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TIME-FREQUENCY ANALYSIS ASSOCIATED WITH K-HANKEL-WIGNER TRANSFORMS

Mohamed Amine Boubatra

Abstract. In this paper, we introduce the k-Hankel-Wigner transform on $\mathbb{R}$ in some problems of time-frequency analysis. As a first point, we present some harmonic analysis results such as Plancherel's, Parseval's and an inversion formulas for this transform. Next, we prove a Heisenberg's uncertainty principle and a Calderón's reproducing formula for this transform. We conclude this paper by studying an extremal function for this transform.

1. Introduction

Dunkl's theory is a far reaching generalization of Fourier analysis and special function theory related to root systems and where the Lebesgue measure is replaced by a weighted measure invariant under the refection group and parameterized by a multiplicity function k.

A deformation of Dunkl's theory has recently investigated by Ben Said et al. (see [5]) by a parameter $a > 0$ which arises from the "interpolation" of two different reductive Lie groups actions on the Weil representation of the metaplectic group $M_p(n; \mathbb{R})$ and the minimal unitary representation of the conformal group $O(n+1; 2)$ and a parameter k arises form Dunkl's theory of differential difference operators [11], namely the (k, a)-generalized Fourier transform $\mathcal{F}_{k,a}$. Various known integral transforms are covered by $\mathcal{F}_{k,a}$; the Fourier transform ($k = 0$ and $a = 2$), the Dunkl transform [12] ($k > 0$ and $a = 2$) and a new unitary operator [2] $\mathcal{F}_k = \mathcal{F}_{k,1}$ ($k > 0$ and $a = 1$) having a rich structure, as much as the Dunkl transform, which we call it the k-Hankel transform.

In the theory of harmonic analysis, time-frequency transforms and theirs properties such as orthogonal relations, inversion formulas and Heisenberg's uncertainty principle are of great interest in the last years.

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Fourier analysis has been applied to many other physical problems. One of the causes of the success of this formalism is that it constitutes a mathematical tool which describes in a fairly natural way many physical situations. As examples, thermal radiation, radio transmissions, x-rays and color rays of the visible spectrum.

However, even if no one will question the usefulness of the Fourier transform as well as its efficiency of implantation, in reality one meets many signals that the Fourier transform describes rather badly. These are in particular so-called non-stationary signals, the frequency of which depends on time. It was therefore necessary to develop new mathematical tools making it possible to process such signals and easily extract useful information.

The time-frequency resolution is associated with the Fourier-Wigner transform also known as Gabor transform, or the short-time Fourier transform. Recently, a considerable attention has been made to develop a new characterization of the uncertainty principle for the Fourier-Wigner transform, see for examples ([6, 10, 16]) and the references therein. The most famous of them is the sharp Heisenberg-type uncertainty inequality (see [6], Theorem 5.1). An analogue fundamental tool in time-frequency analysis in our paper is the k-Hankel-Wigner transform.

Several results of uncertainty principles have already been proved for the generalized Fourier transform $\mathcal{F}_{k,a}$, associated to a Dunkl-type operator by H. Mejjaoli in [20] and by Gorbachev et al. in [15] for the radially generalized Fourier transform $\mathcal{F}_{k,a}$. Motivated by the previous works such as the work of F. Soltani on which he studied the Wigner transform for the Dunkl transform on the real line (see, e.g; [25]). In our paper we will give a new approach of this transform by studying the Wigner transform for the k-Hankel transform denoted by $\mathcal{V}_g$. Note that the k-Hankel transform has a rich structure, as much as the Dunkl transform, and recently has been gaining a lot of attention (see, e.g., [1–5, 7–9, 15, 17, 20, 21]).

Let us define the measure

$$d\mu_k(t) = \frac{1}{2\Gamma(2k)}|t|^{2k-1}dt. \tag{1}$$

Let $g \in L^2(\mathbb{R}, \mu_k)$. The k-Hankel-Wigner transform $\mathcal{V}_g$ is the mapping defined for $f \in L^2(\mathbb{R}, \mu_k)$ by

$$\mathcal{V}_g(f)(x, y) = \int_{\mathbb{R}} f(t)\overline{\tau_x^k g_{k,y}(t)}d\mu_k(t),$$

where

$$g_{k,y}(z) = \mathcal{F}_k\left(\sqrt{\tau_y^k|\mathcal{F}_k(g)|^2}\right)(z).$$

This paper has been divided into four parts. In the second part of this paper, we provide some background materials associated with the k-Hankel transform. The third part deals with the study of the k-Hankel-Wigner transforms $\mathcal{V}_g$ for which we give a Heisenberg uncertainty principle and a Calderón's reproducing formula for this transform. The last part is reserved to study the extremal function of the problem

$$\inf_{f\in H^s(\mathbb{R},\mu_k)}\{\eta\|f\|^2_{H^s(\mathbb{R},\mu_k)}+\|h-\mathcal{V}_g(f)\|^2_{L^2(\mathbb{R}^2,\nu_k)}\}, \tag{2}$$

where $\nu_k=\mu_k\otimes\mu_k$.

For an unknown function f, here $h\in L^2(\mathbb{R}^2,\nu_k)$ is a given function and $\eta>0$, $s>k$, and $H^s(\mathbb{R},\mu_k)$ is the k-Sobolev-Hankel space of fractional order s. We show an analysis of the minimizer $f^*_{\eta,h}$ of problem (2). More precisely we characterize this minimizer by integral representations associated to the theory of the k-Hankel transform; and we study the convergence rates of these representations.

The problem (2) reduces to the Tikhonov regularization problem

$$\inf_{f\in H^s(\mathbb{R},\mu_k)}\{\|h-\mathcal{V}_g(f)\|^2_{L^2(\mathbb{R}^2,\nu_k)}\}\quad\text{when}\quad\eta\to 0.$$

2. Background for the k-Hankel transform

In this section, we provide some background materials associated with k-Hankel transform that we need thereafter.

