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A NOTE ON STATIC MANIFOLDS AND ALMOST RICCI SOLITONS

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Abstract. In this short paper, we investigate the existence of non-trivial almost Ricci solitones on static manifolds. As a result we show any compact nontrivial static manifold is isometric to a Euclidean sphere.

1. Introduction

In [2], Corvino studied localized scalar curvature deformation of a Riemannian metric and introduced the following definition:

Definition 1.1. A Riemannian metric g is called static on a manifold M if the linearized scalar curvature map at g has a nontrivial cokernel, i.e., if there exists a nontrivial function f on M such that

$$-\Delta(f)g + \nabla^2 f - fRic = 0. \tag{1.1}$$

Here ∇^2, Δ and Ric denote the Hessian, the Laplacian and the Ricci curvature of g, respectively.

A nontrivial solution f to (1.1) has been called a static potential if it exists. It is proved that a static metric (as defined above) must have constant scalar curvature [2]. When this constant is zero (which is always the case for an asymptotically flat, static metric), (1.1) becomes

$$\nabla^2 f = fRic \quad \text{and} \quad \Delta f = 0. \tag{1.2}$$

Some investigations around static manifolds can be found in [3, 5, 6].

On the other hand, the concept of almost Ricci soliton was introduced in a recent paper due to Pigola et al. [4], where essentially they modified the definition of Ricci solitons by adding the condition on the parameter λ to be a variable function. More precisely:

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Definition 1.2. A Riemannian manifold (M, g) is an almost Ricci soliton if there exist a complete vector filed X and a smooth soliton function $\lambda : M \to \mathbb{R}$ satisfying:

$$Ric + \frac{1}{2}L_X g = \lambda g, \tag{1.3}$$

where L_X denotes the Lie derivative in the direction of X.

Almost Ricci soliton will be called expanding, steady or shrinking, respectively if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. When the vector filed X is a gradient of a smooth function $f : M \to \mathbb{R}$, the manifold will be called a gradient almost Ricci soliton. In this case the solition equation (1.3) turns out to be:

$$Ric + \nabla^2 f = \lambda g. \tag{1.4}$$

Moreover, when X is a Killing vector filed, the almost Ricci soliton will be called trivial, otherwise it will be a nontrivial almost Ricci soliton [4].

Barros et al. proved that:

Theorem 1.3 ([1]). *Every compact nontrivial almost Ricci soliton with constant scaler curvature is gradient.*

2. Main results

In this section, we investigate the existence of gradient almost Ricci solitones on static manifolds. As a result we prove any compact nontrivial static manifold is isometric to a Euclidean sphere. At first, we show:

Theorem 2.1. *Almost Ricci solitons on static manifold (M, g) are gradient.*

Proof. Since static manifold (M, g) has constant scaler curvature, the result is obtained from Theorem 1.3. □

Finally, we obtain the following theorem:

Theorem 2.2. *Every compact static manifold (M, g) with static potential f has a gradient almost Ricci solitons.*

Proof. Let (M, g) be a compact static manifold with static potential f. Taking the trace of (1.1), we have

$$\Delta f - n\Delta f - fR = 0, \tag{2.1}$$

where R is the scaler curvature of g. Thus we have

$$(1 - n)\Delta f - fR = 0. \tag{2.2}$$

Consequently

$$\Delta f = \frac{R}{1 - n} f. \tag{2.3}$$

On the other hand, computing the trace of equation (1.4) yields that

$$R + \Delta f = n\lambda. \tag{2.4}$$

Hence,

$$\Delta f = n\lambda - R. \tag{2.5}$$

Then, from equations (2.5) and (2.3) we arrive at

$$n\lambda - R + \frac{R}{n-1} f = 0. \tag{2.6}$$

Therefore we acquire the smooth function λ as follows:

$$\lambda = \left(\frac{1}{n} - \frac{f}{n(n-1)} \right) R. \tag{2.7}$$

So we proved on the compact static manifold (M, g) there exist a gradient almost Ricci soliton with potential function f and a soliton function λ as given by equation (2.7). □

Corollary 2.3. *Any compact nontrivial static manifold is isometric to a Euclidean sphere.*

Proof. Since we have shown every compact static manifold (M, g) with static potential f has a gradient almost Ricci solitons, applying Corollary 1 in [1] we conclude that the every compact nontrivial static manifold isometric to a Euclidean sphere. □

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