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SOME RESULTS ON S-ACCR PAIRS

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Abstract. Let $R \subseteq T$ be an extension of a commutative ring and $S \subseteq R$ a multiplicative subset. We say that (R, T) is an S-accr (a commutative ring R is said to be S-accr if every ascending chain of residuals of the form $(I : B) \subseteq (I : B^2) \subseteq (I : B^3) \subseteq \cdots$ is S-stationary, where I is an ideal of R and B is a finitely generated ideal of R) pair if every ring A with $R \subseteq A \subseteq T$ satisfies S-accr. Using this concept, we give an S-version of several different known results.

1. Introduction

Rings and modules satisfying the accr condition were introduced by Lu in [7]: an R-module M *satisfies the accr conditions* (*resp., accr**) if every ascending chain of submodules of M of the form $(N : B) \subseteq (N : B^2) \subseteq (N : B^3) \subseteq \cdots$ terminates for every submodule N of M and every finitely generated (resp., principal) ideal B of R. The ring R *satisfies the accr condition* if the R-module R does. Note that if M is a Noetherian module, then M satisfies the accr condition. Later, Hamed and Hizem [2] generalize this notion by introducing the definition of modules and rings satisfying the S-accr condition. First let us recall the following definition. Let R be a commutative ring, S a multiplicative subset of R such that $1 \in S$ and $0 \notin S$ and M an R-module. According to [2] an increasing sequence $(N)_{n\in\mathbb{N}}$ of submodules of M is called *S-stationary* if there exist a positive integer k and some $s \in S$ such that for each $n \geq k$, $sN_n \subseteq N_k$. We say that M satisfies *the S-accr condition* if any ascending chain of residuals of the form $(N : B) \subseteq (N : B^2) \subseteq (N : B^3) \subseteq \cdots$ is S-stationary where N is a submodule of M and B is a finitely generated ideal of R. The ring R satisfies *the S-accr condition* if the R-module R does.

On the other hand, let $R \subseteq T$ be an extension of commutative rings. Recall from [11] that the extension (R, T) is called an *accr* (*resp.,* accr*) *pair* if every ring A with $R \subseteq A \subseteq T$ satisfies accr (resp., *accr**). In [11], the author studied the accr pair property. He showed that $(R, R[X])$ is an accr pair if and only if R is Artinian. Let $F_1 \subseteq F_2$ be an extension of fields. The author proved that the following assertions are equivalent: (1) $(F_1[X], F_2[X])$ is an accr pair, (2)

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F_2 is algebraic over F_1, (3) $F_2[X]$ is integral over $F_1[X]$ and (4) $(F_1[X], F_2[X])$ is an LP. Let T be a ring and M be a T-module. M is said to be a *Laskerian T-module* if M is a finitely generated T-module and every proper submodule N of M is a finite intersection of primary submodules of M. T is said to be a *Laskerian ring* if T is Laskerian as a T-module [4].

In this paper, we generalize the concept of an accr pair by introducing the notion of an S-accr (resp., S-accr*) pair. For a pair of rings $R \subseteq T$ and S a multiplicative subset of R, (R,T) is called an *S-accr* (*resp., S*-accr*) pair if every ring A with $R \subseteq A \subseteq T$ satisfies S-accr (resp., *S-accr**). Note that (R,T) is an S-accr pair if and only if (R,T) is an S-accr* pair. We show that if $(R,R[X])$ is an S-accr pair, then $R \setminus Z(R) \subseteq S$ where S is a saturated multiplicative subset of R. In the particular case when S consists of units of R, we find the following well-known result. If $(R,R[X])$ is an accr pair, then every non zero-divisor of R must be a unit of R. Also, we prove that for (R,T) an S-accr pair, the following statements hold:

(1) For each proper ideal A of T disjointed with S, $(R/(A \cap R),\ T/A)$ is an $\overline{S}$-accr pair.
(2) Assume that S does not contain zero-divisors of T. Then $(S^{-1}R,\ S^{-1}T)$ is an accr pair.

