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A NEW EXPLICIT EXTRAGRADIENT METHOD FOR SOLVING EQUILIBRIUM PROBLEMS WITH CONVEX CONSTRAINTS

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Abstract. The purpose of this research is to formulate a new proximal-type algorithm to solve the equilibrium problem in a real Hilbert space. A new algorithm is analogous to the famous two-step extragradient algorithm that was used to solve variational inequalities in the Hilbert spaces previously. The proposed iterative scheme uses a new step size rule based on local bifunction details instead of Lipschitz constants or any line search scheme. The strong convergence theorem for the proposed algorithm is well-proven by letting mild assumptions about the bifunction. Applications of these results are presented to solve the fixed point problems and the variational inequality problems. Finally, we discuss two test problems and computational performance is explicating to show the efficiency and effectiveness of the proposed algorithm.

1. Introduction

Suppose that $\mathbb{C}$ is a nonempty, closed and convex subset of a real Hilbert space $\mathbb{H}$. Assume that $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is a bifunction having $f(y, y) = 0$ for each $y \in \mathbb{C}$ and a *equilibrium problem* (EP) for f on $\mathbb{C}$ is considered in the following form: Find $\rho^* \in \mathbb{C}$ in such a way that

$$f(\rho^*, y) \geq 0, \ \ \forall\, y \in \mathbb{C}. \tag{EP}$$

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In this research work, the problem (EP) is studied based on the following conditions. Let $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a bifunction (see for further details [4, 6]) satisfies the following conditions:

(C1) f is *pseudomonotone* on $\mathbb{C}$, that is, if $f(y_1, y_2) \geq 0$ then

$$f(y_2, y_1) \leq 0, \;\; \forall\, y_1, y_2 \in \mathbb{C}. \tag{1.1}$$

(C2) f is *Lipschitz-type continuous* [19] on $\mathbb{C}$, that is, if there exist $c_1, c_2 > 0$ such that

$$f(y_1, y_3) \leq f(y_1, y_2) + f(y_2, y_3) + c_1\|y_1 - y_2\|^2 + c_2\|y_2 - y_3\|^2, \tag{1.2}$$

for all $y_1, y_2, y_3 \in \mathbb{C}$.

(C3) for any weakly convergent $\{y_n\} \subset \mathbb{C}$ ($y_n \rightharpoonup y^*$) the following inequality holds

$$\limsup_{n\to\infty} f(y_n, y) \leq f(y^*, y), \;\; \forall\, y \in \mathbb{C}. \tag{1.3}$$

(C4) $f(y, \cdot)$ is convex and sub-differentiable on $\mathbb{H}$ for every fixed $y \in \mathbb{H}$.

The general format of the problem of equilibrium draws a great deal of interest to the researcher as it involves a variety of mathematical problems, for example the fixed point problems, scalar and vector minimization problems, the complementarity problems, the variational inequalities problems, the Nash equilibrium problems in non-cooperative games, the saddle point problems and the inverse minimization problems [1, 2, 6, 13, 14, 20, 23, 30, 33, 34] with applications in economics [8] or the dynamics of offer and demand [3], continuing to exploit the theoretical structure of non-cooperative games and Nashs equilibrium idea [24, 25]. In the literature, as best of our knowledge the term "equilibrium problem" was primarily introduced in 1992 by Muu and Oettli [23] and studied extensively by Blum and Oettli [6] and other iterative methods in [12, 21, 22, 26, 29, 31, 32, 35, 36, 37, 38, 40].

By applying the approach of Korpelevich extragradient algorithm [15], Flam et al. [9] and Quoc et al. [27] suggested the following algorithm for dealing with equilibrium problem containing pseudomonotone and Lipschitz-type bifunction: Select a random starting point $x_0 \in \mathbb{C}$; looking at the given iterate x_n, pick up the next iteration under the following scheme:

$$\begin{cases} y_n = \arg\min_{y\in\mathbb{C}}\{\chi f(x_n, y) + \frac{1}{2}\|x_n - y\|^2\}, \\ x_{n+1} = \arg\min_{y\in\mathbb{C}}\{\chi f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\}, \end{cases} \tag{1.4}$$

where $0 < \chi < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$ and c_1, c_2 are two Lipschitz-type constants of a bifunction (1.2).

