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FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. Existence and uniqueness for fractional differential equations satisfying a general nonlocal initial or boundary condition are proven by means of Schauder's fixed point theorem. The nonlocal condition is given as an integral with respect to a signed measure, and includes the standard initial value condition and multi-point boundary value condition.

1. Introduction

In the past few years, there have been many studies on the initial value problem for fractional differential equation

$$\begin{cases} D^\alpha x(t) = f(t, x(t)), \\ x(0) = x_0, \end{cases} \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$, $0 < \alpha < 1$ and

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}\, x'(s)\, \mathrm{d}s \tag{2}$$

is the Caputo's derivative of order $\alpha \in (0, 1)$. It is defined to be the usual derivative when $\alpha = 1$.

The existence and uniqueness for the above initial value problem have been proven in [4]. Related papers [2, 10] prove the existence and uniqueness for the initial value problem where the derivative is given in the Riemann-Liouville sense. Other papers which consider the initial value or boundary value problem for higher-order fractional derivative include [1, 6–9, 11]. For a comprehensive study on fractional differential equation and its interpretations, we refer the reader to [3]. For a physical application of fractional differential equations, refer to [5].

The purpose of this paper is to study the fractional differential equation for $0 < \alpha < 1$ but with a more general integral (nonlocal) condition, which could

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various types of initial value condition, terminal value condition, or multi-point boundary value condition. More precisely, we will prove the existence and uniqueness of the solution for the problem on $[0, b]$:

$$
\begin{cases} D^\alpha x(t) = f(t, x(t)), \\ \int_0^b x(t)\,\mathrm{d}\nu = x_0, \end{cases} \tag{3}
$$

where $f : [0, b] \times \mathbb{R}^n \to \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$, $0 < \alpha < 1$, and $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, and each ν_i is a signed measure on $[0, b]$ such that $\nu_i[0, b] \neq 0$, and ν_i satisfies certain bounds.

The above integral boundary condition is quite natural for applications since the value of physical quantities (velocity, temperature, etc.) is experimentally just the mean (integral) of measurements taken over a time interval. We are interested to determine the flow of the system given such integral conditions.

Note that the above integral boundary condition covers many types of common initial/boundary conditions. For instance, by taking ν_i to be Delta measure supported at 0, we recover the results for initial value problem. By taking ν_i to be Delta measure supported at b, we recover the terminal boundary condition. If ν_i are linear combinations of Delta measures supported at various points, we recover the results for boundary conditions of Cauchy-Nicoletti or interpolation type such as $x_i(t_i) = d_i$ for $i = 1, \ldots, n$, where $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = b$. Also, by taking ν_i to be appropriate linear combinations of Delta and Lebesgue measures, we can include integrated initial condition such as $x(0) = \int_0^b x(t)\,\mathrm{d}t$. This list is certainly not exhaustive, and can include various types of multi-point conditions.

Subsequently, let $B = \{x \in \mathbb{R}^n : \|x - x_0\| \leq c\}$, where c is a constant. Throughout, let $\nu = (\nu_1, \ldots, \nu_n)$ in the problem (3) be signed measures on $[0, b]$, with $\nu_i[0, b] \neq 0$, satisfying a growth condition: there is a constant $C > 0$ such that for any $t \in [0, b]$,

$$
\int_t^b (s - t)^{\alpha - 1}\,\mathrm{d}|\nu_i|(s) \leq C(b - t)^{\alpha - 1}. \tag{4}
$$

This growth condition is certainly satisfied by the Delta and Lebesgue measure. We will prove the following main theorems. The first one is the existence of at least one solution.

Theorem 1.1. *Suppose that $f : [0, b] \times B \to \mathbb{R}^n$ satisfies the Carathéodory-type conditions*:

(1) *$f(t, \cdot)$ is continuous on B for each fixed t, and*
(2) *there is a constant $\beta \in (0, \alpha)$ and a real-valued function $g \in L^{\frac{1}{\beta}}([0, b])$ such that $\|f(t, x)\| \leq g(t)$ for almost every $t \in [0, b]$ and all $x \in B$.*

Then the problem (3) *has at least one absolutely continuous solution on* $[0, b]$.

The second one is on existence and uniqueness.