Recall from [6], that a multiplicative subset S of R is called a *strongly-multiplicative set* if for each family $(s_\alpha)_{\alpha\in\Lambda}$ of elements of S we have $(\cap_{\alpha\in\Lambda} s_\alpha R) \cap S \neq \emptyset$. Also, according to [8,9], R is said to be *S-Artinian* if every descending chain of ideals $I_0 \supseteq I_1 \supseteq \cdots$ there exist $s \in S$ and $n \in \mathbb{N}$ such that for each $k \geq n$, $sI_n \subseteq I_k$. Let R be a commutative ring and S a strongly multiplicative set. We show that if R is an S-Artinian ring, then $(R,\ R[X])$ is an S-accr pair. We end this part by showing, under some hypothesis, that R is S-Artinian if and only if $(R,\ R[X])$ is an S-accr pair. Let $F_1 \subseteq F_2$ fields. Let X_1 be indeterminate over F_2. Let $R = F_1[X_1]$ (resp., $R_1 = F_1[[X_1]]$), $T = F_2[X_1]$ (resp., $T_1 = F_2[[X_1]]$) and S a multiplicative subset of F_1. We prove that the following assertions are equivalent:

(1) (R,T) is an S-accr pair.
(2) F_2 is an algebraic extension over F_1.
(3) (R,T) is an accr pair.

We end this work by giving an example of an S-accr pair which is not an accr pair.

2. Main results

Let R be a commutative ring, S be a multiplicative subset of R and M an R-module. Recall from [2] that an increasing sequence $(N_n)_{n\in\mathbb{N}}$ of submodules of M is called *S-stationary* if there exist a positive integer k and $s \in S$ such that for each $n \geq k$, $sN_n \subseteq N_k$. Also, we say that M satisfies *S-accr* (resp., *S-accr**) if any ascending chain of residuals of the form $(N : B) \subseteq (N : B^2) \subseteq (N : B^3) \subseteq \cdots$ is S-stationary where N is a submodule of M and B is a finitely generated (resp., principal) ideal of R. It was shown in [2], that the properties S-accr and S-accr* are equivalent.

Definition. Let $R \subseteq T$ be an extension of a commutative ring and $S \subseteq R$ a multiplicative subset. We say that (R, T) is an S-accr (resp., *S-accr**) pair if every ring A with $R \subseteq A \subseteq T$ satisfies S-accr (resp., *S-accr**).

Remark 2.1. Let $S \subseteq R$ be a multiplicative subset.

(1) (R, T) is an S-accr pair if and only if it is S-accr*.
(2) If S consists of units of R, then (R, T) is an S-accr pair if and only if it is an accr pair
(3) If (R, T) is an accr pair, then (R, T) is an S-accr pair. In Section 3, we give an example of an S-accr pair which is not an accr pair.

Let A be a commutative ring. We denote by $Z(A)$ the set of all zero-divisors of R.

Proposition 2.2. *Let R be a commutative ring and $S \subseteq R$ a saturated multiplicative set. If $(R, R[X])$ is an S-accr pair, then $R \setminus Z(R) \subseteq S$.*

Proof. Let $\alpha \in R \setminus Z(R)$. We will show that $\alpha \in S$. Let $T = R + (1+\alpha X)R[X]$. Note that $R \subseteq T \subseteq R[X]$ and T is a subring of $R[X]$. Now, $\alpha X = -1 + (1 + \alpha X) \in T$; so $(\alpha X)^n \in T$ for all $n \geq 1$. Consider the ascending sequence of ideals of T,

$$(1+\alpha X)T : \alpha \subseteq (1+\alpha X)T : \alpha^2 \subseteq \cdots .$$

Since $(R, R[X])$ is an S-accr pair, T satisfies S-accr; so there exist $m \geq 1$ and $s \in S$ such that $s((1+\alpha X)T : \alpha^h) \subseteq (1+\alpha X)T : \alpha^m$ for each $h \geq m$. Now, since $(\alpha X)^{m+1} \in T$ this implies that $(1+\alpha X)(\alpha X)^{m+1} \in (1+\alpha X)T$. Thus $(1+\alpha X)X^{m+1} \in (1+\alpha X)T : \alpha^{m+1}$; so $s((1+\alpha X)X^{m+1}) \in (1+\alpha X)T : \alpha^m$. Hence $s(1+\alpha X)X^{m+1}\alpha^m \in (1+\alpha X)T$. This implies that $sX^{m+1}\alpha^m \in T$, because $1+\alpha X$ is non zero-divisor.