It is noteworthy to point out that the above well-proven method carries two serious drawbacks, the first is the constant step size that involves the knowledge or approximation of the Lipschitz constant of the related bifunction and it only converges weakly in Hilbert spaces. From the computational point of regard, it might be problematic to determine the Lipschitz constant previously, and hence the convergence rate and appropriateness of the method could be affected. So a natural question arises:

Question: Is it possible to develop a new strongly convergent extragradient algorithm independent of the Lipschitz-type constants with a monotone step size rule to determine the numerical solution of the problem (EP) involving a pseudomonotone bifunction?

In this study, we study about the positive answer to this question, that is, the gradient methods still hold in case of monotonic step size rule for solving equilibrium problems associated with pseudomonotone functions and retain a strong convergence. Inspired by the works of [18, 27], we introduce a new extragradient-type algorithm to figure out the problem (EP) in the context of infinite-dimensional real Hilbert spaces.

- (i) We introduce a self-adaptive subgradient extragradient algorithm by using a monotone step size rule to figure out equilibrium problems and also prove that generated sequence is strongly convergent. This results seen as the modification of the method (1.4).
- (ii) The implementations of our main findings are studied in order to solve particular classes of equilibrium problems in real a real Hilbert space.
- (iii) The numerical study of Algorithm 1 with Algorithm 3.2 (Alg3.2) in [10] and Algorithm 4.1 in [11] (Alg4.1). The numerical results has shown that the proposed algorithms are useful and performed better compared to the existing ones.

The rest of the study has been drawn up as follows: Section 2 comprises basic definitions and key lemmas used throughout the manuscript. Section 3 consists of proposed iterative scheme with variable step size rule and a theorem of convergence analysis. Section 4 sets out the application of the proposed results to solve the variational inequality problems and fixed point problems. Section 5 given numerical results to illustrate the performance of the new algorithms and equate them with two existing algorithms.

2. Preliminaries

Assume that $\mathbb{C}$ is a nonempty, closed and convex subset of a real Hilbert space $\mathbb{H}$. The *metric projection* $P_{\mathbb{C}}(x)$ of $x \in \mathbb{H}$ onto a closed and convex subset $\mathbb{C}$ of $\mathbb{H}$ is defined by

$$P_{\mathbb{C}}(x) = \arg\min_{y\in\mathbb{C}}\|y - x\|. \tag{2.1}$$

Next, some useful properties of the metric projection are given.

Lemma 2.1. [16] *A metric projection $P_{\mathbb{C}} : \mathbb{H} \to \mathbb{C}$ satisfy the following.*

(i)
$$\|y_1 - P_{\mathbb{C}}(y_2)\|^2 + \|P_{\mathbb{C}}(y_2) - y_2\|^2 \le \|y_1 - y_2\|^2, \; y_1 \in \mathbb{C}, y_2 \in \mathbb{H}.$$

(ii) $y_3 = P_{\mathbb{C}}(y_1)$ *if and only if*
$$\langle y_1 - y_3, y_2 - y_3\rangle \le 0, \; \forall y_2 \in \mathbb{C}.$$

(iii)
$$\|y_1 - P_{\mathbb{C}}(y_1)\| \le \|y_1 - y_2\|, \; y_2 \in \mathbb{C}, y_1 \in \mathbb{H}.$$

Definition 2.2. Let $\mathbb{C}$ be a subset of a real Hilbert space $\mathbb{H}$ and $\varkappa : \mathbb{C} \to \mathbb{R}$ a given convex function.

(1) The *subdifferential* of $\varkappa$ at $x \in \mathbb{C}$ is defined by
$$\partial\varkappa(x) = \{z \in \mathbb{H} : \varkappa(y) - \varkappa(x) \ge \langle z, y - x\rangle, \forall\, y \in \mathbb{C}\}. \tag{2.2}$$

(2) The *normal cone* at $x \in \mathbb{C}$ is defined by
$$N_{\mathbb{C}}(x) = \{z \in \mathbb{H} : \langle z, y - x\rangle \le 0, \forall\, y \in \mathbb{C}\}. \tag{2.3}$$

Lemma 2.3. ([28]) *Suppose that $\varkappa : \mathbb{C} \to \mathbb{R}$ is a sub-differentiable, lower semi-continuous function on $\mathbb{C}$. An element $x \in \mathbb{C}$ is a minimizer of a function $\varkappa$ if and only if*
$$0 \in \partial\varkappa(x) + N_{\mathbb{C}}(x),$$
where $\partial\varkappa(x)$ stands for the sub-differential of $\varkappa$ at $x \in \mathbb{C}$ and $N_{\mathbb{C}}(x)$ the normal cone of $\mathbb{C}$ at x.

