b(y)||f(y)||\mathrm{g}(2x+2t-y)-\mathrm{g}(2x-y)|dy\\
&\leq C\|b\|_{L^\infty(\mathbb{R}^d)}BM_\alpha(f,\tau_{2t}\mathrm{g})(x).
\end{aligned}
$$

This finishes the proof. □

3. Proof of Theorems 1.4-1.5

This section is devoted to proving Theorems 1.4-1.5, some proof ideals are inspired by [2], [30] and [31].

Proof of Theorem 1.4. We only prove $\mathcal{M}^1_{\alpha,b}$ and $\mathcal{M}_{\alpha,\vec{b}}$ are compact, as the proof of $\mathcal{M}^2_{\alpha,b}$ can be get similarly.

Since

$$|\mathcal{M}^1_{\alpha,b_1}(f,\mathrm{g})(x)-\mathcal{M}^1_{\alpha,b_2}(f,\mathrm{g})(x)|\le \mathcal{M}^1_{\alpha,b_1-b_2}(f,\mathrm{g})(x)$$

and $b_1,b_2\in \mathrm{CMO}(\mathbb{R}^d)\subset \mathrm{BMO}(\mathbb{R}^d)$, then by Remark 1.1, the commutator $\mathcal{M}^1_{\alpha,b}$ is continuous on $L^q(v^q_{\vec{w}})$. Hence, for any bounded set $F\subset L^{p_1}(w_1^{p_1})$ and $G\subset L^{p_2}(w_2^{p_2})$, where $f\in F$ with $\|f\|_{L^{p_1}(w_1^{p_1})}\le C$, and $\mathrm{g}\in G$ with $\|\mathrm{g}\|_{L^{p_2}(w_2^{p_2})}\le C$, it's sufficient to prove

$$\mathcal{F}=\{\mathcal{M}^1_{\alpha,b}(f,\mathrm{g}): f\in F,\ \mathrm{g}\in G\}$$

is a sequentially compact subset for $b\in \mathrm{CMO}(\mathbb{R}^d)$. According to a density argument, if $b\in \mathrm{CMO}(\mathbb{R}^d)$, then there exists a sequence of functions $b_\epsilon\in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that

$$\|b-b_\epsilon\|_{\mathrm{BMO}(\mathbb{R}^d)}<\epsilon.$$

Thus, by Remark 1.1, we show that

$$\begin{aligned}\|\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})-\mathcal{M}^1_{\alpha,b_\epsilon}(f,\mathrm{g})\|_{L^q(v^q_{\vec{w}})}&\le \|\mathcal{M}^1_{\alpha,b-b_\epsilon}(f,\mathrm{g})\|_{L^q(v^q_{\vec{w}})}\\&\le C\|b-b_\epsilon\|_{\mathrm{BMO}(\mathbb{R}^d)}\|f\|_{L^{p_1}(w_1^{p_1})}\|\mathrm{g}\|_{L^{p_2}(w_2^{p_2})}\\&\le C\epsilon.\end{aligned}$$

Therefore, by Lemma 2.2, it is enough to prove that $\mathcal{F}$ is sequentially compact. Next, we will verify $\mathcal{F}$ satisfies the conditions (i)-(iii) in Lemma 2.5 for $b\in \mathcal{C}_c^\infty(\mathbb{R}^d)$. The proof is divided into two steps since the condition (i) is immediate obtained by Remark 1.1.

Step I. $\mathcal{F}$ satisfies condition (ii).

Since the uncentered maximal operator and the centered maximal operator can be controlled each other, it only need to consider the centered fractional maximal operator. Choose R large so that supp $b\subset B(0,R)$, then for $x>N\ge \max\{2R,1\}$, we have

$$\begin{aligned}|\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x)|&\le C\sup_{r>0}\frac{1}{|B(x,r)|^{2-\frac{\alpha}{d}}}\int_{B(x,r)}\int_{B(x,r)}|b(y)||f(y)||\mathrm{g}(z)|dydz\\&\le C\sup_{r>0}\frac{\|b\|_{L^\infty(\mathbb{R}^d)}}{|B(x,r)|^{2-\frac{\alpha}{d}}}\int_{B(0,R)}|f(y)|dy\int_{B(x,r)}|\mathrm{g}(z)|dz.\end{aligned}$$

Since $|x|>N\ge\max\{2R,1\}$ and $B(x,r)\cap \mathrm{supp} b\ne\emptyset$ it implies that $r>|x|-R>|x|/2$, this give us that

$$\begin{aligned}&|\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x)|\\&\le C\sup_{r>0}\frac{\|f\|_{L^{p_1}(w_1^{p_1})}\|\mathrm{g}\|_{L^{p_2}(w_2^{p_2})}}{|B(x,r)|^{2-\frac{\alpha}{d}}}\left(\int_{B(0,R)}w_1^{-p_1'}(y)dy\right)^{\frac{1}{p_1'}}\left(\int_{B(x,r)}w_2^{-p_2'}(z)dz\right)^{\frac{1}{p_2'}}.\end{aligned}$$