Now, we have $s\alpha^{m-1}X^m = (1+\alpha X)s\alpha^{m-1}X^m - s\alpha^m X^{m+1} \in T$, because $(1+\alpha X)s\alpha^{m-1}X^m \in (1+\alpha X)R[X] \subseteq T$ and $s\alpha^m X^{m+1} \in T$. Proceeding like this one can show that $sX \in T$. Thus $sX = y + (1+\alpha X)P$ for some $P \in R[X]$ and $y \in R$. Since α is non zero-divisor, we find that $P \in R$. Comparing the coefficients of X in the tow parts, we obtain $s = \alpha a$ for some $a \in R$. Finally, since S is saturated, then $\alpha \in S$, and the proof is completed. ☐

In the particular case when S consists of units of R we find the following result.

Corollary 2.3 ([11, Proposition 1.3])**.** *Let R be a commutative ring. If $(R, R[X])$ is an accr pair, then every non zero-divisor of R must be a unit of R.*

Let R be a commutative ring and $S \subseteq R$ a multiplicative set. Our next results study the transfer of the S-accr pair property to the localization and the quotient ring. To prove this we need the following results.

Lemma 2.4 ([8, Example 3.1(3)]). *Let $S \subseteq R$ be a multiplicative subset of R. If R satisfies the S-accr condition, then $S^{-1}R$ satisfies accr.*

Lemma 2.5 ([8, Theorem 3.2]). *Let N be a submodule of an R-module M and let $S \subseteq R$ be a multiplicative subset. Then M satisfies S-accr if and only if N and M/N satisfy S-accr.*

Lemma 2.6. *Let R and T be commutative rings with identity, $f : R \to T$ a ring homomorphism and S a multiplicative subset of R such that $S \cap ker f = \emptyset$. If R satisfies the S-accr condition, then $f(R)$ satisfies $f(S)$-accr.*

Proof. Let I be an ideal of $f(R)$ and y be an element of $f(R)$. We will show that $(I : y) \subseteq (I : y^2) \subseteq \cdots$ is $f(S)$-stationary. We have $y = f(x)$ for some $x \in R$. It is easy to see that $(f^{-1}(I) : x) \subseteq (f^{-1}(I) : x^2) \subseteq \cdots$. Since R satisfies S-accr, there exist $s \in S$ and $n \in \mathbb{N}$ such that $s(f^{-1}(I) : x^k) \subseteq (f^{-1}(I) : x^n)$ for each $k \geq n$. Now, let $k \geq n$ and $b = f(a) \in (I : y^k)$. Since $by^k \in I$, then $f(ax^k) \in I$; so $ax^k \in f^{-1}(I)$. This implies that $sa \in (f^{-1}(I) : x^n)$. This equivalent to $f(s)by^n \in I$. Hence the sequence $(I : y) \subseteq (I : y^2) \subseteq \cdots$ is $f(S)$-stationary. □

Let R be a commutative ring, S a multiplicative subset of R and I an ideal of R disjointed with S. Let $\pi : R \to R/I$ be the canonical surjection. Then $\pi(S)$ is a multiplicative subset of R/I. The next result improves [11, Lemma 15].

Theorem 2.7. *Let S be a multiplicative subset of R and (R, T) an S-accr pair. Then the following statements hold.*

(1) *For each proper ideal A of T disjointed with S, $(R/(A \cap R), T/A)$ is an $\overline{S}$-accr pair.*

(2) *Assume that S does not contain zero-divisors of T. Then $(S^{-1}R, S^{-1}T)$ is an accr pair.*

Proof. (1) Let B be a commutative ring such that $R/(A \cap R) \subseteq B \subseteq T/A$. Let $\pi : T \to T/A$ be the canonical surjection. Since (R, T) is an S-accr pair and $R \subseteq \pi^{-1}(B) \subseteq T$, then $\pi^{-1}(B)$ satisfies S-accr. Hence by Lemma 2.6, $B = \pi(\pi^{-1}(B))$ satisfies $\overline{S}$-accr.