For $k\geq\frac{1}{2}$ and $1\leq p\leq\infty$, let $L^p(\mathbb{R},\mu_k)$ to be the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{L^p(\mathbb{R},\mu_k)}=\left(\int_{\mathbb{R}}|f(x)|^pd\mu_k(x)\right)^{\frac{1}{p}}\quad\text{for}\quad 1\leq p<\infty,$$
$$\|f\|_{L^\infty(\mathbb{R},\mu_k)}=ess\sup_{x\in\mathbb{R}}|f(x)|<\infty,$$

where $d\mu_k$ is the measure defined by (1).

For $p=2$, we provide this space with the scalar product

$$\langle f,g\rangle_{L^2(\mathbb{R},\mu_k)}:=\int_{\mathbb{R}}f(x)\overline{g(x)}d\mu_k(x).$$

The k-Hankel transform of $f\in L^1(\mathbb{R},\mu_k)$ is defined by

$$\mathcal{F}_k(f)(\lambda)=\int_{\mathbb{R}}f(x)B_k(\lambda,x)d\mu_k(x),\quad\lambda\in\mathbb{R}, \tag{3}$$

where $B_k(\lambda,x)$ is the k-Hankel kernel given by

$$B_k(\lambda,x)=\jmath_{2k-1}(2\sqrt{|\lambda x|})-\frac{\lambda x}{2k(2k+1)}\jmath_{2k+1}(2\sqrt{|\lambda x|}).$$

Here

$$\jmath_\alpha(u)=\Gamma(\alpha+1)\left(\frac{u}{2}\right)^{-\alpha}J_\alpha(u)=\Gamma(\alpha+1)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(\alpha+m+1)}\left(\frac{u}{2}\right)^{2m},$$

denotes the normalized Bessel function of index α.

The definition (3) makes sense as

$$|B_k(x,y)|\leq 1. \tag{4}$$

For all $x, y \in \mathbb{R}$, and $f \in L^1(\mathbb{R}, \mu_k)$,

$$\|\mathcal{F}_k f\|_{L^\infty(\mathbb{R},\mu_k)} \leq \|f\|_{L^1(\mathbb{R},\mu_k)}.$$

Theorem 2.1 (See [5])**.** *Assume that* $k \geq \frac{1}{2}$.

(i) (*Plancherel's theorem*) $\mathcal{F}_k$ *is an isometric isomorphism on* $L^2(\mathbb{R}, \mu_k)$

$$\int_{\mathbb{R}} |f(x)|^2 d\mu_k(x) = \int_{\mathbb{R}} |\mathcal{F}_k(f)(\lambda)|^2 d\mu_k(\lambda).$$

(ii) (*Parseval's formula*) *For all* $f, g \in L^2(\mathbb{R}, \mu_k)$, *we have*

$$\int_{\mathbb{R}} f(x)\overline{g(x)} d\mu_k(x) = \int_{\mathbb{R}} \mathcal{F}_k(f)(\lambda)\overline{\mathcal{F}_k(g)(\lambda)} d\mu_k(\lambda).$$

(iii) (*Inversion formula*) $\mathcal{F}_k$ *satisfies*

$$\mathcal{F}_k^{-1} = \mathcal{F}_k.$$

For $f \in L^2(\mathbb{R}, \mu_k)$ and $x, y \in \mathbb{R}$ we have

$$\mathcal{F}_k(\tau_x^k f)(y) = B_k(x, y)\mathcal{F}_k(f)(y). \tag{5}$$

By means of k-Hankel transform we can define the translation operator on the space $L^p(\mathbb{R}, \mu_k), 1 \leq p \leq \infty$, by the following theorem.

Theorem 2.2 (See [2])**.** *For* λ, x, $y \in \mathbb{R}$, *we have*

$$\tau_x^k f(y) = \int_{\mathbb{R}} f(z)\, d\sigma_{x,y}^k(z),$$

where

$$d\sigma_{x,y}^k(z) = \begin{cases} \mathcal{K}_k(x, y, z) d\mu_k(z) & \text{if} \quad xy \neq 0, \\ d\delta_x(z) & \text{if} \quad y = 0, \\ d\delta_y(z) & \text{if} \quad x = 0. \end{cases}$$

For more details about the generalized translation operator $\tau_{(.)}^k$ we refer the reader to [2].

We recall some properties that concern the translation operator in the following proposition, (see [2]).

Let $L_e^p(\mathbb{R}, \mu_k)$ to be the space of even functions in $L^p(\mathbb{R}, \mu_k)$.

Proposition 2.3.

(i) *For all nonnegative function* $f \in L_e^1(\mathbb{R}, \mu_k)$, *for all* $x \in \mathbb{R}$ *we have*

$$\tau_x^k f \geq 0, \quad \tau_x^k f \in L^1(\mathbb{R}, \mu_k)$$

and

$$\int_{\mathbb{R}} \tau_x^k f(y) d\mu_k(y) = \int_{\mathbb{R}} f(y) d\mu_k(y). \tag{6}$$

(ii) *For all* f *in* $L_e^p(\mathbb{R}, \mu_k)$, $1 \leq p \leq \infty$, *there exists a positive constant* A_k *such that*

$$\|\tau_x^k f\|_{L^p(\mathbb{R},\mu_k)} \leq A_k \|f\|_{L^p(\mathbb{R},\mu_k)} \quad \text{for all} \quad x \in \mathbb{R}. \tag{7}$$

According to the translation operator, we can define the convolution product $*_k$ as (see [2]):

Definition. For $f, g \in L^1(\mathbb{R}, \mu_k)$, we define the generalized convolution product $*_k$, by

$$f *_k g(x) = \int_{\mathbb{R}} f(y)\tau_x^k g(y) d\mu_k(y), \quad x \in \mathbb{R}.$$

In particular

$$f *_k g = g *_k f \quad \text{and} \quad (f *_k g) *_k h = f *_k (g *_k h).$$

Proposition 2.4 ([2]). *The following statements hold true.*

(i) *Let $f \in L^2(\mathbb{R}, \mu_k)$ and $g \in L^1(\mathbb{R}, \mu_k)$. Then the relation convolution $f *_k g$ is defined almost everywhere on $\mathbb{R}$ by*

$$f *_k g(y) = \int_{\mathbb{R}} f(x)\tau_y^k g(x) d\mu_k(x), \quad y \in \mathbb{R}.$$

Moreover, the function $f *_k g$ belongs to $L^2(\mathbb{R}, \mu_k)$.

