Commun. Korean Math. Soc. **37** (2022), No. 2, pp. 327–335
https://doi.org/10.4134/CKMS.c210103
pISSN: 1225-1763 / eISSN: 2234-3024

ALMOST WEAKLY FINITE CONDUCTOR RINGS AND WEAKLY FINITE CONDUCTOR RINGS

Hanan Choulli, Haitham El Alaoui, and Hakima Mouanis

Abstract. Let R be a commutative ring with identity. We call the ring R to be an almost weakly finite conductor if for any two elements a and b in R, there exists a positive integer n such that $a^nR \cap b^nR$ is finitely generated. In this article, we give some conditions for the trivial ring extensions and the amalgamated algebras to be almost weakly finite conductor rings. We investigate the transfer of these properties to trivial ring extensions and amalgamation of rings. Our results generate examples which enrich the current literature with new families of examples of non-finite conductor weakly finite conductor rings.

1. Introduction

We assume throughout that all rings are commutative with $1 \neq 0$ and that all modules are unital. Recall that a ring R is said to be coherent if every finitely generated ideal of R is finitely presented. In commutative algebra, a coherent ring is a relevant topic. Due to its importance, not only coherent rings but also several kinds of rings related to coherent rings have been studied by many mathematicians. Finite conductor rings and weakly coherent rings are examples of rings related to coherent rings. For more details, see [4, 15].

In 1960, according to Chase [5], R is a coherent domain if and only if the intersection of any two finitely generated ideals is again finitely generated. In 1973, Dobbs [12] introduced the concept of "finite conductor domain" in which every intersection of two principal ideals is a finitely generated ideal. Coherent domains and Greatest Common Divisor (GCD) domains (such that the intersection of any two principal ideals is again principal) are trivial examples of finite conductor domains. In 2000, Glaz extended the definition of a finite conductor domains to rings with zero divisors, that is, the intersection of two principal ideals is a finitely generated ideal and $ann_R(a)$ is finitely generated for every element a of R [15]. Also, in the same paper, Glaz shows that R is a finite conductor ring if and only if any ideal I of R with $\mu(I) \leq 2$ is finite

Received March 25, 2021; Revised May 19, 2021; Accepted May 26, 2021.
2010 *Mathematics Subject Classification.* 15A03, 13A15, 13B25, 13D05, 13F05.
Key words and phrases. Almost weakly finite conductor rings, weakly finite conductor rings, trivial rings extension, amalgamated algebra along an ideal.

presented, where $\mu(I)$ denotes the cardinality of a minimal set of generators of I.

For a ring A and an A-module E, the trivial ring extension of A by E is the ring $R := A \ltimes E$ where the underlying group is $A \times E$ and the multiplication is defined by $(a, e)(b, f) = (ab, af + be)$. The ring R is also called the *Nagata idealization* of E over A and is denoted by $A(+)E$. This construction was first introduced, in 1962, by Nagata [20] in order to facilitate interaction between rings and their modules and also to provide various families of examples of commutative rings containing zero-divisors. The literature abounds of papers on trivial extensions dealing with the transfer of ring-theoretic notions in various settings of these constructions (see, for instance, [2–4, 6, 16, 18]). For more details on commutative trivial extensions (*or idealizations*), we refer the reader to Glaz's and Huckaba's respective books [14,16], and also D. D. Anderson and Winders relatively recent and comprehensive survey paper [2].

The *amalgamation* algebras along an ideal, introduced and studied by D'Anna, Finocchiaro and Fontana in [7–9] and defined as follows:

Let A and B be two rings, J an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) : a \in A, j \in J\}$$

called the amalgamation of A and B along J with respect to f. In particular, they have studied amalgamations in the frame of pullbacks which allowed them to establish numerous (prime) ideal and ring-theoretic basic properties for this new construction. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [10,11]). See for instance [7–9,13,17]. Moreover, other classical constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation [7, Examples 2.5 and 2.6].