(2) Let B be a commutative ring such that $S^{-1}R \subseteq B \subseteq S^{-1}T$. Since R and T satisfy S-accr, then by Lemma 2.4, $S^{-1}R$ and $S^{-1}T$ satisfy accr. Let I be an ideal of B and $b \in B$. We show that $(I : b^k)_{k \in \mathbb{N}}$ is stationary. Since $b \in B \subseteq S^{-1}T$, $b = t/s$ for some $t \in T$ and $s \in S$. Then $t = \frac{t}{s}\frac{s}{1} \in B$ since $S^{-1}R \subseteq B$. Consider the ascending sequence of ideals of $B \cap T$,

$$(I \cap T : t) \subseteq (I \cap T : t^2) \subseteq \cdots .$$

We have $R \subseteq S^{-1}R \cap R \subseteq B \cap T \subseteq T$. Since (R, T) is an S-accr pair, then the sequence $(I \cap T : t^k)_{k \in \mathbb{N}}$ is S-stationary; so there exist $c \in S$ and $n \in \mathbb{N}$ such that for all $k \geq n$, $c(I \cap T : t^k) \subseteq (I \cap T : t^n)$. We will prove that $(I : b^{n+1}) = (I : b^n)$. Let $x = \frac{\alpha}{\beta} \in (I : b^{n+1})$, where $\alpha \in T$ and $\beta \in S$. Since $xb^{n+1} \in I$, there exist $\gamma \in T$ and $t' \in S$ such that $xb^{n+1} = \frac{\gamma}{t'} \in I$. So $\frac{\alpha}{\beta}\frac{t^{n+1}}{s^{n+1}} = \frac{\gamma}{t'}$. Thus

$t'\alpha t^{n+1} = \beta s^{n+1}\gamma$. Then $\alpha t^{n+1} = \frac{\alpha t^{n+1}}{1}\frac{t'}{t'} = \frac{\beta s^{n+1}\gamma}{t'} = \frac{\beta s^{n+1}}{1}\frac{\gamma}{t'} \in I$. This implies that $\alpha t^{n+1} \in I \cap T$; so $\alpha \in I \cap T : t^{n+1}$. Thus $c\alpha t^n \in I \cap T \subseteq I$. As $S^{-1}R \subseteq B$, $c\alpha t^n \frac{1}{c\beta s^n} \in I$ which implies that $xb^n \in I$. Hence B satisfies accr. □

Let R be a commutative ring and S a multiplicative subset of R. Recall from [8, 9] that R is said to be *S-Artinian* if for every descending chain of ideals $I_0 \supseteq I_1 \supseteq \cdots$ there exist $s \in S$ and $n \in \mathbb{N}$ such that for each $k \geq n$, $sI_n \subseteq I_k$. In [9], the authors showed that if R is an S-Artinian ring, then $S^{-1}R$ is an Artinian ring. Our next proposition gives another proof to this result. We also study when $S^{-1}R$ is Artinian implies that R is S-Artinian. First, we need to collect some necessary notions. For an ideal I of R, $\mathrm{Sat}_S(I)$ denotes the S-saturation of I, that is, $\mathrm{Sat}_S(I) = S^{-1}I \cap R$. A multiplicative set S of a ring R is called strongly *anti-Archimedean* if

$$\cap_{i\geq 1} s_i R \cap S \neq \emptyset$$

for every sequence $(s_i)_{i\geq 1} \in S$. Note that every strongly anti-Archimedean multiplicative set is anti-Archimedean. The converse is not true as was observed in [5, Example 2.7] and [10, Example 4.7]. Let M be an R-module. According to [3], the module M is called *S-finite* if $sM \subseteq F$ for some finitely generated submodule F of M and some $s \in S$. The module M is called *S-Noetherian* if each submodule of M is S-finite. A ring R is said to be *S-Noetherian* if it is S-Noetherian as an R-module.

Proposition 2.8. *Let R be a commutative ring and S a multiplicative subset of R.*

(1) *If R is an S-Artinian ring, then $S^{-1}R$ is an Artinian ring.*
(2) *Assume that R is an S-Noetherian ring, with S is strongly anti-Archimedean which does not contain zero-divisors. If $S^{-1}R$ is Artinian, then R is an S-Artinian ring.*

Proof. (1) Let $(I_k)_{k\in\mathbb{N}}$ be a descending chain of ideals of $S^{-1}R$. For each $k \in \mathbb{N}$, we can find an ideal J_k of R such that I_k is the localization of J_k. Consider the descending chain of ideals of R

$$J_1 \supseteq J_1 \cap J_2 \supseteq J_1 \cap J_2 \cap J_3 \supseteq \cdots .$$

Since R is S-Artinian, there exist $s \in S$ and $n \in \mathbb{N}^*$ such that for each $k \geq n$, $s(J_1 \cap J_2 \cap \cdots \cap J_n) \subseteq (J_1 \cap J_2 \cap \cdots \cap J_k) \subseteq J_k$. This implies that $S^{-1}(J_1 \cap J_2 \cap \cdots \cap J_n) = S^{-1}J_1 \cap \cdots \cap S^{-1}J_n = I_1 \cap \cdots \cap I_n \subseteq S^{-1}J_k = I_k$. Hence $S^{-1}R$ is Artinian.