(ii) *(Young's inequality) For p, q, r such that $1 \le p, q, r \le \infty$ and $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$, and for $f \in L^p(\mathbb{R}, \mu_k)$ and $g \in L^q(\mathbb{R}, \mu_k)$, the convolution product $f *_k g$ is a well defined element in $L^r(\mathbb{R}, \mu_k)$ and*

$$\|f *_k g\|_{L^r(\mathbb{R},\mu_k)} \le A_k \|f\|_{L^p(\mathbb{R},\mu_k)} \|g\|_{L^q(\mathbb{R},\mu_k)},$$

where A_k is the same constant as in (ii) *of Proposition 2.3.*

(iii) *For $f \in L^2(\mathbb{R}, \mu_k)$ and $g \in L^1(\mathbb{R}, \mu_k)$, we have*

$$\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(g).$$

As a consequence of the Plancherel theorem and the last equality is the following relation

$$\int_{\mathbb{R}} |f *_k g(x)|^2 d\mu_k(x) = \int_{\mathbb{R}} |\mathcal{F}_k f(\lambda)|^2 |\mathcal{F}_k g(\lambda)|^2 d\mu_k(\lambda), \quad x \in \mathbb{R}, \tag{8}$$

where both sides are finite.

3. k-Hankel-Wigner transforms

In this section, we define and study the k-Hankel-Wigner transforms, then we give a Plancherel's and an inversion formulas for it. Moreover, we study a Heisenberg's uncertainty principle and a Calderón's reproducing formula associated with $\mathcal{V}_g$.

Let $g \in L^2(\mathbb{R}, \mu_k)$ and $y \in \mathbb{R}$. The modulation of g by y is the function $g_{k,y}$ defined by

$$g_{k,y}(z) = \mathcal{F}_k\left(\sqrt{\tau_y^k |\mathcal{F}_k(g)|^2}\right)(z), \qquad z \in \mathbb{R}. \tag{9}$$

Thus,

$$\|g_{k,y}\|_{L^2(\mathbb{R},\mu_k)} = \|g\|_{L^2(\mathbb{R},\mu_k)}. \tag{10}$$

Furthermore, we have

$$\overline{\mathcal{F}_k(g_{k,y})(z)} = \mathcal{F}_k\overline{(g_{k,y})}(z) = \sqrt{\tau_y^k|\mathcal{F}_k(g)|^2(z)}. \tag{11}$$

For $x, y \in \mathbb{R}$, we consider the family $m_{k,y,x}$ defined by

$$m_{k,y,x}(t) = \tau_x^k g_{k,y}(t), \quad t \in \mathbb{R}.$$

By using relations (7) and (10) we obtain

$$\|m_{k,y,x}\|_{L^2(\mathbb{R},\mu_k)} \leq A_k\|g\|_{L^2(\mathbb{R},\mu_k)}. \tag{12}$$

Let $g \in L^2(\mathbb{R}, \mu_k)$. The k-Hankel-Wigner transform denoted by $\mathcal{V}_g$ is defined for $f \in L^2(\mathbb{R}, \mu_k)$ and $x, y \in \mathbb{R}$ by

$$\mathcal{V}_g(f)(x, y) = \int_{\mathbb{R}} f(t)\overline{m_{k,y,x}(t)}d\mu_k(t). \tag{13}$$

(13) can be written as

$$\mathcal{V}_g(f)(x, y) = \overline{g_{k,y}} *_k f(x), \tag{14}$$

where $g_{k,y}$ is the function given by (9).

Proposition 3.1. *Let $(f, g) \in L^2(\mathbb{R}, \mu_k) \times L^2(\mathbb{R}, \mu_k)$. Then we have the followings.*

(i) $\mathcal{V}_g(f)(x, y) = \int_{\mathbb{R}} B_k(x, z)\mathcal{F}_k(f)(z)\sqrt{\tau_y^k|\mathcal{F}_k(g)|^2(z)}d\mu_k(z).$

(ii) *The function $\mathcal{V}_g(f)$ belongs to $L^\infty(\mathbb{R}^2, \nu_k)$, and*

$$\|\mathcal{V}_g(f)\|_{L^\infty(\mathbb{R}^2,\nu_k)} \leq A_k\|f\|_{L^2(\mathbb{R},\mu_k)}\|g\|_{L^2(\mathbb{R},\mu_k)}.$$

Proof. (i) By (ii) of Theorem 2.1 and relation (5) we have

$$\mathcal{V}_g(f)(x, y) = \int_{\mathbb{R}} B_k(x, z)\mathcal{F}_k(f)(z)\overline{\mathcal{F}_k(g_{k,y})(z)}d\mu_k(z),$$

then, by (11) we infer the result.