In 2004, Mahdou established necessary and sufficient conditions for the trivial ring extensions to be weakly finite conductor rings [19]. In 2018, we gave some characterizations of the property weakly finite conductor to amalgamated algebras along an ideal [13]. The purpose of this paper is to study the possible transfer of the property weakly finite conductor to trivial ring extensions and amalgamated. Also, we investigate a new class of rings called, almost weakly finite conductor rings, over which every two elements a and b in R, there exists a positive integer n such that $a^nR \cap b^nR$ is finitely generated. Examples of almost weakly finite conductor rings are weakly finite conductor rings and almost (GCD) domains (i.e., for all $x, y \in R$ there exists an $n \in \mathbb{N}$ such that $x^nR \cap y^nR$ is principal). The latter introduced by Zafrullah in [21] as a generalization of (GCD) domains.

The goal of Section 2 of this article is to provide necessary and sufficient conditions for trivial ring extensions and amalgamated algebras along an ideal to be an almost weakly finite conductor ring. Our aim is to prove that almost weakly finite conductor rings are not weakly finite conductor rings, in general. The third section deals with the transfer of the notion of weakly finite conductor property to the pre-mentioned ring extensions. In order to give new results to enrich the current literature with new families of examples of non-finite conductor weakly finite conductor rings.

2. Almost weakly finite conductor rings

Definition 2.1. Let R be a ring. R is called an almost weakly finite conductor ring, simply (AWFC)-ring, if, for any two elements a and b in R, there exists a positive integer n such that the ideal $a^nR \cap b^nR$ is finitely generated.

The first purpose of this section is to investigate the possible transfer of the almost weakly finite conductor property to various trivial extension contexts. Recall that a module over a domain is divisible if each element of the module is divisible by every nonzero element of the domain.

Theorem 2.1. *Let A be a ring, E be a nonzero A-module, and $R := A \ltimes E$.*

1. *If R is an (AWFC)-ring, then A is an (AWFC)-ring.*
2. *Suppose that A is a domain and E is a divisible A-module. Then, R is an (AWFC)-ring if and only if so is A.*
3. *Let A be a local ring with a maximal ideal M, and E be an A-module such that $M = \sqrt{Ann(E)}$. Then, R is an (AWFC)-ring if and only if so is A.*

Proof. (1) Suppose that R is an (AWFC)-ring, and let a and $b \in A$. Then there exists a positive integer n such that the ideal $R(a,0)^n \cap R(b,0)^n = R(a^n,0) \cap R(b^n,0)$ is a finitely generated ideal of R. Therefore, the ideal $a^nA \cap b^nA$ is a finitely generated ideal of A, and hence A is an (AWFC)-ring.

(2) Suppose that A is a domain and E is a divisible A-module. If R is an (AWFC)-ring, then so is A by (1). Conversely, let (a,e) and $(b,f) \in R$. We will show that there exists a positive integer n such that $R(a,e)^n \cap R(b,f)^n$ is finitely generated. If $a = 0$ or $b = 0$; it suffices to take $n = 2$. Suppose on the contrary, $a \neq 0$ and $b \neq 0$. First, notice that $(a,e)^n = (a^n, na^{n-1}e)$ for all $n \geq 1$. Now, assume that A is an (AWFC)-ring. Then, there exists a positive integer n such that $a^nA \cap b^nA = \sum_{i=1}^m Ac_i$ where $c_i \in A$ for every $i \in \{1,\dots,m\}$. Let $y \in R(a,e)^n \cap R(b,f)^n$. It is easily seen that y is written in the form $y = (\sum_{i=1}^m \alpha_i c_i, z) = (\alpha_1 c_1, z) + (\alpha_2 c_2, 0) + \cdots + (\alpha_m c_m, 0)$ where $\alpha_i \in A$ for every $i \in \{1,\dots,m\}$ and $z \in E$. By divisibility assumption, we obtain $z = c_1\beta$ for some $\beta \in E$. Hence $y = (\alpha_1,\beta)(c_1,0) + (\alpha_2 c_2, 0) + \cdots + (\alpha_m c_m, 0)$. Therefore $R(a,e)^n \cap R(b,f)^n \subset \sum_{i=1}^m R(c_i,0)$. For the reverse inclusion, we have $c_i \in a^nA \cap b^nA$ for all $i \in \{1,\dots,m\}$. So, there exist $\alpha_1,\dots,\alpha_m$ and $\beta_1,\dots,\beta_m$ such that $c_i = \alpha_i a^n = \beta_i b^n$ for all $i \in \{1,\dots,m\}$. Further, by divisibility, we obtain, for each $i \in \{1,\dots,m\}$, $0 = \alpha_i n a^{n-1} e - k_i a^n$ and $0 = \beta_i n b^{n-1} f - t_i b^n$ for some $k_i, t_i \in E$. Hence, $(c_i, 0) = (\alpha_i, -k_i)(a,e)^n =$