Note that for $z \in B(x,r)$, we have $|z| \le |x| + r \le 3r$. Thus, the following inequality holds

$$
\begin{aligned}
&|\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x)| \\
\le\ & \frac{C}{|x|^{2d-\alpha}} \left(\int_{B(0,R)} w_1^{-p_1'}(y)dy \right)^{\frac{1}{p_1'}} \left(\int_{B(0,|x|)} w_2^{-p_2'}(z)dz \right)^{\frac{1}{p_2'}} \\
&+ C \left(\int_{B(0,R)} w_1^{-p_1'}(y)dy \right)^{\frac{1}{p_1'}} \left(\int_{|z|>|x|} \frac{w_2^{-p_2'}(z)}{|z|^{(2d-\alpha)p_2'}} dz \right)^{\frac{1}{p_2'}} \\
\le\ & \frac{C}{|x|^{2d-\alpha}} \left(\int_{B(0,R)} w_1^{-p_1'}(y)dy \right)^{\frac{1}{p_1'}} \left(\int_{B(0,|x|)} w_2^{-p_2'}(z)dz \right)^{\frac{1}{p_2'}} \\
&+ C \sum_{l=1}^{\infty} \frac{1}{(2^{l-1}|x|)^{2d-\alpha}} \left(\int_{B(0,R)} w_1^{-p_1'}(y)dy \right)^{\frac{1}{p_1'}} \left(\int_{|z|\le 2^l|x|} w_2^{-p_2'}(z)dz \right)^{\frac{1}{p_2'}} \\
\le\ & C \sum_{l=0}^{\infty} \frac{1}{(2^{l}|x|)^{2d-\alpha}} \left(\int_{B(0,R)} w_1^{-p_1'}(y)dy \right)^{\frac{1}{p_1'}} \left(\int_{B(0,2^l|x|)} w_2^{-p_2'}(z)dz \right)^{\frac{1}{p_2'}}.
\end{aligned}
$$

Therefore, for $q \ge 1$, by Minkowski inequality, we obtain

$$
\begin{aligned}
&\left(\int_{|x|>N} |\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x)|^q v_{\vec{w}}^q(x)dx \right)^{\frac{1}{q}} \\
\le\ & C \sum_{j=1}^{\infty} \left(\int_{2^{j-1}N<|x|\le 2^jN} |\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x)|^q v_{\vec{w}}^q(x)dx \right)^{\frac{1}{q}} \\
\le\ & C \sum_{j=1}^{\infty} \sum_{l=0}^{\infty} \frac{1}{(2^{j+l-1}N)^{2d-\alpha}} \left(\int_{B(0,2^jN)} v_{\vec{w}}^q(x)dx \right)^{\frac{1}{q}} \\
&\times \left(\int_{B(0,R)} w_1^{-p_1'}(y)dy \right)^{\frac{1}{p_1'}} \left(\int_{B(0,2^{j+l}N)} w_2^{-p_2'}(z)dz \right)^{\frac{1}{p_2'}}.
\end{aligned}
$$

Using Lemma 2.2 and Lemma 2.3, we have

$$
\begin{aligned}
&\left(\int_{|x|>N} |\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x)|^q v_{\vec{w}}^q(x)dx \right)^{\frac{1}{q}} \\
\le\ & C \sum_{j=1}^{\infty} \sum_{l=0}^{\infty} \frac{1}{(2^{j+l-1}N)^{2d-\alpha}} \frac{1}{2^{\frac{ld\theta}{q}}} \frac{R^{\frac{d\theta}{p_1'}}}{(2^{j+l}N)^{\frac{d\theta}{p_1'}}} \left(\int_{B(0,2^{j+l}N)} v_{\vec{w}}^q(x)dx \right)^{\frac{1}{q}} \\
&\times \left(\int_{B(0,2^{j+l}N)} w_1^{-p_1'}(y)dy \right)^{\frac{1}{p_1'}} \left(\int_{B(0,2^{j+l}N)} w_2^{-p_2'}(z)dz \right)^{\frac{1}{p_2'}}
\end{aligned}
$$

$$\leq C\sum_{j=1}^{\infty}\frac{1}{2^{\frac{d\theta j}{p_1'}}}\sum_{l=0}^{\infty}\frac{1}{2^{\frac{d\theta l}{p_1'}+\frac{d\theta l}{q}}}\left(\frac{R}{N}\right)^{\frac{d\theta}{p_1'}}\leq C\left(\frac{R}{N}\right)^{\frac{d\theta}{p_1'}}.$$

For $q<1$, since $(\sum_{l=0}^{\infty}a_l)^q\leq\sum_{l=0}^{\infty}a_l^q$, similar to the above estimates, we can obtain

$$\begin{aligned}&\int_{|x|>N}|\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x)|^q v_{\vec{w}}^q(x)dx\\ &\leq C\sum_{j=1}^{\infty}\int_{2^{j-1}N<|x|\leq 2^jN}|\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x)|^q v_{\vec{w}}^q(x)dx\\ &\leq C\left(\frac{R}{N}\right)^{\frac{dq\theta}{p_1'}}.\end{aligned}$$

Then, for $1/2<q<\infty$, we have

$$\lim_{N\to\infty}\int_{|x|>N}|\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x)|^q v_{\vec{w}}^q(x)dx=0$$

holds, where $f\in F$ and $\mathrm{g}\in G$.

Step II. $\mathcal{F}$ satisfies condition (iii).