The assertion (ii) follows immediately from Hölder's inequality, and relations (13) and (12), which furnishes the proposition. □

Theorem 3.2. *Let $g \in L^2(\mathbb{R}, \mu_k)$.*

(i) (*Plancherel's formula*): *For every $f \in L^2(\mathbb{R}, \mu_k)$, we have*

$$\|\mathcal{V}_g(f)\|_{L^2(\mathbb{R}^2,\nu_k)} = \|g\|_{L^2(\mathbb{R},\mu_k)}\|f\|_{L^2(\mathbb{R},\mu_k)}.$$

(ii) (*Parseval's formula*): *For every $f, h \in L^2(\mathbb{R}, \mu_k)$, we have*

$$\int_{\mathbb{R}^2} \mathcal{V}_g(f)(x, y)\overline{\mathcal{V}_g(h)(x, y)}d\nu_k(x, y) = \|g\|^2_{L^2(\mathbb{R},\mu_k)} \int_{\mathbb{R}} f(x)\overline{h(x)}d\mu_k(x).$$

(iii) (*Inversion formula*): *For all $f \in L^1 \cap L^2(\mathbb{R}, \mu_k)$ such that $\mathcal{F}_k(f) \in L^1(\mathbb{R}, \mu_k)$, we have*

$$f(z) = \frac{1}{\|g\|^2_{L^2(\mathbb{R},\mu_k)}} \int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{V}_g(f)(x, y)\overline{m_{k,y,z}(x)}d\mu_k(x)d\mu_k(y).$$

Proof. (i) By (i) of Theorem 2.1, relations (6), (8) and (14), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} |\mathcal{V}_g(f)(x,y)|^2 d\mu_k(x)d\mu_k(y)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |\overline{g_{k,y}} *_k f(x)|^2 d\mu_k(x)d\mu_k(y)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |\mathcal{F}_k(\overline{g_{k,y}})(z)|^2 |\mathcal{F}_k(f)(z)|^2 d\mu_k(z)d\mu_k(y)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \tau_y^k |\mathcal{F}_k(g)|^2(z) |\mathcal{F}_k(f)(z)|^2 d\mu_k(z)d\mu_k(y)\\
&= \|g\|^2_{L^2(\mathbb{R},\mu_k)} \int_{\mathbb{R}} |\mathcal{F}_k(f)(z)|^2 d\mu_k(z).
\end{aligned}$$

(ii) According the assertions (i) and (ii) of Theorem 2.1 we obtain,

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{V}_g(f)(x,y)\overline{\mathcal{V}_g(h)(x,y)} d\mu_k(x)d\mu_k(y)\\
&= \|g\|^2_{L^2(\mathbb{R},\mu_k)} \int_{\mathbb{R}} \mathcal{F}_k(f)(z)\overline{\mathcal{F}_k(h)(z)} d\mu_k(z)\\
&= \|g\|^2_{L^2(\mathbb{R},\mu_k)} \int_{\mathbb{R}} f(x)\overline{h(x)} d\mu_k(x).
\end{aligned}$$

(iii) By relation (8), and (iii) of Proposition 2.4, one can assert that

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{V}_g(f)(x,y)\overline{m_{k,y,z}(x)} d\mu_k(x)d\mu_k(y)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \tau_y^k |\mathcal{F}_k(g)|^2(t)\mathcal{F}_k(f)(t)B_k(z,t)d\mu_k(t)d\mu_k(y).
\end{aligned}$$

Then, by Fubini's theorem, (i) of Theorem 2.1 and relation (6), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{V}_g(f)(x,y)\overline{m_{k,y,z}(x)} d\mu_k(x)d\mu_k(y)\\
&= \|g\|^2_{L^2(\mathbb{R},\mu_k)} \int_{\mathbb{R}} \mathcal{F}_k(f)(t)B_k(z,t)d\mu_k(t)\\
&= \|g\|^2_{L^2(\mathbb{R},\mu_k)} f(z),
\end{aligned}$$

which furnishes the theorem. □

Remark. For $T>0$, we define the dilation operator of $f \in L^2(\mathbb{R},\mu_k)$ by

$$f_T(x) = T^{-2k} f\left(\frac{x}{T}\right), \quad x \in \mathbb{R}.$$

Then

$$\begin{aligned}
&\mathcal{F}_k(f_T)(z) = T^{2k}\mathcal{F}_k(f)(Tz),\\
&\tau_x^k(f_T)(y) = T^{-2k}\tau^k_{\frac{x}{T}} f\left(\frac{y}{T}\right).
\end{aligned} \tag{15}$$

Lemma 3.3. *Let $T > 0$ and let $g \in L^2(\mathbb{R}, \mu_k)$, $g \neq 0$. Then, for $f \in L^2(\mathbb{R}, \mu_k)$, one has*

$$\mathcal{V}_{g_T}(f_T)(x, y) = \mathcal{V}_g(f)\left(\frac{x}{T^4}, Ty\right), \quad x, y \in \mathbb{R}.$$

Proof. From (i) of Proposition 3.1, we have

$$\mathcal{V}_{g_T}(f_T)(x, y) = \int_{\mathbb{R}} B(x, z)\mathcal{F}_k(f_T)(z)\sqrt{\tau_y^k |\mathcal{F}_k(g_T)|^2(z)}d\mu_k(z).$$

Using (15), we get

$$\tau_y^k |\mathcal{F}_k(g_T)|^2(z) = T^{4k}\tau_{T_y}^k |\mathcal{F}_k(g)|^2(Tz).$$

Then, by a change of variable we obtain

$$\begin{aligned}\mathcal{V}_{g_T}(f_T)(x, y) &= T^{4k}\int_{\mathbb{R}} B_k(x, z)\mathcal{F}_k(f)(Tz)\sqrt{\tau_{Ty}^k |\mathcal{F}_k(g)|^2(Tz)}d\mu_k(z)\\ &= \int_{\mathbb{R}} B_k(x, \frac{z}{T^4})\mathcal{F}_k(f)(z)\sqrt{\tau_{Ty}^k |\mathcal{F}_k(g)|^2(z)}d\mu_k(z)\\ &= \mathcal{V}_g(f)\left(\frac{x}{T^4}, Ty\right),\end{aligned}$$

which furnishes the lemma. □

Let us recall the Heisenberg uncertainty principle for the k-Hankel transform.