$(\beta_i, -t_i)(b, f)^n \in R(a, e)^n \cap R(b, f)^n$. Thus, $R(a, e)^n \cap R(b, f)^n = \sum_{i=1}^m R(c_i, 0)$, and hence R is an (AWFC)-ring.

(3) By (1), it suffices to prove the "if" assertion. Let (a, e) and $(b, f) \in R$. If a (resp., b) is a unit of A, then (a, e) (resp., (b, f)) is a unit of R by [16, Theorem 25.1(6)], and so $R(a, e) \cap R(b, f) = R$. Thus, we may assume, without loss of generality, that both a and b are in M. As $M = \sqrt{Ann(E)}$, there exist positive integers n and m such that $a^n \in Ann(E)$ and $b^m \in Ann(E)$. Then we get

$$(a, e)^{nm+1} = (a^{nm+1}, (nm + 1)a^{nm}e) = (a^{nm+1}, 0),$$
$$(b, f)^{nm+1} = (b^{nm+1}, (nm + 1)b^{nm}f) = (b^{nm+1}, 0).$$

Since A is an (AWFC)-ring, there exists a positive integer p such that $Aa^{(nm+1)p} \cap Ab^{(nm+1)p}$ is finitely generated. Therefore, $R(a, e)^{(nm+1)p} \cap R(b, f)^{(nm+1)p}$ is finitely generated, as desired. □

The following corollaries are an immediate consequence of Theorem 2.1.

Corollary 2.1. *Let A be a domain, $K := qf(A)$, E be a K-vector space, and $R := A \ltimes E$ the trivial ring extension of A by E. Then, R is an (AWFC)-ring if and only if so is A.*

Corollary 2.2. *Let A be a local ring with a maximal ideal M, and E be an A-module such that $ME = 0$, and $R := A \ltimes E$ the trivial ring extension of A by E. Then, R is an (AWFC)-ring if and only if so is A.*

Now, we are able to construct a non-weakly finite conductor ring which is an (AWFC)-ring.

Example 2.1. Let (A, M) be a nondiscrete valuation domain. Then, $R := A \ltimes A/M$ satisfies the following statements:

(1) R is an (AWFC)-ring, by Corollary 2.2;
(2) R is not a weakly finite conductor ring, by [19, Theorem 2.1] since M is not a finitely generated ideal of A.

Next, we study the transfer of the almost weakly finite conductor property to amalgamation of rings.