Lemma 2.4. ([39]) *Suppose that $\{a_n\} \subset (0, +\infty)$ is a sequence satisfying*
$$a_{n+1} \le (1 - b_n)a_n + b_n\eta_n, \; \textit{for all } n \in \mathbb{N}.$$
Moreover, $\{b_n\} \subset (0, 1)$ and $\{\eta_n\} \subset \mathbb{R}$ are sequences such that $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n = +\infty$ and $\limsup_{n\to\infty} \eta_n \le 0$. Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. ([17]) *Assume that $\{a_n\} \subset \mathbb{R}$ be a sequence and there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_{i+1}}$ for all $i \in \mathbb{N}$. Then, there is a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ as $k \to \infty$, and the subsequent conditions are fulfilled by all (sufficiently large) numbers $k \in \mathbb{N}$:*
$$a_{m_k} \le a_{m_{k+1}} \textit{ and } a_k \le a_{m_{k+1}},$$
where $m_k = \max\{j \le k : a_j \le a_{j+1}\}$.

Lemma 2.6. ([5]) *For all* $y_1, y_2 \in \mathbb{H}$ *and* $\Im \in \mathbb{R}$, *the subsequent relationship hold.*

(i) $\|\Im y_1 + (1-\Im)y_2\|^2 = \Im\|y_1\|^2 + (1-\Im)\|y_2\|^2 - \Im(1-\Im)\|y_1 - y_2\|^2$.

(ii) $\|y_1 + y_2\|^2 \leq \|y_1\|^2 + 2\langle y_2, y_1 + y_2\rangle$.

3. Main Results

Next, we introduce a variant of algorithm (1.4) in which the constant step size χ is chosen adaptively and thus yield a sequence χ_n that does not require the knowledge of the Lipschitz-like parameters of the bifunction f.

Algorithm 1 (Strongly convergent extragradient-type method)

Step 0: Choose $x_0 \in \mathbb{C}$, $\mu \in (0,1)$, $\chi_0 > 0$, $\{\gamma_n\} \subset (a,b) \subset (0, 1-\delta_n)$ and $\{\delta_n\} \subset (0,1)$ satisfies the conditions, that is,

$$\lim_{n\to+\infty} \delta_n = 0 \text{ and } \sum_{n=1}^{+\infty} \delta_n = +\infty.$$

Step 1: Compute

$$y_n = \arg\min_{y\in\mathbb{C}}\{\chi_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2\}.$$

If $x_n = y_n$, then STOP. Otherwise go to **Step 2**.

Step 2: Firstly choose $\omega_n \in \partial_2 f(x_n, y_n)$ satisfying $x_n - \chi_n\omega_n - y_n \in N_{\mathbb{C}}(y_n)$ and create a half-space

$$\mathbb{H}_n = \{z \in \mathbb{H} : \langle x_n - \chi_n\omega_n - y_n, z - y_n\rangle \leq 0\}$$

and compute $z_n = \arg\min_{y\in\mathbb{H}_n}\{\chi_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\}$.

Step 3: Compute

$$x_{n+1} = (1 - \gamma_n - \delta_n)x_n + \gamma_n z_n.$$

Step 4: Compute

$$\chi_{n+1} = \begin{cases} \min\left\{\chi_n, \ \frac{\mu\|x_n - y_n\|^2 + \mu\|z_n - y_n\|^2}{2[f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n)]}\right\} \\ \quad \text{if} \quad f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n) > 0, \\ \chi_n, \qquad\qquad\qquad\qquad \textit{else}. \end{cases}$$

Set $n := n + 1$ and move back to **Step 1**.

Lemma 3.1. *Let* $\{\chi_n\}$ *be the sequence in the Algorithm* 1. *Then it is decreasing monotonically with a lower bound value* $\min\left\{\frac{\mu}{2\max\{c_1, c_2\}}, \chi_0\right\}$.