Proposition 3.4 (see [5, 13, 14])**.** *For $s, a > 0$, there exists a positive constant $c(s, a)$ such that for every $f \in L^2(\mathbb{R}, \mu_k)$, the following inequality holds*

$$\| |\xi|^s \mathcal{F}_k(f)(\xi)\|_{L^2(\mathbb{R},\mu_k)}^{\frac{a}{s+a}} \| |x|^a f(x)\|_{L^2(\mathbb{R},\mu_k)}^{\frac{s}{s+a}} \geq c(s, a)\|f\|_{L^2(\mathbb{R},\mu_k)}.$$

Theorem 3.5 (Heisenberg-type uncertainty principle for $\mathcal{V}_g$)**.** *Let $s, a > 0$. Then there exists a constant $c(s, a) > 0$ such that, for all $f \in L^2(\mathbb{R}, \mu_k)$, $g \in L^2(\mathbb{R}, \mu_k)$, we have*

$$\begin{aligned}&\left(\int_{\mathbb{R}^2} |x|^{2a}|\mathcal{V}_g(f)(x, y)|^2 d\nu_k(x, y)\right)^{\frac{s}{s+a}} \left(\int_{\mathbb{R}} |\xi|^{2s}|\mathcal{F}_k(f)(\xi)|^2 d\mu_k(\xi)\right)^{\frac{a}{s+a}}\\ &\geq c^2(s, a)\|f\|^2_{L^2(\mathbb{R},\mu_k)}\|g\|^{\frac{2s}{s+a}}_{L^2(\mathbb{R},\mu_k)}.\end{aligned} \tag{16}$$

The constant $c(s, a)$ is the same constant as in Proposition 3.4.

Proof. We consider the non-trivial case where both integrals on the left hand side of (16) are finite. Fix y arbitrary, Proposition 3.4 gives

$$\begin{aligned}&\left(\int_{\mathbb{R}} |\xi|^{2s}|\mathcal{F}_k(\mathcal{V}_g(f)(\cdot, y))(\xi)|^2 d\mu_k(\xi)\right)^{\frac{s}{s+a}} \left(\int_{\mathbb{R}} |x|^{2a}|\mathcal{V}_g(f)(x, y)|^2 d\mu_k(\xi)\right)^{\frac{s}{s+a}}\\ &\geq c^2(s, a)\int_{\mathbb{R}} |\mathcal{V}_g(f)(x, y)|^2 d\mu_k(\xi).\end{aligned}$$

We integrate both sides over y with respect to the measure $d\mu_k(y)$, and using Hölder's inequality, we obtain

$$\left(\int_{\mathbb{R}^2}|\xi|^{2s}|\mathcal{F}_k(\mathcal{V}_g(f)(\cdot,y))(\xi)|^2 d\nu_k(\xi,y)\right)^{\frac{s}{s+a}}\left(\int_{\mathbb{R}^2}|x|^{2a}|\mathcal{V}_g(f)(x,y)|^2 d\nu_k(x,y)\right)^{\frac{s}{s+a}}$$
$$\geq c^2(s,a)\int_{\mathbb{R}^2}|\mathcal{V}_g(f)(x,y)|^2 d\nu_k(x,y).$$

Moreover, by using the fact that

$$\int_{\mathbb{R}^2}|\xi|^{2s}|\mathcal{F}_k(\mathcal{V}_g(f)(\cdot,y))(\xi)|^2 d\nu_k(\xi,y)=\|g\|^2_{L^2(\mathbb{R},\mu_k)}\int_{\mathbb{R}}|\xi|^{2s}|\mathcal{F}_k(f)(\xi)|^2 d\mu_k(\xi),$$

we deduce

$$\|g\|^{\frac{2a}{s+a}}_{L^2(\mathbb{R},\mu_k)}\left(\int_{\mathbb{R}}|\xi|^{2s}|\mathcal{F}_k(f)(\xi)|^2 d\mu_k(\xi)\right)^{\frac{a}{s+a}}\left(\int_{\mathbb{R}^2}|x|^{2a}|\mathcal{V}_g(f)(x,y)|^2 d\nu_k(x,y)\right)^{\frac{s}{s+a}}$$
$$\geq c^2(s,a)\int_{\mathbb{R}^2}|\mathcal{V}_g(f)(x,y)|^2 d\nu_k(x,y)$$
$$=c^2(s,a)\|f\|^2_{L^2(\mathbb{R},\mu_k)}\|g\|^2_{L^2(\mathbb{R},\mu_k)},$$

which furnishes the proof of the theorem. □

Now, we're in position to give our main result of this section which is represented by the following reproducing formula of Calderón's type for $\mathcal{V}_g$.

Theorem 3.6 (Calderón's reproducing formula for $\mathcal{V}_g$)**.** *Let $a,b\in\mathbb{R}$ such that $a<b$ and let $g\in L^2(\mathbb{R},\mu_k)$, $g\neq 0$ such that $\mathcal{F}_k(g)\in L^\infty(\mathbb{R},\mu_k)$. Then, for $f\in L^2(\mathbb{R},\mu_k)$, the following function*

$$f_{a,b}(z)=\frac{1}{\|g\|_{L^2(\mathbb{R},\mu_k)}}\int_a^b\int_{\mathbb{R}}\mathcal{V}_g(f)(x,y)\overline{m_{k,y,z}(x)}d\mu_k(x)d\mu_k(y)$$

belongs to $L^2(\mathbb{R},\mu_k)$ and satisfies

$$\lim_{\substack{a\to-\infty\\ b\to+\infty}}\|f_{a,b}-f\|_{L^2(\mathbb{R},\mu_k)}=0. \tag{17}$$