Proposition 2.1. *Let A be a domain, $f : A \longrightarrow B$ be a ring homomorphism and J be a non-finitely generated regular ideal of B such that $f(A) \cap J = (0)$ (e.g., let $A := \mathbb{K}$, $B := \mathbb{K}[[X_1, \dots]] = \mathbb{K} + M$ denote the ring of formal power series over the field $\mathbb{K}$ in the indeterminates $(X_i)_{i=1,\dots,\infty}$, $J := M$ and f the inclusion map of A into B). Then, $R := A \bowtie^f J$ is an (AWFC)-ring if and only if $f^{-1}(J) = 0$ and $f(A) + J$ is an (AWFC)-ring.*

Proof. Assume that $R := A \bowtie^f J$ is an (AWFC)-ring and $f^{-1}(J) \neq 0$. Let $i \in f^{-1}(J) \setminus \{0\}$ and j be a regular element of J. Then there exists a positive integer n such that the ideal $R(i, j)^n \cap R(0, j)^n$ is a finitely generated ideal of R. Set $R(i^n, j^n) \cap R(0, j^n) = \sum_{i=1}^m R(c_i, f(c_i) + d_i)$ where $c_i \in A$ and $d_i \in J$ for all $i \in \{1, \dots, m\}$. Then there exist $(\alpha_i, f(\alpha_i) + e_i)$ and $(\beta_i, f(\beta_i) + f_i)$

such that $(c_i, f(c_i) + d_i) = (\alpha_i, f(\alpha_i) + e_i)(i^n, j^n) = (\beta_i, f(\beta_i) + f_i)(0, j^n)$ for each $i \in \{1, \ldots, m\}$. Hence, $c_i = \alpha_i i^n = 0$ for each $i \in \{1, \ldots, m\}$. Since A is a domain and $0 \neq i^n$, we get that $\alpha_i = 0$ and $f(\alpha_i) = 0$ for each $i \in \{1, \ldots, m\}$. Therefore, $d_i = e_i j^n$ for each $i \in \{1, \ldots, m\}$. On the other hand, let $x \in J$. Then, it's clear that $(0, xj^n) \in R(i,j)^n \cap R(0,j)^n$ so there exists $(\gamma_i, f(\gamma_i) + k_i)_{1 \leq i \leq m} \in R^m$ such that $(0, xj^n) = \sum_{i=1}^m (\gamma_i, f(\gamma_i) + k_i)(c_i, f(c_i) + d_i) = \sum_{i=1}^m (0, (f(\gamma_i) + k_i)d_i)$. Thus, $xj^n = \sum_{i=1}^m (f(\gamma_i) + k_i)d_i = \sum_{i=1}^m (f(\gamma_i) + k_i)e_i j^n$. Hence $J = \sum_{i=1}^m (f(A) + J)e_i$, which is absurd since J is a non-finitely generated ideal of B. It follows that $f^{-1}(J) = 0$. Moreover $f(A) + J$ is an (AWFC)-ring by [7, Proposition 5.1(3)]. Conversely, since $f^{-1}(J) = 0$, we have from [7, Proposition 5.1(3)] $A \bowtie^f J \cong f(A) + J$. Therefore $A \bowtie^f J$ is an (AWFC)-ring. $\square$

We use the notation $\mathrm{Nilp}(B)$ to denote the set of all nilpotent elements of B.

Proposition 2.2. *Let A and B be a pair of rings and $f : A \longrightarrow B$ be a ring homomorphism. Suppose that A is a local ring with a maximal ideal M, and J is a proper ideal of B such that $f(M)J = 0$ and $J \subseteq Nilp(B)$. Then $R := A \bowtie^f J$ is an (AWFC)-ring if and only if so is A.*

The proof of this proposition requires the following preparatory lemma which is an immediate consequence of [7, Proposition 5.1(3)] and the fact that if A is an (AWFC)-ring and I is an ideal of A, then A/I is an (AWFC)-ring.