It's enough to show that

$$\lim_{|t|\to 0}\|\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(\cdot+t)-\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(\cdot)\|_{L^q(v_{\vec{w}}^q)}=0.$$

For two fixed points $x,t\in\mathbb{R}^d$ with $|t|<1$, without loss of generality, we may assume that

$$\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x+t)\leq\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x)\quad\text{and}\quad\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x)<\infty.$$

Thus, for any $\epsilon\in(0,1)$, there is a ball $B_1=B(x_0,r)\ni x$ such that

$$\text{(3.1)}\quad\frac{1}{|B_1|^{2-\frac{\alpha}{d}}}\int_{B_1}\int_{B_1}|b(x)-b(y)||f(y)||\mathrm{g}(z)|dydz\geq(1-\epsilon)\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x).$$

Since $x+t\in B(x_0,r+|t|)=:B_2$, we infer that

$$\text{(3.2)}\quad\frac{1}{|B_2|^{2-\frac{\alpha}{d}}}\int_{B_2}\int_{B_2}|b(x+t)-b(y)||f(y)||\mathrm{g}(z)|dydz\leq\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x+t).$$

Using inequalities (3.1) and (3.2), we obtain

$$\begin{aligned}&(1-\epsilon)\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x)-\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x+t)\\ &\leq\frac{1}{|B_1|^{2-\frac{\alpha}{d}}}\int_{B_1}\int_{B_1}|b(x)-b(y)||f(y)||\mathrm{g}(z)|dydz\\ &\quad-\frac{1}{|B_2|^{2-\frac{\alpha}{d}}}\int_{B_2}\int_{B_2}|b(x+t)-b(y)||f(y)||\mathrm{g}(z)|dydz\\ &\leq\frac{|b(x+t)-b(x)|}{|B_1|^{2-\frac{\alpha}{d}}}\int_{B_1}\int_{B_1}|f(y)||\mathrm{g}(z)|dydz\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|B_1|^{2-\frac{\alpha}{d}}}\int_{B_1}\int_{B_1}|b(x+t)-b(y)||f(y)||\mathrm{g}(z)|dydz\\
&-\frac{1}{|B_2|^{2-\frac{\alpha}{d}}}\int_{B_2}\int_{B_2}|b(x+t)-b(y)||f(y)||\mathrm{g}(z)|dydz\\
\leq\ & C|t|\|\nabla b\|_{L^\infty(\mathbb{R}^d)}\mathcal{M}_\alpha(f,\mathrm{g})(x)\\
&+\int_{B_2}\int_{B_2}\left|\frac{\chi_{B_1}(y)\chi_{B_1}(z)}{|B_1|^{2-\frac{\alpha}{d}}}-\frac{\chi_{B_2}(y)\chi_{B_2}(z)}{|B_2|^{2-\frac{\alpha}{d}}}\right||b(x+t)-b(y)||f(y)||\mathrm{g}(z)|dydz.
\end{aligned}
$$

It follows from Lemma 2.7 and letting $\epsilon=|t|$, which implies that for any $k>1$

$$
\begin{aligned}
&|\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x+t)-\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x)|\\
&\lesssim |t|\mathcal{M}_\alpha(f,\mathrm{g})(x)+|t|^{\frac{1}{k}}\mathcal{M}_{L(\log L),\alpha}(f,\mathrm{g})(x)+|t|\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(x).
\end{aligned}
$$

Hence, by Lemma 2.4 and Remark 1.1, we have that for any $k>1$

$$
\begin{aligned}
&\|\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(\cdot+t)-\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(\cdot)\|_{L^q(v_{\vec{w}}^q)}\\
&\lesssim |t|\|\mathcal{M}_\alpha(f,\mathrm{g})\|_{L^q(v_{\vec{w}}^q)}+|t|^{\frac{1}{k}}\|\mathcal{M}_{L(\log L),\alpha}(f,\mathrm{g})\|_{L^q(v_{\vec{w}}^q)}+|t|\|\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})\|_{L^q(v_{\vec{w}}^q)}\\
&\lesssim (|t|+|t|^{\frac{1}{k}})\|f\|_{L^{p_1}(w_1^{p_1})}\|\mathrm{g}\|_{L^{p_2}(w_2^{p_2})}\lesssim |t|+|t|^{\frac{1}{k}}.
\end{aligned}
$$

Thus the desired result holds

$$
\lim_{|t|\to 0}\|\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(\cdot+t)-\mathcal{M}^1_{\alpha,b}(f,\mathrm{g})(\cdot)\|_{L^q(v_{\vec{w}}^q)}=0.
$$

Next, we will prove $\mathcal{M}_{\alpha,\vec{b}}$ is compact. If $\vec{b}\in \mathrm{CMO}(\mathbb{R}^d)\times\mathrm{CMO}(\mathbb{R}^d)$, then for any $\epsilon>0$, there exists $\vec{b}^\epsilon=(b_1^\epsilon,b_2^\epsilon)\in\mathcal{C}_c^\infty(\mathbb{R}^d)\times\mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $\|b_j-b_j^\epsilon\|_{\mathrm{BMO}(\mathbb{R}^d)}<\epsilon$ $(j=1,2)$, then by Theorem A, we have

$$
\begin{aligned}
&\|\mathcal{M}_{\alpha,\vec{b}}(f,\mathrm{g})-\mathcal{M}_{\alpha,\vec{b}^\epsilon}(f,\mathrm{g})\|_{L^q(v_{\vec{w}}^q)}\\
&\leq \|\mathcal{M}_{\alpha,(b_1-b_1^\epsilon,b_2)}(f,\mathrm{g})\|_{L^q(v_{\vec{w}}^q)}+\|\mathcal{M}_{\alpha,(b_1,b_2-b_2^\epsilon)}(f,\mathrm{g})\|_{L^q(v_{\vec{w}}^q)}\\
&\leq C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^d)}\|b_1-b_1^\epsilon\|_{\mathrm{BMO}(\mathbb{R}^d)}\|f\|_{L^{p_1}(w_1^{p_1})}\|\mathrm{g}\|_{L^{p_2}(w_2^{p_2})}\\
&\quad +C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^d)}\|b_2-b_2^\epsilon\|_{\mathrm{BMO}(\mathbb{R}^d)}\|f\|_{L^{p_1}(w_1^{p_1})}\|\mathrm{g}\|_{L^{p_2}(w_2^{p_2})}\\
&\leq C\epsilon.
\end{aligned}
$$