Theorem 1.2. *Suppose that $f : [0,b] \times B \to \mathbb{R}^n$ satisfies the Lipschitz-type condition*:

(1) *there is a constant $\gamma \in (0,\alpha)$ and a function $h \in L^{\frac{1}{\gamma}}([0,b])$ such that $\|f(t,x) - f(t,y)\| \leq h(t)\|x-y\|$ for almost every $t \in [0,b]$ and all $x, y \in B$,*

and the following bound on b:

$$\frac{(1+C)\|h\|_{L^{\frac{1}{\gamma}}}}{\Gamma(\alpha)}\left(\frac{1-\gamma}{\alpha-\gamma}\right)^{1-\gamma} b^{\alpha-\gamma} < 1.$$

Then the problem (3) *has a unique absolutely continuous solution on* $[0,b]$.

Proofs of these theorems will be given in the next section.

2. Proofs of the main theorems

First, note that an absolutely continuous function x satisfies the equation $D^\alpha x(t) = f(t,x(t))$ if and only if it satisfies the following nonlinear Volterra integral equation of the second kind for $t \geq t_0$ (see Lemma 2.1 of [4] or the discussion in [3]):

$$x(t) = x(t_0) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-w)^{\alpha-1} f(w, x(w))\,\mathrm{d}w.$$

We will use the above result to prove the main theorems. First, we prove Theorem 1.1.

Proof of Theorem 1.1. To simplify notation we can normalise $\nu[0,b] = 1$. Note that x is a solution if and only if

$$\begin{aligned}
&x(t) - x_0\\
&= x(t) - \int_0^b x(s)\,\mathrm{d}\nu(s)\\
&= \int_0^b x(t) - x(s)\,\mathrm{d}\nu(s)\\
&= \int_0^b \left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-w)^{\alpha-1} f(w,x(w)) - \frac{1}{\Gamma(\alpha)}\int_0^s (s-w)^{\alpha-1} f(w,x(w))\right)\mathrm{d}w\,\mathrm{d}\nu(s)\\
&= \frac{1}{\Gamma(\alpha)}\left(\int_0^b\int_0^t (t-w)^{\alpha-1} f(w,x)\,\mathrm{d}w\,d\nu(s) - \int_0^b\int_0^s (s-w)^{\alpha-1} f(w,x)\,\mathrm{d}w\,\mathrm{d}\nu(s)\right)\\
&= \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-w)^{\alpha-1} f(w,x)\,\mathrm{d}w - \int_0^b\int_w^b (s-w)^{\alpha-1} f(w,x)\,\mathrm{d}\nu(s)\,\mathrm{d}w\right).
\end{aligned}$$

Now let the set $U := \{x \in C[0,b] : \|x - x_0\|_\infty \leq d\}$, which is closed, bounded, and convex, and the operator T defined on U as:

$$Tx(t) = x_0 + \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-w)^{\alpha-1} f(w,x)\,\mathrm{d}w - \int_0^b\int_w^b (s-w)^{\alpha-1} f(w,x)\,\mathrm{d}\nu(s)\,\mathrm{d}w\right).$$

We will prove that T has at least a fixed point by Schauder's fixed point theorem. First, we will show that $Tx \in U$ for any $x \in U$. By Hölder inequality and the assumptions,

$$\begin{aligned}
&\|Tx(t) - x_0\|_{L^\infty} \\
&\le \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-w)^{\alpha-1} m(w)\,\mathrm{d}w + \int_0^b \int_w^b (s-w)^{\alpha-1} m(w)\,\mathrm{d}\nu(s)\,\mathrm{d}w \right) \\
&\le \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-w)^{\alpha-1} m(w)\,\mathrm{d}w + C \int_0^b (b-w)^{\alpha-1} m(w)\,\mathrm{d}w \right) \\
&\le \frac{1}{\Gamma(\alpha)} \left[C' \|m\|_{L^{\frac{1}{\beta}}} \|(b-w)^{\alpha-1}\|_{L_w^{\frac{1}{1-\beta}}} \right] \\
&\le \frac{C'}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} b^{\alpha-\beta} \|m\|_{L^{\frac{1}{\beta}}} =: d.
\end{aligned}$$

Next, note that T is continuous. Suppose that $x_n \to x$ in $C[0,b]$. Then

$$\|Tx_n - Tx\|_{L^\infty} \le \frac{Cb^\alpha}{\Gamma(\alpha+1)} \sup_{w \in [0,b]} \|f(w, x_n(w)) - f(w, x(w))\| \to 0$$

as $n \to \infty$, by the continuity assumption on f.