Lemma 2.1. *Let A and B be two rings, $f : A \longrightarrow B$ be a ring homomorphism and J a nonzero proper ideal of B. If $A \bowtie^f J$ is an (AWFC)-ring, then so are A and $f(A) + J$.*

Proof of Proposition 2.2. If R is an (AWFC)-ring, then A is an (AWFC)-ring by Lemma 2.1. Conversely, assume that A is an (AWFC)-ring, and let $(a, f(a) + e)$ and $(b, f(b) + k) \in R$. If a or b is a unit of A, then $R(a, f(a) + e) = R$ or $R(b, f(b) + k) = R$ by [17, Remark 2.1(1)]; and so we are done. Thus we may assume that $a, b \in M$. But since $e, k \in J$, there exist positive integers n and m such that $e^n = 0$ and $k^m = 0$. As A is an (AWFC)-ring, there is a positive integer p such that $Aa^{nmp} \cap Ab^{nmp} = \sum_{i=1}^j Ac_i$, where $c_i \in A$ for each $i \in \{1, \ldots, j\}$. By applying binomial theorem (which is valid in any commutative ring), we get that $R(a, f(a) + e)^{nmp} \cap R(b, f(b) + k)^{nmp} = R(a, f(a))^{nmp} \cap R(b, f(b))^{nmp}$. Hence $R(a, f(a) + e)^{nmp} \cap R(b, f(b) + k)^{nmp} = \sum_{i=1}^j (c_i, f(c_i))$; and therefore R is an (AWFC)-ring. $\square$

Now, we give a second example of an (AWFC)-ring which is not a weakly finite conductor ring.

Example 2.2. Let A be a local coherent domain with maximal M and $E := \frac{A}{M}[X]$. Set $B := A \ltimes E$ and $J := 0 \ltimes E$. Consider the homomorphism $f : A \longrightarrow B$ $(f(a) = (a, 0))$. Then:

(1) $A \bowtie^f J$ is an (AWFC)-ring, by Proposition 2.2.
(2) $A \bowtie^f J$ is not a weakly finite conductor, by [13, Theorem 1(2)b)], since E is not of finite rank.

3. Weakly finite conductor rings ((WFC)-rings)

The first aim of this section is to examine the transfer of weakly finite conductor property to the context of trivial extensions of domains by divisible modules. In this vein, we will use Mahdou definition of a weakly finite conductor ring; that is, the intersection of two principal ideals is a finitely generated ideal [19].

Theorem 3.1. *Let A be a Noetherian domain and E be a divisible A-module. Then, $R := A \ltimes E$ is a (WFC)-ring.*

Proof. Let $I = R(a,e)$ and $J = R(b,f)$ be two principal ideals, where $a,b \in A$ and $e,f \in E$. Then there are three possible cases.
Case 1: $a = b = 0$. Hence $I = R(0,e) = 0 \ltimes Ae$ and $J = R(0,f) = 0 \ltimes Af$. Then, $Ae \cap Af \subseteq Ae + Af$ which is a finitely generated A-module. So $Ae \cap Af = \sum_{i=1}^{n} Aa_i$, where $a_i \in Ae \cap Af$ for each $i \in \{1,\dots,n\}$ since A is a Notherian domain. Therefore, $I \cap J = 0 \ltimes Ae \cap Af = 0 \ltimes \sum_{i=1}^{n} Aa_i = \sum_{i=1}^{n} R(0,a_i)$ is a finitely generated ideal of R.
Case 2: $a \neq 0$ and $b = 0$ or $a = 0$ and $b \neq 0$. By symmetry, we may assume that $a \neq 0$ and $b = 0$. Then, $I = R(a,e) = Aa \ltimes E$ by [1, Lemma 2.3] and $J = R(0,f) = 0 \ltimes Af$. Thus, $J \subseteq I$ and $I \cap J = J$ which is a finitely generated ideal of R.
Case 3: $a \neq 0$ and $b \neq 0$. Hence, $I = R(a,e) = Aa \ltimes E$ and $J = R(b,f) = Ab \ltimes E$. Let $Aa \cap Ab = \sum_{i=1}^{n} Aa_i$, where $a_i \in A \setminus \{0\}$ since A is a (WFC)-ring. Hence,

$$\begin{aligned} I \cap J =&(Aa \ltimes E) \cap (Ab \ltimes E) = (Aa \cap Ab) \ltimes E \\ =&(\sum_{i=1}^{n} Aa_i) \ltimes E = \sum_{i=1}^{n} R(a_i,0) \quad \text{(see the proof of [1, Lemma 2.3])}. \end{aligned}$$