Thus, to prove $\mathcal{M}_{\alpha,\vec{b}}$ is compact on $L^q(v_{\vec{w}}^q)$ for any $\vec{b}\in\mathrm{CMO}(\mathbb{R}^d)\times\mathrm{CMO}(\mathbb{R}^d)$, we only need to show that $\mathcal{M}_{\alpha,\vec{b}}$ is compact for any $\vec{b}\in\mathcal{C}_c^\infty(\mathbb{R}^d)\times\mathcal{C}_c^\infty(\mathbb{R}^d)$. For arbitrary bounded sets $F\in L^{p_1}(w_1^{p_1})$ and $G\in L^{p_2}(w_2^{p_2})$, let

$$
\mathcal{G}=\{\mathcal{M}_{\alpha,\vec{b}}(f,\mathrm{g}):f\in F,\ \mathrm{g}\in G\}.
$$

Then, we shall prove that for any $\vec{b}\in\mathcal{C}_c^\infty(\mathbb{R}^d)\times\mathcal{C}_c^\infty(\mathbb{R}^d)$, $\mathcal{G}$ satisfies the conditions (i)-(iii) of Lemma 2.5.

By Theorem A, we can get (i) holds immediately.

Assume that $b_j \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and $\operatorname{supp} b_j \subset B(0,R)$, $j=1,2$. For any $|x| > N \geq \max\{2R,1\}$. Then by Hölder inequality, we have

$$\begin{aligned}
&|\mathcal{M}_{\alpha,\vec{b}}(f,\mathrm{g})(x)| \\
&\leq \sup_{r>0} \frac{1}{|B(x,r)|^{2-\frac{\alpha}{d}}} \int_{B(0,R)}\int_{B(0,R)} |b_1(y)||b_2(z)||f(y)||\mathrm{g}(z)|dydz \\
&\leq \sup_{r>0} \frac{\|b_1\|_{L^\infty(\mathbb{R}^d)}\|b_2\|_{L^\infty(\mathbb{R}^d)}}{|B(x,r)|^{2-\frac{\alpha}{d}}} \int_{B(0,R)}\int_{B(0,R)} |f(y)||\mathrm{g}(z)|dydz \\
&\lesssim \frac{\|f\|_{L^{p_1}(w_1^{p_1})}\|\mathrm{g}\|_{L^{p_2}(w_2^{p_2})}}{|x|^{2d-\alpha}} \left(\int_{B(0,R)} w^{-p_1'}(y)dy\right)^{\frac{1}{p_1'}} \left(\int_{B(0,R)} w^{-p_2'}(z)dz\right)^{\frac{1}{p_2'}},
\end{aligned}$$

where the last step is due to $r > |x| - R > |x|/2$. Thus, it follows that

$$\int_{|x|>N} |\mathcal{M}_{\alpha,\vec{b}}(f,\mathrm{g})(x)| v_{\vec{w}}^q(x)dx \leq C\int_{|x|>N} \frac{v_{\vec{w}}^q(x)}{|x|^{(2d-\alpha)q}}dx,$$

by Lemma 2.2, $\frac{1}{2} < q < \infty$ and $1 < 1 + q(2-\frac{1}{p}) = q(2-\frac{\alpha}{d}) < \infty$ we have $v_{\vec{w}}^q \in A_{q(2-\frac{\alpha}{d})}$. Thus, together with Lemma 2.3(ii) yields that

$$\lim_{N\to\infty}\int_{|x|>N} |\mathcal{M}_{\alpha,\vec{b}}(f,\mathrm{g})(x)| v_{\vec{w}}^q(x)dx = 0,$$

whenever $f \in F$ and $\mathrm{g} \in G$, that is, the condition (ii) is satisfied.

It remains to show that the set $\mathcal{G}$ is uniformly equicontinuous. It suffices to verify that for any $\epsilon \in (0,1)$, if $|t|$ is sufficiently small and dependent only on ϵ, then

$$\|\mathcal{M}_{\alpha,\vec{b}}(f,\mathrm{g})(\cdot+t) - \mathcal{M}_{\alpha,\vec{b}}(f,\mathrm{g})(\cdot)\|_{L^q(v_{\vec{w}}^q)} \leq C\epsilon, \tag{3.3}$$

holds uniformly for $f \in F$ and $\mathrm{g} \in G$.