Clearly, the family $T(U)$ is uniformly bounded since $\|Tx\| \le \|x_0\| + d$. Finally, we will prove equicontinuity. For $t_1 < t_2 \in [0,b]$,

$$\begin{aligned}
\|Tx(t_2) - Tx(t_1)\| &\le \frac{1}{\Gamma(\alpha)} \left[\int_0^{t_1} \|((t_2-w)^{\alpha-1} - (t_1-w)^{\alpha-1}) f(w,x)\|\,\mathrm{d}w \right. \\
&\qquad \left. + \int_{t_1}^{t_2} \|(t_2-w)^{\alpha-1} f(w,x)\|\,\mathrm{d}w \right] \\
&\le \frac{1}{\Gamma(\alpha)} \left(\int_0^{t_1} (t_2-w)^{\frac{\alpha-1}{1-\beta}} - (t_1-w)^{\frac{\alpha-1}{1-\beta}}\,\mathrm{d}w \right)^{1-\beta} \|m\|_{L^{\frac{1}{\beta}}} \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\int_{t_1}^{t_2} (t_2-w)^{\frac{\alpha-1}{1-\beta}}\,\mathrm{d}w \right)^{1-\beta} \|m\|_{L^{\frac{1}{\beta}}} \\
&\le \frac{\|m\|_{L^{\frac{1}{\beta}}}}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} \left(t_2^{\alpha-\beta} - (t_2-t_1)^{\alpha-\beta} - t_1^{\alpha-\beta} \right)^{1-\beta} \\
&\qquad + \frac{\|m\|_{L^{\frac{1}{\beta}}}}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} (t_2-t_1)^{\alpha-\beta} \\
&\le \frac{2\|m\|_{L^{\frac{1}{\beta}}}}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} (t_2-t_1)^{\alpha-\beta},
\end{aligned}$$

which proves equicontinuity of the family $T(U)$ on $[0,b]$. Therefore, the family $T(U)$ is pre-compact by the Arzelà-Ascoli theorem. By Schauder's fixed point theorem, T has at least one fixed point. This completes the proof. $\square$

Next, we will prove Theorem 1.2.

Proof of Theorem 1.2. Let the set $U := \{x \in C[0,b] : \|x - x_0\|_\infty \le d\}$, and the operator T defined on U as:

$$Tx(t) = x_0 + \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-w)^{\alpha-1} f(w,x)\,\mathrm{d}w - \int_0^b \int_w^b (s-w)^{\alpha-1} f(w,x)\,\mathrm{d}\nu(s)\,\mathrm{d}w\right).$$

We will show that T is a contraction map on U. For any $x, y \in U$, using the assumption (c), we get

$$\begin{aligned}
&\|Tx(t) - Ty(t)\| \\
&\le \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-w)^{\alpha-1} h(w)\,\mathrm{d}w + C\int_0^b (b-w)^{\alpha-1} h(w)\,\mathrm{d}w\right)\|x-y\|_{L^\infty} \\
&\le \frac{1}{\Gamma(\alpha)}\left(\|h\|_{L^{\frac{1}{\gamma}}}\|(t-w)^{\alpha-1}\|_{L^{\frac{1}{1-\gamma}}} + C\|h\|_{L^{\frac{1}{\gamma}}}\|(b-w)^{\alpha-1}\|_{L^{\frac{1}{1-\gamma}}}\right)\|x-y\|_{L^\infty} \\
&\le \frac{(1+C)\|h\|_{L^{\frac{1}{\gamma}}}}{\Gamma(\alpha)}\left(\frac{1-\gamma}{\alpha-\gamma}\right)^{1-\gamma} b^{\alpha-\gamma}\|x-y\|_{L^\infty}.
\end{aligned}$$

Therefore, T is a contraction when

$$\frac{(1+C)\|h\|_{L^{\frac{1}{\gamma}}}}{\Gamma(\alpha)}\left(\frac{1-\gamma}{\alpha-\gamma}\right)^{1-\gamma} b^{\alpha-\gamma} < 1$$

and in this case yielding the unique fixed point x, which solves the problem (3). □

3. Concluding remarks

We have thus proven the existence and uniqueness of solution to (3) under certain conditions. By taking the measure $\nu = \delta_{t_0}$, the Dirac measure supported at t_0, we recover the existence and uniqueness result for the initial value problem. By taking the measure $\nu = \sum_{i=1}^n a_i\delta_{t_i}$ with $\sum_{i=1}^n a_i \neq 0$, we have the existence and uniqueness result for the problem with Cauchy-Nicoletti or interpolation-type boundary condition. By taking the measure ν to be a probability measure, we have the existence and uniqueness result for the problem equipped with a given mean value (which is widely applicable in practice since many measurements are actually mean/integral value of some quantities).

The extension of the results to other type of fractional differential equation involving different types of fractional derivatives will be explored in the future.

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