Proof. According to (i) of Theorem 2.1, relations (8) and (14), we have

$$f_{a,b}(z)=\frac{1}{\|g\|^2_{L^2(\mathbb{R},\mu_k)}}\int_{a,b}\int_{\mathbb{R}}\tau^k_y|\mathcal{F}_k(g)|^2(t)\mathcal{F}_k(f)(t)B_k(z,t)d\mu_k(t)d\mu_k(y).$$

By Fubini's theorem, we get

$$f_{a,b}(z)=\int_{\mathbb{R}}K_{a,b}(t)\mathcal{F}_k(f)(t)B_k(z,t)d\mu_k(t), \tag{18}$$

where

$$K_{a,b}(t)=\frac{1}{\|g\|^2_{L^2(\mathbb{R},\mu_k)}}\int_a^b\tau^k_y|\mathcal{F}_k(g)|^2(t)d\mu_k(y).$$

By using relation (7), we can assert that $\|K_{a,b}\|_{L^\infty(\mathbb{R},\mu_k)} \leq A_k$. On the other hand, by Hölder's inequality, we deduce that

$$|K_{a,b}(t)|^2 \leq \frac{\mu_k(a,b)}{\|g\|^4_{L^2(\mathbb{R},\mu_k)}} \int_a^b |\tau_y^k|\mathcal{F}_k(g)|^2(t)|^2 d\mu_k(y).$$

Hence, by relation (7) we find

$$\begin{aligned}\|K_{a,b}\|^2_{L^2(\mathbb{R},\mu_k)} &\leq \frac{(\mu_k(a,b))^2 A_k^2}{\|g\|^4_{L^2(\mathbb{R},\mu_k)}} \int_{\mathbb{R}} |\mathcal{F}_k(g)(t)|^4 d\mu_k(t) \\ &\leq \frac{(\mu_k(a,b))^2 A_k^2 \|\mathcal{F}_k(g)\|^2_{L^\infty(\mathbb{R},\mu_k)}}{\|g\|^2_{L^2(\mathbb{R},\mu_k)}}.\end{aligned}$$

Thus $K_{a,b} \in L^\infty \cap L^2(\mathbb{R},\mu_k)$.

And by (18), we obtain

$$\mathcal{F}_k(f_{a,b})(t) = K_{a,b}(t)\mathcal{F}_k(f)(t).$$

According the last equality and (i) of Theorem 2.1, it follows that $f_{a,b} \in L^2(\mathbb{R},\mu_k)$ and

$$\|f_{a,b} - f\|^2_{L^2(\mathbb{R},\mu_k)} = \int_{\mathbb{R}} |\mathcal{F}_k(f)(t)|^2 (1 - K_{a,b}(t))^2 d\mu_k(t).$$

By relation (6), we have

$$\lim_{\substack{a\to-\infty \\ b\to+\infty}} K_{a,b}(t) = 1 \quad \text{for all } t \in \mathbb{R},$$

and

$$|\mathcal{F}_k(f)(t)|^2 (1 - K_{a,b}(t))^2 \leq (A_k + 1)^2 |\mathcal{F}_k(f)(t)|^2 \ \text{ for all } t \in \mathbb{R}.$$

The dominated convergence theorem ensure the relation (17), which furnishes the theorem. □

4. Extremal functions for the k-Hankel-Wigner transform

In this section, by using the theory associated with the k-Hankel transform and the ideas of Saitoh ([22–24]), we study the extremal function of the problem (2).

Let $s \geq 0$. We define the k-Sobolev-Hankel space of order s, that will be denoted $H^s(\mathbb{R},\mu_k)$, as the set of all $f \in L^2(\mathbb{R},\mu_k)$ such that $(1+|z|)^{s/2}\mathcal{F}_k(f) \in L^2(\mathbb{R},\mu_k)$. The space $H^s(\mathbb{R},\mu_k)$ equipped with the inner product on the space $H^s(\mathbb{R},\mu_k)$

$$\langle f, g\rangle_{H^s(\mathbb{R},\mu_k)} = \int_{\mathbb{R}} (1+|z|)^s \mathcal{F}_k(f)(z)\overline{\mathcal{F}_k(g)(z)} d\mu_k(z),$$

and the norm

$$\|f\|_{H^s(\mathbb{R},\mu_k)} = \left[\int_{\mathbb{R}} (1+|z|)^s |\mathcal{F}_k(f)(z)|^2 d\mu_k(z)\right]^{1/2}.$$

Let $\eta > 0$. The inner product $\langle \cdot, \cdot \rangle_{\eta, H^s(\mathbb{R},\mu_k)}$ on the space $H^s(\mathbb{R}, \mu_k)$ is defined by

$$\langle f, h\rangle_{\eta, H^s(\mathbb{R},\mu_k)} = \eta\langle f, h\rangle_{H^s(\mathbb{R},\mu_k)} + \langle \mathcal{V}_g(f), \mathcal{V}_g(h)\rangle_{L^2(\mathbb{R}^2,\nu_k)},$$

and the norm is defined by $\|f\|_{\eta, H^s(\mathbb{R},\mu_k)} = \sqrt{\langle f, f\rangle_{\eta, H^s(\mathbb{R},\mu_k)}}$.