For two fixed points $x, t \in \mathbb{R}^d$ with $|t| < 1$, without loss of generality, we may assume that

$$\mathcal{M}_{\alpha,\vec{b}}(f,\mathrm{g})(x+t) \leq \mathcal{M}_{\alpha,\vec{b}}(f,\mathrm{g})(x) \quad \text{and} \quad \mathcal{M}_{\alpha,\vec{b}}(f,\mathrm{g})(x) < \infty.$$

Thus, for any $\epsilon \in (0,1)$, there is a ball $B_1 = B(x_0, r) \ni x$ such that

$$\begin{aligned}
&\frac{1}{|B_2|^{2-\frac{\alpha}{d}}}\int_{B_2}\int_{B_2} |b_1(x)-b_1(y)||b_2(x)-b_2(z)||f(y)||\mathrm{g}(z)|dydz \\
&\geq (1-\epsilon)\mathcal{M}_{\alpha,\vec{b}}(f,\mathrm{g})(x).
\end{aligned}\tag{3.4}$$

Since $x + t \in B(x_0, r+|t|) =: B_2$, we infer that

$$\begin{aligned}
&\mathcal{M}_{\alpha,\vec{b}}(f,\mathrm{g})(x+t) \\
&\geq \frac{1}{|B_2|^{2-\frac{\alpha}{d}}}\int_{B_2}\int_{B_2} |b_1(x+t)-b_1(y)||b_2(x+t)-b_2(z)||f(y)||\mathrm{g}(z)|dydz.
\end{aligned}\tag{3.5}$$

Using inequalities (3.4) and (3.5), we obtain

$$(1-\epsilon)\mathcal{M}_{\alpha,\vec{b}}(f,\mathrm{g})(x) - \mathcal{M}_{\alpha,\vec{b}}(f,\mathrm{g})(x+t)$$

$$\leq \frac{1}{|B_1|^{2-\frac{\alpha}{d}}}\int_{B_1}\int_{B_1}|b_1(x)-b_1(y)||b_2(x)-b_2(z)||f(y)||g(z)|dydz$$
$$-\frac{1}{|B_2|^{2-\frac{\alpha}{d}}}\int_{B_2}\int_{B_2}|b_1(x+t)-b_1(y)||b_2(x+t)-b_2(z)||f(y)||g(z)|dydz$$
$$\leq \frac{|b_1(x+t)-b_1(x)||b_2(x+t)-b_2(x)|}{|B_1|^{2-\frac{\alpha}{d}}}\int_{B_1}\int_{B_1}|f(y)||g(z)|dydz$$
$$+\frac{|b_1(x+t)-b_1(x)|}{|B_1|^{2-\frac{\alpha}{d}}}\int_{B_1}\int_{B_1}|b_2(x+t)-b_2(z)||f(y)||g(z)|dydz$$
$$+\frac{|b_2(x+t)-b_2(x)|}{|B_1|^{2-\frac{\alpha}{d}}}\int_{B_1}\int_{B_1}|b_1(x+t)-b_1(y)||f(y)||g(z)|dydz$$
$$+\frac{1}{|B_1|^{2-\frac{\alpha}{d}}}\int_{B_1}\int_{B_2}|b_1(x+t)-b_1(y)||b_2(x+t)-b_2(z)||f(y)||g(z)|dydz$$
$$-\frac{1}{|B_2|^{2-\frac{\alpha}{d}}}\int_{B_2}\int_{B_2}|b_1(x+t)-b_1(y)||b_2(x+t)-b_2(z)||f(y)||g(z)|dydz.$$

Note that the fact $|b_i(x+t)-b_i(x)|\leq C\|\nabla b_i\|_{L^\infty(\mathbb{R}^d)}|t|$ for $i=1,2$, it follows that

$$(1-\epsilon)\mathcal{M}_{\alpha,\vec{b}}(f,g)(x)-\mathcal{M}_{\alpha,\vec{b}}(f,g)(x+t)$$
$$\lesssim |t|^2\|\nabla b_1\|_{L^\infty(\mathbb{R}^d)}\|\nabla b_2\|_{L^\infty(\mathbb{R}^d)}\mathcal{M}_\alpha(f,g)(x)$$
$$+|t|\|\nabla b_1\|_{L^\infty(\mathbb{R}^d)}\|b_2\|_{L^\infty(\mathbb{R}^d)}\mathcal{M}_\alpha(f,g)(x)$$
$$+|t|\|b_1\|_{L^\infty(\mathbb{R}^d)}\|\nabla b_2\|_{L^\infty(\mathbb{R}^d)}\mathcal{M}_\alpha(f,g)(x)$$
$$+\int_{B_2}\int_{B_2}\left|\frac{\chi_{B_1}(y)\chi_{B_1}(z)}{|B_1|^{2-\frac{\alpha}{d}}}-\frac{\chi_{B_2}(y)\chi_{B_2}(z)}{|B_2|^{2-\frac{\alpha}{d}}}\right||b_1(x+t)-b_2(y)||b_2(x+t)-b_2(z)|$$
$$\times|f(y)||g(z)|dydz.$$

Similarly as in the proof of Lemma 2.7, let $\epsilon=|t|$, then for any $k>1$, we have

$$|\mathcal{M}_{\alpha,\vec{b}}(f,g)(x+t)-\mathcal{M}_{\alpha,\vec{b}}(f,g)(x)|$$
$$\lesssim |t|\mathcal{M}_\alpha(f,g)(x)+|t|^{\frac{1}{k}}\mathcal{M}_{L(\log L),\alpha}(f,g)(x)+|t|\mathcal{M}_{\alpha,\vec{b}}(f,g)(x).$$

Therefore, by Lemma 2.4 and Theorem A, we give that for any $k>1$

$$\|\mathcal{M}^1_{\alpha,b}(f,g)(\cdot+t)-\mathcal{M}^1_{\alpha,b}(f,g)(\cdot)\|_{L^q(v^q_{\vec{w}})}\lesssim |t|+|t|^{\frac{1}{k}}.$$

Thus, we conclude that (3.3) holds for whenever $f\in F$ and $g\in G$. This completes the proof of Theorem 1.4. □