Thus, $I \cap J$ is a finitely generated ideal of R. □

Theorem 3.1 enriches the literature with original examples of (WFC)-rings which are not finite conductor rings. Recall that a ring R is a finite conductor ring if $aR \cap bR$ and $(0 : c)$ are finitely generated for any $a,b,c \in R$ [15].

Example 3.1. Let $A := \mathbb{Z}$ and $E := \mathbb{Q}[X]$. Then:
(1) $A \ltimes E$ is a (WFC)-ring;
(2) $A \ltimes E$ is not a finite conductor ring.

Proof. (1) It follows from Theorem 3.1.
(2) Let $c := (0,1) \in A \ltimes E$. It can easily be seen that $(0 : c) = 0 \ltimes E$. So, $A \ltimes E$ is not a finite conductor ring. □

In our main results of this section in which we give a new characterization of the weakly finite conductor property to amalgamation of rings.

Proposition 3.1. *Let A be a domain, $f : A \longrightarrow B$ be a ring homomorphism and J be a non-finitely generated regular ideal of B such that $f(A) \cap J = (0)$. Then, $R := A \bowtie^f J$ is a (WFC)-ring if and only if $f^{-1}(J) = 0$ and $f(A) + J$ is a (WFC)-ring.*

Proof. By [7, Proposition 5.1(3)], we need only prove to that if R is a (WFC)-ring, then $f^{-1}(J) = 0$. Suppose on the contrary, i.e., $f^{-1}(J)$ is a nonzero ideal of A. Let $i \in f^{-1}(J) \setminus \{0\}$ and j be a regular element of J. Set $R(i,j) \cap R(0,j) = \sum_{i=1}^{n} R(c_i, f(c_i) + d_i)$ where $c_i \in A$ and $d_i \in J$ for all $i \in \{1, \ldots, n\}$. With a similar argument of the proof of Proposition 2.1, we get that J is a finitely generated ideal of B, which is a contradiction since J is a non-finitely generated ideal of B. Hence $f^{-1}(J) = 0$, as desired. □

Theorem 3.2. *Let A be a domain, $f : A \longrightarrow B$ be a ring homomorphism and J be a non-finitely generated ideal of B.*

(1) *Suppose that, $f(A \setminus \{0\}) \subseteq Reg(B)$. If $A \bowtie^f J$ is a (WFC)-ring, then $f(A) \cap J = 0$.*
(2) *Suppose that, J is regular. If $A \bowtie^f J$ is a (WFC)-ring, then f is injective.*

Proof. (1) Suppose the statement is false, i.e., $f(A) \cap J \neq 0$, and choose an element $a \in A$ such that $f(a) \in J \setminus \{0\}$. Then $(0, f(a))$ is an element of $A \bowtie^f J$. Since $A \bowtie^f J$ is a (WFC)-ring, the ideal $R(0, f(a)) \cap R(a, f(a))$ is finitely generated. Set $R(0, f(a)) \cap R(a, f(a)) = \sum_{i=1}^{m} A \bowtie^f J(d_i, f(d_i) + e_i)$ for some $d_1, \ldots, d_m \in A$ and $e_1, \ldots, e_m \in J$. Then for each $i \in \{1, \ldots, m\}$ there exist $(\alpha_i, f(\alpha_i) + k_i)$ and $(\beta_i, f(\beta_i) + \gamma_i)$ in $A \bowtie^f J$ such that

$$(d_i, f(d_i) + e_i) = (\alpha_i, f(\alpha_i) + k_i)(0, f(a)) = (\beta_i, f(\beta_i) + \gamma_i)(a, f(a)).$$

