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A NOTE ON DISCRETE SEMIGROUPS OF BOUNDED LINEAR OPERATORS ON NON-ARCHIMEDEAN BANACH SPACES

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Abstract. Let $A \in B(X)$ be a spectral operator on a non-archimedean Banach space over an algebraically closed field. In this note, we give a necessary and sufficient condition on the resolvent of A so that the discrete semigroup consisting of powers of A is uniformly-bounded.

1. Introduction and preliminaries

In the archimedean operator theory, necessary and sufficient conditions on the resolvent of a densely defined closed linear operator are given in order to be the infinitesimal generator of a strongly continuous semigroup $(T(s))_{s\in\mathbb{R}^+}$ such that there is $M \geq 1$, $\|T(s)\| \leq M$. For more details, we refer to [2, 4]. In particular, we have the following theorem and its corollary.

Theorem 1.1 ([6]). *A necessary and sufficient condition for a closed linear operator A with dense domain to be the infinitesimal generator of a strongly continuous semigroup $(T(s))_{s\in\mathbb{R}^+}$ such that for all $s \in \mathbb{R}^+$, $\|T(s)\| \leq M$ is that*

$$\|R_\lambda(A)^n\| \leq \frac{M}{\lambda^n}$$

for $\lambda > 0$ and $n \in \mathbb{N}$, where $R_\lambda(A) = (\lambda I - A)^{-1}$.

Corollary 1.2 ([6]). *A necessary and sufficient condition for a closed linear operator A with dense domain to be the infinitesimal generator of a strongly continuous semigroup $(T(s))_{s\in\mathbb{R}^+}$ such that for all $s \in \mathbb{R}^+$, $\|T(s)\| \leq 1$ is that*

$$\|R_\lambda(A)\| \leq \frac{M}{\lambda}$$

for $\lambda > 0$.

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Throughout this paper, X is a non-archimedean (n.a) Banach space over a (n.a) non trivially complete valued field $\mathbb{K}$ of characteristic zero which is also algebraically closed with valuation $|\cdot|$, $B(X)$ denotes the set of all bounded linear operators on X. $\mathbb{Q}_p$ is the field of p-adic numbers ($p \geq 2$ being a prime) equipped with p-adic valuation $|\cdot|_p$, $\mathbb{Z}_p$ denotes the ring of p-adic integers of $\mathbb{Q}_p$ and it is the unit ball of $\mathbb{Q}_p$. For more details and related issues, we refer to [5, 8]. We denote the completion of the algebraic closure of $\mathbb{Q}_p$ under the p-adic absolute value $|\cdot|_p$ by $\mathbb{C}_p$ (see [5]). Let $r > 0$ and Ω_r be the clopen ball of $\mathbb{K}$ centred at 0 with radius $r > 0$, that is $\Omega_r = \{t \in \mathbb{K} : |t| < r\}$. For more details on non-archimedean operators theory, we refer to [1, 2, 7].

Definition ([9])**.** For $A \in B(X)$, let $\nu(A) = \inf_n \|A^n\|^{\frac{1}{n}} = \lim_n \|A^n\|^{\frac{1}{n}}$. A is said to be a spectral operator if $\sup\{|\lambda| : \lambda \in \sigma(A)\} = \nu(A)$. For $A \in B(X)$, set

$$U_A = \{\lambda \in \mathbb{K} : (I - \lambda A)^{-1} \in B(X)\}$$

$\Big(U_A$ is open and $0 \in U_A\Big)$ and

$$C_A = \{\alpha \in \mathbb{K} : B(0, |\beta|) \subset U_A \text{ for some } \beta \in \mathbb{K},\ |\beta| > |\alpha|\}.$$

We have the following proposition.

Proposition 1.3 ([9])**.** *Let $A \in B(X)$. Then the following are equivalent.*

(i) *A is a spectral operator.*
(ii) *For all $\lambda \in C_A$, $(I - \lambda A)^{-1} = \sum_{n=0}^{\infty} \lambda^n A^n$.*
(iii) *For each $\alpha \in C_A^*$, the function $\lambda \mapsto (I - \lambda A)^{-1}$ is analytic on $B(0, |\alpha|)$.*

We begin with the following definition.

Definition ([3])**.** Let X be a non-archimedean Banach space over $\mathbb{K}$. A family $(T(n))_{n\in\mathbb{N}}$ of bounded linear operators is said to be a discrete semigroup of bounded linear operators on X if

(i) $T(0) = I$, where I is the unit operator of X,
(ii) For all $m, n \in \mathbb{N}$, $T(m + n) = T(m)T(n)$.

Remark 1.4. Let $A \in B(X)$. Then, $T(n) = A^n$ is a discrete semigroup of bounded linear operators on X, and its generator is A.

Definition ([3])**.** Let X be a non-archimedean Banach space over $\mathbb{K}$. A discrete semigroup $(T(n))_{n\in\mathbb{N}}$ is said to be uniformly bounded if $\sup_{n\in\mathbb{N}} \|T(n)\|$ is finite.