Proof of Theorem 1.5. We use the similar method in Theorem 1.4, only prove $BM^1_{\alpha,b}$ and $BM_{\alpha,\vec{b}}$ are compact, since the proof of the commutator $BM^2_{\alpha,b}$ can be get similarly. By a change of variables, this maximal commutators can be rewritten as

$$BM^1_{\alpha,b}(f,g)(x)=\sup_{r>0}\frac{1}{|B(0,r)|^{1-\frac{\alpha}{d}}}\int_{B(x,r)}|b(x)-b(y)||f(y)||g(2x-y)|dy.$$

Similarly, we only need to prove that for $b \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $\mathfrak{F} = \{BM^1_{\alpha,b}(f,\mathrm{g}) : f \in F,\ \mathrm{g} \in G\}$ satisfies the conditions (ii)-(iii) of Lemma 2.5.

Suppose that b have a compact support set $B(0,R)$ and $|x| > N \geq \max\{2R, 1\}$. Then, for $\operatorname{supp} b \cap B(x,r) \neq \emptyset$, we have $r > |x| - R \geq |x|/2$ and $|2x-y| \leq |x-y| + |x| \leq 3r$. It implies that

$$|BM^1_{\alpha,b}(f,\mathrm{g})(x)| \leq C \int_{B(0,R)} \frac{|f(y)||\mathrm{g}(2x-y)|}{|2x-y|^{d-\alpha}} dy. \tag{3.6}$$

For $1/2 < q < 1$, by Hölder's inequality, we obtain that

$$\begin{aligned}
&\int_{|x|>N} |BM^1_{\alpha,b}(f,\mathrm{g})(x)|^q v^q_{\vec{w}}(x)dx \\
&\lesssim \sum_{j=1}^\infty \int_{2^{j-1}N<|x|\leq 2^jN} \left| \int_{B(0,R)} \frac{|f(y)||\mathrm{g}(2x-y)|}{|2x-y|^{d-\alpha}} dy \right|^q v^q_{\vec{w}}(x)dx \\
&\lesssim \sum_{j=1}^\infty \left(\int_{2^{j-1}N<|x|\leq 2^jN} \int_{B(0,R)} \frac{|f(y)||\mathrm{g}(2x-y)|}{|2x-y|^{d-\alpha}} dydx \right)^q \\
&\quad \times \left(\int_{2^{j-1}N<|x|\leq 2^jN} v_{\vec{w}}^{\frac{q}{1-q}}(x)dx \right)^{1-q}.
\end{aligned}$$

Since $|x| > 2^{j-1}N$ and $|2x-y| > 2|x| - |y| > |x| > 2^{j-1}N$, by a change of variables, we give that

$$\begin{aligned}
&\int_{|x|>N} |BM^1_{\alpha,b}(f,\mathrm{g})(x)|^q v^q_{\vec{w}}(x)dx \\
&\lesssim \sum_{j=1}^\infty \left(\int_{|z|>2^{j-1}N} \int_{B(0,R)} \frac{|f(y)||\mathrm{g}(z)|}{|z|^{d-\alpha}} dydz \right)^q \left(\int_{B(0,2^jN)} v_{\vec{w}}^{\frac{q}{1-q}}(x)dx \right)^{1-q} \\
&\lesssim \sum_{j=1}^\infty \sum_{l=0}^\infty \frac{1}{(2^{j+l-1}N)^{(d-\alpha)q}} \left(\int_{B(0,2^jN)} v_{\vec{w}}^{\frac{q}{1-q}}(x)dx \right)^{1-q} \\
&\quad \times \left(\int_{B(0,2^{j+l}N)} |\mathrm{g}(z)|dz \right)^q \left(\int_{B(0,R)} |f(y)|dy \right)^q.
\end{aligned}$$

Using Hölder's inequality and (iii) of Lemma 2.3, we have that for $1 < a < \min\{p_1, p_2\}$

$$\begin{aligned}
&\int_{|x|>N} |BM^1_{\alpha,b}(f,\mathrm{g})(x)|^q v^q_{\vec{w}}(x)dx \\
&\lesssim \sum_{j=1}^\infty \sum_{l=0}^\infty \frac{\|f\|_{L^{p_1}(w_1^{p_1})} \|\mathrm{g}\|_{L^{p_2}(w_2^{p_2})}}{(2^{j+l-1}N)^{(d-\alpha)q}} \left(\int_{B(0,2^jN)} v_{\vec{w}}^{\frac{q}{1-q}}(x)dx \right)^{1-q}
\end{aligned}$$

$$\times \left(\int_{B(0,2^{j+l}N)} w_2^{-p_2'}(x)dx\right)^{\frac{q}{p_2'}} \left(\int_{B(0,R)} w_1^{-p_1'}(x)dx\right)^{\frac{q}{p_1'}}$$

$$\lesssim \sum_{j=1}^{\infty}\sum_{l=0}^{\infty} \frac{1}{2^{d\theta(1-q)l}} \left(\frac{R}{2^{j+l}N}\right)^{\frac{d\theta q}{p_1'}} \left(\frac{1}{|B(0,2^{j+l}N)|}\int_{B(0,2^{j+l}N)} v_{\vec{w}}^{\frac{aq}{1-q}}(x)dx\right)^{\frac{1-q}{a}}$$

$$\times \left(\frac{1}{|B(0,2^{j+l}N)|}\int_{B(0,2^{j+l}N)} w_2^{-(\frac{p_2}{a})'}(x)dx\right)^{\frac{q}{(\frac{p_2}{a})'}}$$