Next, we suppose that $g \in L^2(\mathbb{R}, \mu_k)$. By (ii) of Theorem 3.2, the inner product $\langle \cdot, \cdot \rangle_{\eta, H^s(\mathbb{R},\mu_k)}$ can be expressed as

$$\langle f, h\rangle_{\eta, H^s(\mathbb{R},\mu_k)} = \eta\langle f, h\rangle_{H^s(\mathbb{R},\mu_k)} + \|g\|^2_{L^2(\mathbb{R},\mu_k)}\langle f, h\rangle_{L^2(\mathbb{R},\mu_k)}. \tag{19}$$

Theorem 4.1. *Let $\eta > 0$ and $s > k$ and let $g \in L^2(\mathbb{R}, \mu_k)$. The space $H^s(\mathbb{R}, \mu_k)$ equipped with the norm $\|\cdot\|_{\eta, H^s(\mathbb{R},\mu_k)}$ has the reproducing kernel*

$$K_{\eta,g}(x, y) = \int_{\mathbb{R}} \frac{B_k(x, z)B_k(y, z)d\mu_k(z)}{\eta(1 + |z|)^s + \|g\|^2_{L^2(\mathbb{R},\mu_k)}}, \tag{20}$$

that is

(i) *For all $y \in \mathbb{R}$, the function $x \mapsto K_s(x, y)$ belongs to $H^s(\mathbb{R}, \mu_k)$.*

(ii) *The reproducing property: for all $f \in H^s(\mathbb{R}, \mu_k)$ and $y \in \mathbb{R}$,*

$$\langle f, K_{\eta,g}(\cdot, y)\rangle_{\eta, H^s(\mathbb{R},\mu_k)} = f(y).$$

Proof. (i) By (4) the function $\Phi_y : z \mapsto \frac{B_k(y,z)}{\eta(1+|z|)^s+\|g\|^2_{L^2(\mathbb{R},\mu_k)}}$ belongs to $L^1 \cap L^2(\mathbb{R}, \mu_k)$. Then, the function $K_{\eta,g}$ is well defined and by (iii) of Theorem 2.1, we have

$$K_{\eta,g}(x, y) = \mathcal{F}_k^{-1}(\Phi_y)(x), \quad x \in \mathbb{R}.$$

From (i) of Theorem 2.1, it follows that $K_{\eta,g}(\cdot, y) \in L^2(\mathbb{R}, \mu_k)$, and we have

$$\mathcal{F}_k(K_{\eta,g}(\cdot, y))(z) = \frac{B_k(y, z)}{\eta(1 + |z|)^s + \|g\|^2_{L^2(\mathbb{R},\mu_k)}}, \quad z \in \mathbb{R}. \tag{21}$$

Then by (4), we obtain

$$|\mathcal{F}_k(K_{\eta,g}(\cdot, y))(z)| \leq \frac{1}{\eta(1 + |z|)^s}, \quad z \in \mathbb{R},$$

and

$$\|K_{\eta,g}(\cdot, y)\|^2_{H^s(\mathbb{R},\mu_k)} \leq \frac{1}{\eta^2}\int_{\mathbb{R}} \frac{d\mu_k(z)}{(1 + |z|)^s} < \infty.$$

Hence, the function $K_{\eta,s}(\cdot, y)$ belongs to $H^s(\mathbb{R}, \mu_k)$ for all $y \in \mathbb{R}$.

(ii) Let $f \in H^s(\mathbb{R}, \mu_k)$ and $y \in \mathbb{R}$. From (19) and (21), we have

$$\begin{aligned}\langle f, K_{\eta,g}(\cdot, y)\rangle_{\eta, H^s(\mathbb{R},\mu_k)} &= \int_{\mathbb{R}} \mathcal{F}_k(f)(z)B_k(y, z)d\mu_k(z)\\ &= f(y).\end{aligned}$$

The last equality is called the reproducing property, which furnishes the proof of the theorem. □

Theorem 4.2. *Let $s > k$ and $g \in L^2(\mathbb{R}, \mu_k)$. For any $h \in L^2(\mathbb{R}^2, \nu_k)$ and for any $\eta > 0$, the problem*

$$\inf_{f\in H^s(\mathbb{R},\mu_k)} \{\eta\|f\|^2_{H^s(\mathbb{R},\mu_k)} + \|h - \mathcal{V}_g(f)\|^2_{L^2(\mathbb{R}^2,\nu_k)}\} \tag{22}$$

has a unique extremal function $f^*_{\eta,h}$ given by

$$f^*_{\eta,h}(y) = \int_{\mathbb{R}}\int_{\mathbb{R}} h(x,t)P_s(x,y,t)d\mu_k(t)d\mu_k(x),$$

where

$$P_s(x,y,t) = \int_{\mathbb{R}} \frac{B_k(x,z)B_k(y,z)\sqrt{\tau^k_t|\mathcal{F}_k(g)|^2(z)}d\mu_k(z)}{\eta(1+|z|)^s + \|g\|^2_{L^2(\mathbb{R},\mu_k)}}.$$

Proof. Firstly, the existence and unicity of the minimizer function $f^*_{\eta,h}$ that satisfies (22) are given by Kimeldorf et al. [18], Matsuura et al. [19] and Saitoh [23]. More precisely, $f^*_{\eta,h}$ is given by the reproducing kernel of $H^s(\mathbb{R}, \mu_k)$ with $\|\cdot\|_{\eta,H^s(\mathbb{R},\mu_k)}$ is a norm as

$$f^*_{\eta,h}(y) = \langle h, \mathcal{V}_g(K_{\eta,g}(\cdot, y))\rangle_{L^2(\mathbb{R}^2,\nu_k)},$$

where $K_{\eta,g}$ is the reproducing kernel given by (20).

By (i) of Proposition 3.1 and the last equality, we have

$$\begin{aligned}\mathcal{V}_g(K_{\eta,g}(\cdot, y))(x,t) &= \int_{\mathbb{R}} B_k(x,z)\mathcal{F}_k(K_{\eta,g}(\cdot, y))(z)\sqrt{\tau^k_t|\mathcal{F}_k(g)|^2(z)}d\mu_k(z)\\ &= \int_{\mathbb{R}} \frac{B_k(x,z)B_k(y,z)\sqrt{\tau^k_t|\mathcal{F}_k(g)|^2(z)}}{\eta(1+|z|)^s + \|g\|^2_{L^2(\mathbb{R},\mu_k)}}d\mu_k(z).\end{aligned}$$

This clearly yields the result. □

Theorem 4.3. *Let $s > k$ and $g \in L^2(\mathbb{R}, \mu_k)$. For any $h \in L^2(\mathbb{R}^2, \nu_k)$ and for any $\eta > 0$, we have*