Example 1.5 ([3])**.** Let $\mathbb{K} = \mathbb{Q}_p$. If

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then A generates a discrete semigroup of bounded linear operators $(T(n))_{n\in\mathbb{N}}$ given by:

$$T(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \ \forall n \in \mathbb{N}.$$

We have the following definition.

Definition ([3])**.** Let $(T(n))_{n\in\mathbb{N}}$ be a discrete semigroup of bounded linear operators on X. $(T(n))_{n\in\mathbb{N}}$ is said to be a semigroup of contractions if $\|T(n)\| \leq 1$ for all $n \in \mathbb{N}$.

Definition ([1])**.** Let $\omega = (\omega_i)_i$ be a sequence of non-zero elements of $\mathbb{K}$. We define $\mathbb{E}_\omega$ by

$$\mathbb{E}_\omega = \{x = (x_i)_i : \forall i \in \mathbb{N},\ x_i \in \mathbb{K}, \text{ and } \lim_{i\to\infty} |\omega_i|^{\frac{1}{2}}|x_i| = 0\},$$

and it is equipped with the norm

$$\forall x \in \mathbb{E}_\omega : x = (x_i)_i,\ \|x\| = \sup_{i\in\mathbb{N}}(|\omega_i|^{\frac{1}{2}}|x_i|).$$

Remark 1.6 ([1]). The space $\big(\mathbb{E}_\omega, \|\cdot\|\big)$ is a non-archimedean Banach space.

Example 1.7. Let $X = \mathbb{E}_\omega$ with $\omega_i = p^i$ for all $i \in \mathbb{N}$. Let A be a unilateral shift given by

$$Ae_i = e_{i+1} \text{ for all } i \in \mathbb{N}.$$

Then $A^n e_i = e_{n+i}$ for all $n \in \mathbb{N}$, hence, $\frac{\|A^n e_i\|}{\|e_i\|} = p^{\frac{-n}{2}} \leq 1$ for all $i, n \in \mathbb{N}$. Consequently, $\|A^n\| \leq 1$ for all $n \in \mathbb{N}$. Moreover, $(A^n)_{n\in\mathbb{N}}$ is a discrete semigroup of contractions on $\mathbb{E}_\omega$.

Lemma 1.8 ([3])**.** *Let $(T(n))_{n\in\mathbb{N}}$ be a discrete semigroup on X such that $\sup\limits_{n\in\mathbb{N}} \|T(n)\| \leq M$. Then there exists an equivalent norm on X such that T becomes a contraction.*

In the rest of this paper, we let $A \in B(X)$ be a spectral operator such that $\sup\limits_{n\in\mathbb{N}} \|A^n\|$ is finite, and assume that $U_A = \Omega_1$ where $\Omega_1 = \{\lambda \in \mathbb{K} : |\lambda| < 1\}$, and for all $\lambda \in U_A$, $R(\lambda, A) = (I - \lambda A)^{-1}$.

Proposition 1.9 ([3])**.** *Let X be a non-archimedean Banach space over $\mathbb{K}$, and let A be a spectral operator for which there is $M \geq 1$ such that $\sup\limits_{n\in\mathbb{N}} \|A^n\| \leq M$. Then*

$$\|R(\lambda,\ A)\| \leq M \quad \textit{for all } \lambda \in C_A.$$

Proposition 1.10 ([3])**.** *Let $A \in B(X)$ be a spectral operator, and let $(A^n)_{n\in\mathbb{N}}$ be a discrete semigroup of bounded linear operators on X such that $\sup\limits_{n\in\mathbb{N}} \|A^n\|$ is finite and $U_A = B(0, 1)$. Then, for all $\lambda, \mu \in C_A$,*

$$\lambda R(\lambda,\ A) - \mu R(\mu,\ A) = (\lambda - \mu) R(\lambda,\ A) R(\mu,\ A).$$

Proposition 1.11 ([3])**.** *Let $A \in B(X)$ be a spectral operator such that $U_A = \Omega_1$ and let $(A^n)_{n\in\mathbb{N}}$ be a discrete semigroup of contractions on X. Then for all $z \in C_A$, $\|R(\lambda, A) - I\| \leq |\lambda|$.*

As Proposition 2.12 of [3], we have the following proposition.

Proposition 1.12. *Let $A \in B(X)$ be a spectral operator such that for all $k \in \mathbb{N}$, $\|A^k\| \leq M$. Then for all $n \in \mathbb{N}$, $\alpha \in C_A^*$, $\lambda \in \Omega_{|\alpha|}$,*

$$R^{(n)}(\lambda, A) = \frac{n!(R(\lambda, A) - I)^n R(\lambda, A)}{\lambda^n}.$$

We have the following theorem.

Theorem 1.13 ([3]). *Let X be a non-archimedean Banach space over $\mathbb{C}_p$, and $A \in B(X)$ be a spectral operator. Then for all $k \in \mathbb{N}$, $\|A^k\| \leq 1$ if and only if*

$$\|\Big(R(\lambda, A) - I\Big)^n R(\lambda, A)\| \leq |\lambda|_p^n$$

for all $\lambda \in \Omega_{|\alpha|}$ and $n \in \mathbb{N}$ where $\alpha \in C_A^*$ and $R(\lambda, A) = (I - \lambda A)^{-1}$.