$$\times \left(\frac{1}{|B(0,2^{j+l}N)|}\int_{B(0,2^{j+l}N)} w_1^{-(\frac{p_1}{a})'}(x)dx\right)^{\frac{q}{(\frac{p_1}{a})'}}$$

$$\lesssim \sum_{j=1}^{\infty}\sum_{l=0}^{\infty} \frac{1}{2^{d\theta(1-q)l}} \left(\frac{R}{2^{j+l}N}\right)^{\frac{d\theta q}{p_1'}}$$

$$\lesssim \left(\frac{R}{N}\right)^{\frac{d\theta q}{p_1'}}.$$

For $q > 1$, by inequality (3.6) and Hölder's inequality, we have

$$|BM^1_{\alpha,b}(f,\mathrm{g})(x) \lesssim \left(\int_{B(0,R)} w_1^{-k_1}(y)dy\right)^{\frac{1}{k_1}} \left(\int_{B(0,R)} \frac{w_2^{-k_2}(2x-y)}{|2x-y|^{(d-\alpha)k_2}}dy\right)^{\frac{1}{k_2}},$$

where $k_1 = s_1\left(\frac{p_1}{s_1}\right)'$ and $k_2 = s_2\left(\frac{p_2}{s_2}\right)'$. Since $2^{j-1}N < |x| \le 2^jN$ and $|y| < R$ implies that $|2x-y| \ge 2|x| - y > |x|$. Hence, by Minkowski's inequality and a change of variables, we obtain

$$\left(\int_{|x|>N} |BM^1_{\alpha,b}(f,\mathrm{g})(x)|^q v_{\vec{w}}^q(x)dx\right)^{\frac{1}{q}}$$

$$\le \sum_{j=1}^{\infty}\left(\int_{2^{j-1}N<|x|\le 2^jN} |BM^1_{\alpha,b}(f,\mathrm{g})(x)|^q v_{\vec{w}}^q(x)dx\right)^{\frac{1}{q}}$$

$$\le \sum_{j=1}^{\infty}\left(\int_{2^{j-1}N<|x|\le 2^jN} \left(\int_{B(0,R)} w_1^{-k_1}(y)dy\right)^{\frac{q}{k_1}} \left(\int_{|z|>|x|} \frac{w_2^{-k_2}(z)}{|z|^{(d-\alpha)k_2}}dz\right)^{\frac{q}{k_2}} v_{\vec{w}}^q(x)dx\right)^{\frac{1}{q}}$$

$$\lesssim \sum_{j=1}^{\infty}\sum_{l=0} \frac{1}{(2^{j+l-1}N)^{d-\alpha}} \left(\int_{B(0,2^iN)} v_{\vec{w}}^q(x)dx\right)^{\frac{1}{q}}$$

$$\times \left(\int_{B(0,R)} w_1^{-k_1}(y)dy\right)^{\frac{1}{k_1}} \left(\int_{B(0,2^{j+l}N)} w_2^{-k_2}(z)dz\right)^{\frac{1}{k_2}}.$$

Similar to the case for $\frac{1}{2} < q < 1$ and by Lemma 2.6, we have

$$\left(\int_{|x|>N} |BM^1_{\alpha,b}(f,\mathrm{g})(x)|^q v^q_{\vec{w}}(x)dx\right)^{\frac{1}{q}} \lesssim \left(\frac{R}{N}\right)^{\frac{d\theta}{k_1}}.$$

Then, for $q \in (\frac{1}{2},1) \cup (1,\infty)$, we have

$$\lim_{N\to\infty} \int_{|x|>N} |BM^1_{\alpha,b}(f,\mathrm{g})(x)|^q v^q_{\vec{w}}(x)dx = 0$$

holds, where $f \in F$ and $\mathrm{g} \in G$.

It remains to show that the set $\mathfrak{F}$ is uniformly equicontinuous. It suffices to verify that for any $\epsilon \in (0,1)$, if $|t|$ is sufficiently small and dependent only on ϵ, then

$$\|BM^1_{\alpha,b}(f,\mathrm{g})(\cdot+t) - BM^1_{\alpha,b}(f,\mathrm{g})(\cdot)\|_{L^q(v^q_{\vec{w}})} \le C\epsilon,$$

holds uniformly for $f \in F$ and $\mathrm{g} \in G$.

For two fixed points $x, t \in \mathbb{R}^d$ with $|t| < 1$, without loss of generality, we may assume that

$$BM^1_{\alpha,b}(f,\mathrm{g})(x+t) \le BM^1_{\alpha,b}(f,\mathrm{g})(x) \quad \text{and} \quad BM^1_{\alpha,b}(f,\mathrm{g})(x) < \infty.$$

Thus, for any $\epsilon \in (0,1)$, there is a ball $B_1 := B(x,r)$ such that

$$\text{(3.7)} \quad \frac{1}{|B_1|^{1-\frac{\alpha}{d}}} \int_{B_1} |b(x)-b(y)||f(y)||\mathrm{g}(2x-y)|dy \ge (1-\epsilon)BM^1_{\alpha,b}(f,\mathrm{g})(x).$$

Let $B_2 := B(x, r+|t|)$, we infer that

$$\text{(3.8)} \quad \frac{1}{|B_2|^{1-\frac{\alpha}{d}}} \int_{B_2} |b(x+t)-b(y)||f(y)||\mathrm{g}(2x-y)|dy \le BM^1_{\alpha,b}(f,\mathrm{g})(x+t).$$