Thus, $d_i = 0$ for all $i \in \{1, \ldots, m\}$. Also, $\beta_i = 0$ since $a\beta_i = 0$ for all $i \in \{1, \ldots, m\}$ and A is a domain. Therefore $e_i = f(a)\gamma_i$ for all $i \in \{1, \ldots, m\}$. On the other hand, let $x \in J$. Then, one can easily check that $(0, xf(a)) \in R(0, f(a)) \cap R(a, f(a))$. Hence, there exist $(c_i, f(c_i) + b_i)_{i=1}^{m} \in (A \bowtie^f J)^m$ such that

$$\begin{aligned}(0, xf(a)) &= \sum_{i=1}^{m} (c_i, f(c_i) + b_i)(d_i, f(d_i) + e_i) \\ &= \sum_{i=1}^{m} (c_i, f(c_i) + b_i)(0, e_i).\end{aligned}$$

From the previous equalities, we deduce that $xf(a) = \sum_{i=1}^{m} (f(c_i) + b_i)e_i = \sum_{i=1}^{m} (f(c_i) + b_i)\gamma_i f(a)$. As $f(a)$ is regular, we get that $x = \sum_{i=1}^{m} (f(c_i) + b_i)\gamma_i$. We conclude that J is finitely generated, which is absurd.

(2) Assume that $A \bowtie^f J$ is a (WFC)-ring, and suppose that f is not injective. Let $a \in Ker(f) \cap (A \setminus \{0\})$ and u be a regular element of J. From the assumption we can write $R(0,u) \cap R(a,u) = \sum_{i=1}^{m} A \bowtie^f J(d_i, f(d_i) + e_i)$ for some $(d_1, f(d_1)+e_1), \ldots, (d_m, f(d_m)+e_m)$ in $A \bowtie^f J$. With a similar argument as in the statement (1), we get that J is non-finitely generated. This completes the proof of Theorem 3.2. $\square$

Corollary 3.1. *Let A and B be a pair of domains, $f : A \longrightarrow B$ be a ring homomorphism, and J be a non-finitely generated ideal of B. Then $A \bowtie^f J$ is a (WFC)-ring if and only if f is injective, $f(A) + J$ is a (WFC)-ring, and $f(A) \cap J = 0$.*

Proof. By Theorem 3.2(1) and (2), we need only prove the sufficient condition. As f is injective and $f(A) \cap J = 0$, then $A \bowtie^f J \cong f(A) + J$ by the natural projection p_B. But since $f(A) + J$ is a (WFC)-ring, $A \bowtie^f J$ is a (WFC)-ring as desired. $\square$

Acknowledgement. The authors sincerely thank the referees for several comments.

References

[1] K. Adarbeh and S. Kabbaj, *Matlis' semi-regularity in trivial ring extensions of integral domains*, Colloq. Math. **150** (2017), no. 2, 229–241. https://doi.org/10.4064/cm7032-12-2016

[2] D. D. Anderson and M. Winders, *Idealization of a module*, J. Commut. Algebra **1** (2009), no. 1, 3–56. https://doi.org/10.1216/JCA-2009-1-1-3

[3] C. Bakkari, S. Kabbaj, and N. Mahdou, *Trivial extensions defined by Prüfer conditions*, J. Pure Appl. Algebra **214** (2010), no. 1, 53–60. https://doi.org/10.1016/j.jpaa.2009.04.011

[4] C. Bakkari and N. Mahdou, *On weakly coherent rings*, Rocky Mountain J. Math. **44** (2014), no. 3, 743–752. https://doi.org/10.1216/RMJ-2014-44-3-743

[5] S. U. Chase, *Direct products of modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473. https://doi.org/10.2307/1993382

[6] R. Damiano and J. Shapiro, *Commutative torsion stable rings*, J. Pure Appl. Algebra **32** (1984), no. 1, 21–32. https://doi.org/10.1016/0022-4049(84)90011-2

[7] M. D'Anna, C. A. Finocchiaro, and M. Fontana, *Amalgamated algebras along an ideal*, in Commutative Algebra and Its Applications, 155–172, Walter de Gruyter, Berlin, 2009.