(2) Let $I_0 \supseteq I_1 \supseteq \cdots$ be a descending chain of ideals of R. Then the sequence $(S^{-1}I_k)_k$ is a descending chain of ideals of $S^{-1}R$. Since $S^{-1}R$ is Artinian, there exists an $n \in \mathbb{N}$ such that for each $k \geq n$, $S^{-1}I_k = S^{-1}I_n$. This implies that for each $k \geq n$, $\mathrm{Sat}_S(I_k) = \mathrm{Sat}_S(I_n)$. Now, since R is S-Noetherian, then by [3, Proposition 2], for each $k \geq n$, there exists $s_k \in S$ such that $\mathrm{Sat}_S(I_k)$

$= I_k : s_k$; so for each $k \geq n$, $I_k : s_k = I_n : s_n$. Thus $s_k I_n \subseteq I_k$ for each $k \geq n$. Since S is a strongly anti-Archimedean set, then $\cap_{i\geq 1} s_i R \cap S \neq \emptyset$. Let $t \in \cap_{i\geq 1} s_i R \cap S$. Therefore for each $k \geq n$, $tI_n \subseteq s_k I_n \subseteq I_k$. Hence R is an S-Artinian ring. □

Note that every strongly-multiplicative set is strongly anti-Archimedean. This two notions coincide if S is at most countable and there are true when S is finite.

Theorem 2.9. *Let R be a commutative ring and S a strongly multiplicative set. If R is S-Artinian, then $(R, R[X])$ is an S-accr pair.*

Proof. Let $R \subseteq A \subseteq R[X]$ be a commutative ring. Let I be an ideal of A and $x \in A$. We will show that the sequence $I : x \subseteq I : x^2 \subseteq \cdots$ is S-stationary. Since R is S-Artinian, then by Proposition 2.8(1), $S^{-1}R$ is Artinian. So by [11, Theorem 1.1], $(S^{-1}R,\ S^{-1}R[X])$ is an accr pair. This implies that the sequence $S^{-1}I :_{S^{-1}A} \frac{x}{1} \subseteq S^{-1}I :_{S^{-1}A} (\frac{x}{1})^2 \subseteq \cdots$ of ideals of $S^{-1}A$ is stationary. Thus there exists $n \in \mathbb{N}$ such that for all $k \geq n$, $S^{-1}I :_{S^{-1}A} (\frac{x}{1})^k = S^{-1}I :_{S^{-1}A} (\frac{x}{1})^n$. It is easy to show that $S^{-1}I :_{S^{-1}A} (\frac{x}{1})^k = S^{-1}(I :_A x^k)$. Then for all $k \geq n$, $S^{-1}(I :_A x^k) = S^{-1}(I :_A x^n)$. Let $k \geq n$. It is easy to show that for each $\alpha \in (I :_A x^k)$, there exists an $s_\alpha \in S$ such that $s_\alpha \alpha \in (I :_A x^n)$. Now, since S is a strongly multiplicative set, then $(\cap_{\alpha\in(I:_A x^k)} s_\alpha R) \cap S \neq \emptyset$. Let $t \in (\cap_{\alpha\in(I:_A x^k)} s_\alpha R) \cap S$. It is easy to show that $t(I :_A x^k) \subseteq (I :_A x^n)$. Hence A satisfies S-accr. □

Proposition 2.10. *Let R be a commutative ring and S a strongly multiplicative set without zero-divisors. Assume that for all finitely generated ideal I of R, $Sat_S(I) = I : s$ for some $s \in S$. Then the following assertions are equivalent.*

(1) *$(R,\ R[X])$ is an S-accr pair.*
(2) *R is S-Artinian.*

Proof. (1) ⇒ (2). Assume that $(R, R[X])$ is an S-accr pair. By Theorem 2.7(2), $(S^{-1}R, S^{-1}R[X])$ is an accr pair. Then by [11, Theorem 1.1], $S^{-1}R$ is an Artinian ring. Thus $S^{-1}R$ is Noetherian. Now, by [3, Proposition 2(f)], R is an S-Noetherian ring; so by Proposition 2.8(2), R is S-Artinian.