Using inequalities (3.7) and (3.8), we obtain

$$\begin{aligned}
&(1-\epsilon)BM^1_{\alpha,b}(f,\mathrm{g})(x) - BM^1_{\alpha,b}(f,\mathrm{g})(x+t)\\
&\le \frac{1}{|B_1|^{1-\frac{\alpha}{d}}} \int_{B_1} |b(x)-b(y)||f(y)||\mathrm{g}(2x-y)|dy\\
&\quad - \frac{1}{|B_2|^{1-\frac{\alpha}{d}}} \int_{B_2} |b(x+t)-b(y)||f(y)||\mathrm{g}(2x-y)|dy\\
&\le \frac{|b(x+t)-b(x)|}{|B_1|^{1-\frac{\alpha}{d}}} \int_{B_1} |f(y)||\mathrm{g}(2x-y)|dy\\
&\quad + \frac{1}{|B_1|^{1-\frac{\alpha}{d}}} \int_{B_1} |b(x+t)-b(y)||f(y)||\mathrm{g}(2x-y)|dy\\
&\quad - \frac{1}{|B_2|^{1-\frac{\alpha}{d}}} \int_{B_2} |b(x+t)-b(y)||f(y)||\mathrm{g}(2x-y)|dy\\
&\le C|t|\|\nabla b\|_{L^\infty(\mathbb{R}^d)} BM_\alpha(f,\mathrm{g})(x)\\
&\quad + \int_{B_2} \left|\frac{\chi_{B_1}(y)}{|B_1|^{2-\frac{\alpha}{d}}} - \frac{\chi_{B_2}(y)}{|B_2|^{1-\frac{\alpha}{d}}}\right| |b(x+t)-b(y)||f(y)||\mathrm{g}(2x-y)|dy.
\end{aligned}$$

It follows from Lemma 2.8 and letting $\epsilon = |t|$, which implies that for any $k > 1$ and $1 < s < \min\{p_1, p_2\}$

$$\begin{aligned}&|BM^1_{\alpha,b}(f,\mathrm{g})(x+t) - BM^1_{\alpha,b}(f,\mathrm{g})(x)|\\ &\lesssim |t|(BM_\alpha(f,\mathrm{g})(x) + BM_\alpha(f,\tau_{2t}\mathrm{g})(x)) + |t|^{\frac{1}{ks'}}(BM_{\alpha s}(f^s,\mathrm{g}^s)(x))^{\frac{1}{s}}\\ &\quad + BM_\alpha(f,\tau_{2t}\mathrm{g}-\mathrm{g})(x) + |t|BM^1_{\alpha,b}(f,\mathrm{g})(x).\end{aligned}$$

Hence, by weighted boundedness of B_α in [15, 16] and Remark 1.3, for any $k > 1$ and $1 < s < \min\{p_1, p_2\}$, we have

$$\begin{aligned}&\|BM^1_{\alpha,b}(f,\mathrm{g})(\cdot+t) - BM^1_{\alpha,b}(f,\mathrm{g})(\cdot)\|_{L^q(v^q_{\vec{w}})}\\ &\lesssim |t|(\|BM_\alpha(f,\mathrm{g})\|_{L^q(v^q_{\vec{w}})} + \|BM_\alpha(f,\tau_{2t}\mathrm{g})\|_{L^q(v^q_{\vec{w}})} + \|BM^1_{\alpha,b}(f,\mathrm{g})\|_{L^q(v^q_{\vec{w}})})\\ &\quad + |t|^{\frac{1}{ks'}}\|(BM_{\alpha s}(f^s,\mathrm{g}^s)(x))^{1/s}\|_{L^q(v^q_{\vec{w}})} + \|BM_\alpha(f,\tau_{2t}\mathrm{g}-\mathrm{g})\|_{L^q(v^q_{\vec{w}})}\\ &\lesssim (|t| + |t|^{\frac{1}{ks'}})\|f\|_{L^{p_1}(w_1^{p_1})}\|\mathrm{g}\|_{L^{p_2}(w_2^{p_2})} + |t|\|f\|_{L^{p_1}(w_1^{p_1})}\|\tau_{2t}\mathrm{g}\|_{L^{p_2}(w_2^{p_2})}\\ &\quad + \|f\|_{L^{p_1}(w_1^{p_1})}\|\tau_{2t}\mathrm{g}-\mathrm{g}\|_{L^{p_2}(w_2^{p_2})}.\end{aligned}$$

Since $\mathrm{g} \in L^{p_2}(w_2^{p_2})$, for a given $\epsilon \in (0,1)$, we can find $\gamma = \gamma(\epsilon, \mathrm{g}) > 0$ such that $|t| < \gamma$ implies

$$\|\tau_{2t}\mathrm{g} - \mathrm{g}\|_{L^{p_2}(w_2^{p_2})} < \epsilon.$$

Finally, by choosing $|t| < \epsilon^{ks'}$, we can get the following desired result

$$\|BM^1_{\alpha,b}(f,\mathrm{g})(\cdot+t) - BM^1_{\alpha,b}(f,\mathrm{g})(\cdot)\|_{L^q(v^q_{\vec{w}})} \lesssim \epsilon.$$

This finishes the proof of $BM^1_{\alpha,b}$.

For $BM_{\alpha,\vec{b}}$. This case is similar to prove $\mathcal{M}_{\alpha,\vec{b}}$ and $BM^1_{\alpha,b}$, we omit the details. Thus, we complete the proof of Theorem 1.5. □

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