Proof. Consider that $f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n) > 0$, so

$$\frac{\mu(\|x_n - y_n\|^2 + \|z_n - y_n\|^2)}{2[f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n)]} \geq \frac{\mu(\|x_n - y_n\|^2 + \|z_n - y_n\|^2)}{2[c_1\|x_n - y_n\|^2 + c_2\|z_n - y_n\|^2]} \geq \frac{\mu}{2\max\{c_1, c_2\}}. \tag{3.1}$$

Thus, above expression implies that the sequences $\{\chi_n\}$ is bounded below by a value $\min\left\{\frac{\mu}{2\max\{c_1,c_2\}}, \chi_0\right\}$. □

Theorem 3.2. *Assume that the condition* (**C1**)-(**C4**) are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to an element $x* = P_{EP(f,\mathbb{C})}(0)$.

Proof. First, now start to prove the boundedness of the sequence $\{x_n\}$. By Lemma 2.3, we have

$$0 \in \partial_2\left\{\chi_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\right\}(z_n) + N_{\mathbb{H}_n}(z_n).$$

For $\omega \in \partial f(y_n, z_n)$ there exists $\overline{\omega} \in N_{\mathbb{H}_n}(z_n)$ such that

$$\chi_n\omega + z_n - x_n + \overline{\omega} = 0.$$

It follows that

$$\langle x_n - z_n, y - z_n\rangle = \chi_n\langle \omega, y - z_n\rangle + \langle \overline{\omega}, y - z_n\rangle,\ \forall\, y \in \mathbb{H}_n.$$

Due to $\overline{\omega} \in N_{\mathbb{H}_n}(z_n)$, follows that $\langle \overline{\omega}, y - z_n\rangle \leq 0$, for all $y \in \mathbb{H}_n$. Thus

$$\langle x_n - z_n, y - z_n\rangle \leq \chi_n\langle \omega, y - z_n\rangle,\ \forall\, y \in \mathbb{H}_n. \tag{3.2}$$

Moreover, $\omega \in \partial f(y_n, z_n)$, we have

$$f(y_n, y) - f(y_n, z_n) \geq \langle \omega, y - z_n\rangle,\ \forall\, y \in \mathbb{H}. \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\chi_n f(y_n, y) - \chi_n f(y_n, z_n) \geq \langle x_n - z_n, y - z_n\rangle,\ \forall\, y \in \mathbb{H}_n. \tag{3.4}$$

Due to description of $\mathbb{H}_n$, we have

$$\chi_n\langle \omega_n, z_n - y_n\rangle \geq \langle x_n - y_n, z_n - y_n\rangle. \tag{3.5}$$

Now, using $\omega_n \in \partial f(x_n, y_n)$, we obtain

$$f(x_n, y) - f(x_n, y_n) \geq \langle \omega_n, y - y_n\rangle,\ \forall\, y \in \mathbb{H}.$$

By letting $y = z_n$, we have

$$f(x_n, z_n) - f(x_n, y_n) \geq \langle \omega_n, z_n - y_n\rangle,\ \forall\, y \in \mathbb{H}. \tag{3.6}$$

From (3.5) and (3.6), we get

$$\chi_n\left\{f(x_n, z_n) - f(x_n, y_n)\right\} \geq \langle x_n - y_n, z_n - y_n\rangle. \tag{3.7}$$

By replacing $y = \rho^*$ in (3.4), we get

$$\chi_n f(y_n, \rho^*) - \chi_n f(y_n, z_n) \geq \langle x_n - z_n, \rho^* - z_n \rangle. \tag{3.8}$$

Since $\rho^* \in EP(f, \mathbb{C})$, we have $f(\rho^*, y_n) \geq 0$. From the pseudomonotonicity of bifunction f, we get $f(y_n, \rho^*) \leq 0$. Thus

$$\langle x_n - z_n, z_n - \rho^* \rangle \geq \chi_n f(y_n, z_n). \tag{3.9}$$

From description of χ_{n+1}, we get

$$f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n) \leq \frac{\mu \|x_n - y_n\|^2 + \mu \|z_n - y_n\|^2}{2\chi_{n+1}}. \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$\begin{aligned} \langle x_n - z_n, z_n - \rho^* \rangle \geq{}& \chi_n \{f(x_n, z_n) - f(x_n, y_n)\} \\ & - \frac{\mu\chi_n}{2\chi_{n+1}} \|x_n - y_n\|^2 - \frac{\mu\chi_n}{2\chi_{n+1}} \|z_n - y_n\|^2. \end{aligned} \tag{3.11}$$