(2) ⇒ (1). Follows from the previous Theorem 2.9. □

Let $F_1 \subseteq F_2$ be fields. Let $X_1, \ldots, X_n$ be indeterminates over F_2. Let $R = F_1[X_1, \ldots, X_n]$ (resp., $R_1 = F_1[[X_1, \ldots, X_n]]$) and $T = F_2[X_1, \ldots, X_n]$ (resp., $T_1 = F_2[[X_1, \ldots, X_n]]$).

The following theorem improves the result of [11, Proposition 3.1].

Theorem 2.11. *Let S be a multiplicative subset of F_1. If (R, T) is an S-accr pair, then F_2 is algebraic over F_1.*

Proof. Let $\alpha \in F_2$, $\alpha \neq 0$. Put $H = R[\alpha] + X_1 T$. Consider the ascending sequence of ideals of H, $X_1 H : \alpha \subseteq X_1 H : \alpha^2 \subseteq \cdots$. Note that $R \subseteq H \subseteq T$.

Since (R,T) is an S-accr pair, there exist $P \in S$, $m \in \mathbb{N}^*$ such that for all $k \geq m$, $P(X_1H : \alpha^k) \subseteq X_1H : \alpha^m$. Now, $\frac{X_1}{\alpha^{m+1}} \in H$ and $\frac{X_1}{\alpha^{m+1}}\alpha^{m+1} = X_1 \in X_1H$. Then $\frac{X_1}{\alpha^{m+1}} \in X_1H : \alpha^{m+1}$. This implies that $P\frac{X_1}{\alpha^{m+1}} \in X_1H : \alpha^m$; so $\frac{PX_1\alpha^m}{\alpha^{m+1}} \in X_1H$. Hence $\frac{P}{\alpha} \in H = R[\alpha] + X_1T$. Therefore, $\frac{P}{\alpha} = y + X_1t$ for some $y \in R[\alpha]$ and $t \in T$. Note that y can be expressed as $y = f_1(\alpha) + z$ for some $f_1(\alpha) \in F_1[\alpha]$ and $z \in (X_1, \dots, X_n)T$. Thus $\frac{P}{\alpha} = f_1(\alpha) + z + X_1t$. Hence $\frac{P}{\alpha} - f_1(\alpha) = z + X_1t$. Take $X_1 = \cdots = X_n = 0$, we obtain, $\frac{P(0,\dots,0)}{\alpha} - f_1(\alpha) = 0$. Now set $g(X) = Xf_1(X) - p(0,\dots,0) \in F_1[X]$ and $g(\alpha) = 0$. Hence F_2 is algebraic over F_1. □

Remark 2.12. Let S be a multiplicative subset of F_1. If (R_1, T_1) is an S-accr pair, then in a similar way one can show that F_2 is algebraic over F_1.

Corollary 2.13. *Let $n = 1$. Let S be a multiplicative subset of F_1. The following assertions are equivalent*:

(1) *(R,T) is an S-accr pair.*
(2) *F_2 is algebraic over F_1.*
(3) *(R,T) is an accr pair.*

Proof. (1) ⇒ (2). Theorem 2.11.
(2) ⇒ (3). Follows from [11, Proposition 3.4].
(3) ⇒ (1). Obvious. □

3. An example of an S-accr pair which is not an accr pair

In this section we give an example of an S-accr pair which is not an accr pair. To do it we need the following results.

Definition. Let $R \subseteq T$ be a ring extension and S a multiplicative subset of R. We call that (R,T) is an S-Noethrian pair if every ring A with $R \subseteq A \subseteq T$ is S-Noetherian.

Since every S-Noetherian ring satisfies the S-accr condition [2], then every S-Noetherian pair is an S-accr pair. In Example 3.4, we show that the reverse is not true in general.