(i) $f^*_{\eta,h}(y) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{B_k(y,z)\sqrt{\tau^k_t|\mathcal{F}_k(g)|^2(z)}\mathcal{F}_k(h(\cdot,t))(z)}{\eta(1+|z|)^s+\|g\|^2_{L^2(\mathbb{R},\mu_k)}}d\mu_k(t)d\mu_k(z).$

(ii) $\mathcal{F}_k(f^*_{\eta,h})(z) = \frac{1}{\eta(1+|z|)^s+\|g\|^2_{L^2(\mathbb{R},\mu_k)}} \int_{\mathbb{R}} \sqrt{\tau^k_t|\mathcal{F}_k(g)|^2(z)}\mathcal{F}_k(h(\cdot,t))(z)d\mu_k(t).$

(iii) $\|f^*_{\eta,h}\|_{H^s(\mathbb{R},\mu_k)} \le \frac{1}{2\sqrt{\eta}}\|h\|_{L^2(\mathbb{R}^2,\nu_k)}.$

Proof. (i) From Theorem 4.2 and Fubini's theorem, we have

$$\begin{aligned}f^*_{\eta,h}(y) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{B_k(y,z)\sqrt{\tau^k_t|\mathcal{F}_k(g)|^2(z)}}{\eta(1+|z|)^s+\|g\|^2_{L^2(\mathbb{R},\mu_k)}}\left[\int_{\mathbb{R}} h(x,t)B_k(x,z)d\mu_k(x)\right]d\mu_k(t)d\mu_k(z)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{B_k(y,z)\sqrt{\tau^k_t|\mathcal{F}_k(g)|^2(z)}\mathcal{F}_k(h(\cdot,t)(z)}{\eta(1+|z|)^s+\|g\|^2_{L^2(\mathbb{R},\mu_k)}}d\mu_k(t)d\mu_k(z).\end{aligned}$$

(ii) The function

$$z \mapsto \frac{1}{\eta(1+|z|)^s + \|g\|^2_{L^2(\mathbb{R},\mu_k)}} \int_{\mathbb{R}} \sqrt{\tau^k_t|\mathcal{F}_k(g)|^2(z)}\mathcal{F}_k(h(\cdot,t))(z)d\mu_k(t)$$

belongs to $L^1 \cap L^2(\mathbb{R}, \mu_k)$. Then from assertions (i) and (iii) of Theorem 2.1, it follows that $f^*_{\eta,h}$ belongs to $L^2(\mathbb{R}, \mu_k)$, and

$$\mathcal{F}_k(f^*_{\eta,h})(z) = \frac{1}{\eta(1+|z|)^s + \|g\|^2_{L^2(\mathbb{R},\mu_k)}} \int_{\mathbb{R}} \sqrt{\tau^k_t |\mathcal{F}_k(g)|^2(z)} \mathcal{F}_k(h(\cdot,t))(z) d\mu_k(t).$$

(iii) From (ii), Hölder's inequality and (10), we have

$$|\mathcal{F}_k(f^*_{\eta,h})(z)|^2 \leq \frac{\|g\|^2_{L^2(\mathbb{R},\mu_k)}}{\left[\eta(1+|z|)^s + \|g\|^2_{L^2(\mathbb{R},\mu_k)}\right]^2} \int_{\mathbb{R}} |\mathcal{F}_k(h(\cdot,t))(z)|^2 d\mu_k(t).$$

Thus,

$$\begin{aligned}
&\|f^*_{\eta,h}\|^2_{H^s(\mathbb{R},\mu_k)} \\
&\leq \int_{\mathbb{R}} \frac{(1+|z|)^s \|g\|^2_{L^2(\mathbb{R},\mu_k)}}{\left[\eta(1+|z|)^s + \|g\|^2_{L^2(\mathbb{R},\mu_k)}\right]^2} \left[\int_{\mathbb{R}} |\mathcal{F}_k(h(\cdot,t))(z)|^2 d\mu_k(t)\right] d\mu_k(z) \\
&\leq \frac{1}{4\eta} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |\mathcal{F}_k(h(\cdot,t))(z)|^2 d\mu_k(t)\right] d\mu_k(z) \\
&= \frac{1}{4\eta} \|h\|^2_{L^2(\mathbb{R}^2,\nu_k)},
\end{aligned}$$

which furnishes the proof of the theorem. □

Theorem 4.4. *Let $s > k$ and $g \in L^2(\mathbb{R}, \mu_k)$. For any $h \in L^2(\mathbb{R}^2, \nu_k)$ and for any $\eta > 0$, we have*

$$\begin{aligned}
&\mathcal{V}_g(f^*_{\eta,h})(x,y) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{B_k(x,z) \sqrt{\tau^k_t |\mathcal{F}_k(g)|^2(z) \tau^k_y |\mathcal{F}_k(g)|^2(z)} \mathcal{F}_k(h(\cdot,t))(z)}{\eta(1+|z|)^s + \|g\|^2_{L^2(\mathbb{R},\mu_k)}} d\mu_k(t) d\mu_k(z).
\end{aligned}$$

Proof. From (i) of Proposition 3.1, we have

$$\mathcal{V}_g(f^*_{\eta,h})(x,y) = \int_{\mathbb{R}} B_k(x,z) \mathcal{F}_k(f^*_{\eta,h})(z) \sqrt{\tau^k_y |\mathcal{F}_k(g)|^2(z)} d\mu_k(z).$$

Then by (ii) of Theorem 4.3 we infer the result. □

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Mohamed Amine Boubatra
Laboratoire d'Analyse Mathématique et Applications LR11ES11
Faculté des Sciences de Tunis
Université Tunis El Manar
2092 El Manar I, Tunis, Tunisie
Email address: boubatra.amine@yahoo.fr