Remark 1.14 ([8]). Let $x \in \mathbb{K}$ and $n \in \mathbb{N}$, we define $\binom{x}{0} = 1$ and $\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$. If $k \in \mathbb{N}$ such that $k \geq n$, then $|\binom{k}{n}| \leq 1$.

2. Main results

We have the following theorem.

Theorem 2.1. *Let X be a non-archimedean Banach space over $\mathbb{K}$, and let $A \in B(X)$ be a spectral operator with $U_A = \Omega_1$. Then a necessary and sufficient condition that for all $k \in \mathbb{N}$, $\|A^k\| \leq M$ is that*

$$\|\Big(R(\lambda, A) - I\Big)^n R(\lambda, A)\| \leq M|\lambda|_p^n \tag{2.1}$$

for all $\lambda \in \Omega_{|\alpha|}$, $n \in \mathbb{N}$ where $\alpha \in C_A^*$ and $R(\lambda, A) = (I - \lambda A)^{-1}$.

Proof. Assume that for all $k \in \mathbb{N}$, $\|A^k\| \leq M$, and let $\alpha \in C_A^*$. Then by Proposition 1.3, $R(\lambda, A) = (I - \lambda A)^{-1} = \sum_{k=0}^{\infty} \lambda^k A^k$ is analytic on $\Omega_{|\alpha|}$. Using Proposition 1.12, for all $n \in \mathbb{N}$, $\lambda \in \Omega_{|\alpha|}$

$$R^{(n)}(\lambda, A) = \frac{n!(R(\lambda, A) - I)^n R(\lambda, A)}{\lambda^n}, \tag{2.2}$$

and

$$R^{(n)}(\lambda, A) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)\lambda^{k-n} A^k = \sum_{k=n}^{\infty} n! \binom{k}{n} \lambda^{k-n} A^k,$$

then for all $n \in \mathbb{N}$ and $\lambda \in \Omega_{|\alpha|}$,

$$\begin{aligned}
\Big\|\frac{R^{(n)}(\lambda, A)}{n!}\Big\| &= \|\sum_{k=n}^{\infty} \binom{k}{n} \lambda^{k-n} A^k\| \\
&\leq \sup_{k \geq n} \Big|\binom{k}{n}\Big| |\lambda|^{k-n} \|A^k\| \\
&\leq \sup_{k \geq n} |\lambda|^{k-n} \|A^k\| \\
&\leq M.
\end{aligned}$$

Thus, for all $n \in \mathbb{N}$ and $\lambda \in \Omega_{|\alpha|}$,

$$\Big\| \frac{R^{(n)}(\lambda, A)}{n!} \Big\| \leq M. \tag{2.3}$$

From (2.2) and (2.3), we have for all $n \in \mathbb{N}$, $\lambda \in \Omega_{|\alpha|}$,

$$\|(R(\lambda, A) - I)^n R(\lambda, A)\| \leq M|\lambda|_p^n. \tag{2.4}$$

Conversely, let $A \in B(X)$ be a spectral operator, we assume that (2.1) holds, then for all $\lambda \in \Omega_{|\alpha|}$, $R(\lambda, A) = \sum_{n=0}^{\infty} \lambda^n A^n$. Set for all $\lambda \in \Omega_{|\alpha|}$, $k \in \mathbb{N}$, $S_k(\lambda) = \lambda^{-k}(R(\lambda, A) - I)^k R(\lambda, A)$, then for all $\lambda \in \Omega_{|\alpha|}$, $k \in \mathbb{N}$, $\|S_k(\lambda)\| \leq M$. Since A and $R(\lambda, A)$ commute, we have

$$\begin{aligned} S_k(\lambda) &= \lambda^{-k}\Big(\big(I - (I - \lambda A)\big)R(\lambda, A)\Big)^k R(\lambda, A) \\ &= \lambda^{-k}(\lambda A R(\lambda, A))^k R(\lambda, A) \\ &= A^k R(\lambda, A)^{k+1}. \end{aligned}$$

Then for all $\lambda \in \Omega_{|\alpha|}$, $k \in \mathbb{N}$,

$$\begin{aligned} \|A^k\| &= \|(I - \lambda A)^{k+1} S_k(\lambda)\| \\ &\leq \|(I - \lambda A)^{k+1}\| \|S_k(\lambda)\| \\ &\leq M \| \sum_{j=0}^{k+1} \binom{k+1}{j} (-\lambda A)^j \| \\ &\leq M \max\{1, \|\lambda A\|, \|\lambda^2 A^2\|, \dots, \|\lambda^{k+1} A^{k+1}\|\} \end{aligned}$$

for $\lambda \to 0$, we have for all $k \in \mathbb{N}$, $\|A^k\| \leq M$. □

Remark 2.2. For $M = 1$, we conclude Theorem 1.13.

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