[8] M. D'Anna, C. A. Finocchiaro, and M. Fontana, *Properties of chains of prime ideals in an amalgamated algebra along an ideal*, J. Pure Appl. Algebra **214** (2010), no. 9, 1633–1641. https://doi.org/10.1016/j.jpaa.2009.12.008

[9] M. D'Anna, C. A. Finocchiaro, and M. Fontana, *New algebraic properties of an amalgamated algebra along an ideal*, Comm. Algebra **44** (2016), no. 5, 1836–1851. https://doi.org/10.1080/00927872.2015.1033628

[10] M. D'Anna and M. Fontana, *The amalgamated duplication of a ring along a multiplicative-canonical ideal*, Ark. Mat. **45** (2007), no. 2, 241–252. https://doi.org/10.1007/s11512-006-0038-1

[11] M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl. **6** (2007), no. 3, 443–459. https://doi.org/10.1142/S0219498807002326

[12] D. E. Dobbs, *On going down for simple overrings*, Proc. Amer. Math. Soc. **39** (1973), 515–519. https://doi.org/10.2307/2039585

[13] H. El Alaoui, *Weakly Finite Conductor Property in Amalgamated Algebra*, in Homological and combinatorial methods in algebra, 117–126, Springer Proc. Math. Stat., 228, Springer, Cham, 2018. https://doi.org/10.1007/978-3-319-74195-6_12

[14] S. Glaz, *Commutative Coherent Rings*, Lecture Notes in Mathematics, 1371, Springer-Verlag, Berlin, 1989. https://doi.org/10.1007/BFb0084570

[15] S. Glaz, *Finite conductor rings*, Proc. Amer. Math. Soc. **129** (2001), no. 10, 2833–2843. https://doi.org/10.1090/S0002-9939-00-05882-2

[16] J. A. Huckaba, *Commutative rings with zero divisors*, Monographs and Textbooks in Pure and Applied Mathematics, 117, Marcel Dekker, Inc., New York, 1988.

[17] K. A. Ismaili and N. Mahdou, *On (n, d)-property in amalgamated algebra*, Asian-Eur. J. Math. **9** (2016), no. 1, 1650014, 13 pp. https://doi.org/10.1142/S1793557116500145

[18] S. Kabbaj, *Matlis semi-regularity and semi-coherence in trivial ring extensions: a survey*, Moroccan Journal of Algebra and Geometry with Applications, In press.

[19] N. Mahdou, *On weakly finite conductor rings*, Comm. Algebra **32** (2004), no. 10, 4027–4036. https://doi.org/10.1081/AGB-200028231

[20] M. Nagata, *Local Rings*, Interscience Tracts in Pure and Applied Mathematics, No. 13, Interscience Publishers a division of John Wiley & Sons New York, 1962.

[21] M. Zafrullah, *A general theory of almost factoriality*, Manuscripta Math. **51** (1985), no. 1-3, 29–62. https://doi.org/10.1007/BF01168346

Hanan Choulli
Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar Al Mahraz
Laboratory of Geometric and Arithmetic Algebra
Fez, Morocco
Email address: hananchoulli@hotmail.com

Haitham El Alaoui
Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar Al Mahraz
Laboratory of Geometric and Arithmetic Algebra
Fez, Morocco
Email address: elalaoui.haithame@gmail.com

Hakima Mouanis
Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar Al Mahraz
Laboratory of Geometric and Arithmetic Algebra
Fez, Morocco
Email address: hmouanis@yahoo.fr