Theorem 3.1. *Let $R \subseteq T$ be an integral domain and S a multiplicative subset of R. Then the following assertions are equivalent*:

(1) *(R,T) is an S-Noetherian pair.*
(2) *R is S-Noetherian and for all ring A such that $R \subseteq A \subseteq T$, A/I is an S-finite R-module for all I proper ideal of A.*

Proof. (2) ⇒ (1). Let $R \subseteq A \subseteq T$. For $I = (0)$, $A \simeq A/(0)$ which is an S-finite R-module by hypothesis. As R is S-Noetherian, by [1, Corollary 2.1] A is an S-Noetherian ring. Hence (R,T) is an S-Noetherian pair.

(1) ⇒ (2). Suppose That (R,T) is an S-Noetherian pair. Thus R is an S-Noetherian ring. Let $R \subseteq A \subseteq T$ and I an ideal of A.

First case: $S \cap I \neq \emptyset$. It is easy to show that A/I is an S-finite R-module.
Second case: S disjoint with I. Since $R \subseteq R+I \subseteq A \subseteq T$, then $R+I$ is an S-Noetherian ring. For all $a \in I \setminus (0)$, $aA \subseteq (R+I)$ and aA is an ideal of $R+I$. Since $R+I$ is S-Noetherian, there exist $s \in S, r_1+i_1, \ldots, r_n+i_n \in R+I$ such that $saA \subseteq (r_1+i_1)(R+I) + \cdots + (r_n+i_n)(R+I) \subseteq aA$. This implies that $sA \subseteq (\frac{r_1+i_1}{a})(R+I) + \cdots + (\frac{r_n+i_n}{a})(R+I)$. Thus A/I is $\overline{S}$-finite as an $(R+I)/I$-module where $\bar{S} = \{\bar{s},\ s \in S\}$. Moreover, $R+I/I$ is a cyclic R-module generated by $\bar{1}$. Therefore there exist $s \in S$, $x_1, \ldots, x_n \in A$ such that $\bar{s}A/I \subseteq (R+I)/I\bar{x}_1 + \cdots + (R+I)/I\bar{x}_n = R\bar{1}\bar{x}_1 + \cdots + R\bar{1}\bar{x}_n = R\bar{x}_1 + \cdots + R\bar{x}_n$. Hence A/I is an S-finite R-module. ☐

Corollary 3.2. *Let $R \subseteq T$ be an integral domain and S a multiplicative subset of R. Then the following assertions are equivalent*:

(1) $(R[[X]], T[[X]])$ *is an S-Noetherian pair.*
(2) $R[[X]]$ *is an S-Noetherian ring and T is an S-finite R-module.*

Proof. (1) $\Rightarrow$ (2). By Theorem 3.1, $R[[X]]$ is S-Noetherian and for $I = XT[[X]]$ ideal of $T[[X]]$, $T[[X]]/I \simeq T$ is an S-finite $R[[X]]$-module. This implies that T is an S-finite R-module.

(2) $\Rightarrow$ (1). Since T is an S-finite R-module, $T[[X]]$ is an S-finite $R[[X]]$-module. Then by [1, Proposition 2.1], $T[[X]]$ is S-Noetherian as $R[[X]]$-module. Let A be a ring such that $R[[X]] \subseteq A \subseteq T[[X]]$ and I an ideal of A. We show that I is S-finite. Since I is an $R[[X]]$-submodule of $T[[X]]$, there exist $s \in S$, $P_1, \ldots, P_n \in I$ such that $sI \subseteq P_1R[[X]] + \cdots + P_nR[[X]] \subseteq I$. This implies that $sI \subseteq P_1A + \cdots + P_nA \subseteq I$. Hence A is S-Noetherian. ☐

Remark 3.3. In the same way we can show $(R[X], T[X])$ is an S-Noetherian pair if and only if $R[X]$ is an S-Noetherian ring and T is an S-finite R-module.

Example 3.4. Let R be an anti-Archimedean domain which is not Noetherian. Take $S = R \setminus \{0\}$. Then S is an anti-Archimedean multiplicative subset of R. Let T be an S-finite R-module such that $R \subseteq T$ is an extension of an integral domain. Since R is not Noetherian, $R[[X]]$ is not an accr ring. Thus $(R[[X]], T[[X]])$ is not an accr pair. By [3, Corollary 11], $R[[X]]$ is an S-Noetherian ring. Moreover, T is an S-finite R-module. Then by Corollary 3.2, $(R[[X]], T[[X]])$ is an S-Noetherian pair. Hence $(R[[X]], T[[X]])$ is an S-accr pair.

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