From (3.7) and (3.11), we have

$$\begin{aligned} \langle x_n - z_n, z_n - \rho^* \rangle \geq{}& \langle x_n - y_n, z_n - y_n \rangle \\ & - \frac{\mu\chi_n}{2\chi_{n+1}} \|x_n - y_n\|^2 - \frac{\mu\chi_n}{2\chi_{n+1}} \|z_n - y_n\|^2. \end{aligned} \tag{3.12}$$

We have the given formula in place:

$$-2\langle x_n - z_n, z_n - \rho^* \rangle = -\|x_n - \rho^*\|^2 + \|z_n - x_n\|^2 + \|z_n - \rho^*\|^2, \tag{3.13}$$

$$2\langle y_n - x_n, y_n - z_n \rangle = \|x_n - y_n\|^2 + \|z_n - y_n\|^2 - \|x_n - z_n\|^2. \tag{3.14}$$

Combining (3.12) and (3.14), we get

$$\|z_n - \rho^*\|^2 \leq \|x_n - \rho^*\|^2 - \Big(1 - \frac{\mu\chi_n}{\chi_{n+1}}\Big)\|x_n - y_n\|^2 - \Big(1 - \frac{\mu\chi_n}{\chi_{n+1}}\Big)\|z_n - y_n\|^2. \tag{3.15}$$

Since $\chi_n \to \chi$, there is a fixed number $\epsilon \in (0, 1 - \mu)$ such that

$$\lim_{n \to \infty} \Big(1 - \frac{\mu\chi_n}{\chi_{n+1}}\Big) = 1 - \mu > \epsilon > 0, \ \forall\, n \geq n_0.$$

Thus, expression (3.15) implies that

$$\|z_n - \rho^*\|^2 \leq \|x_n - \rho^*\|^2, \ \forall\, n \geq n_0. \tag{3.16}$$

Given that $\rho^* \in EP(f, \mathbb{C})$, we obtain

$$\begin{aligned} \big\|x_{n+1} - \rho^*\big\| &= \big\|(1 - \gamma_n - \delta_n)x_n + \gamma_n z_n - \rho^*\big\| \\ &= \big\|(1 - \gamma_n - \delta_n)(x_n - \rho^*) + \gamma_n(z_n - \rho^*) - \delta_n \rho^*\big\| \\ &\leq \big\|(1 - \gamma_n - \delta_n)(x_n - \rho^*) + \gamma_n(z_n - \rho^*)\big\| + \delta_n\big\|\rho^*\big\|. \end{aligned} \tag{3.17}$$

Next, we estimate the following:

$$\begin{aligned}
&\big\|(1-\gamma_n-\delta_n)(x_n-\rho^*)+\gamma_n(z_n-\rho^*)\big\|^2 \\
&= (1-\gamma_n-\delta_n)^2\big\|x_n-\rho^*\big\|^2+\gamma_n^2\big\|z_n-\rho^*\big\|^2 \\
&\quad + 2\big\langle(1-\gamma_n-\delta_n)(x_n-\rho^*),\gamma_n(z_n-\rho^*)\big\rangle \\
&\le (1-\gamma_n-\delta_n)^2\big\|x_n-\rho^*\big\|^2+\gamma_n^2\big\|z_n-\rho^*\big\|^2 \\
&\quad + 2\gamma_n(1-\gamma_n-\delta_n)\big\|x_n-\rho^*\big\|\big\|z_n-\rho^*\big\| \\
&\le (1-\gamma_n-\delta_n)^2\big\|x_n-\rho^*\big\|^2+\gamma_n^2\big\|z_n-\rho^*\big\|^2 \\
&\quad + \gamma_n(1-\gamma_n-\delta_n)\big\|x_n-\rho^*\big\|^2+\gamma_n(1-\gamma_n-\delta_n)\big\|z_n-\rho^*\big\|^2 \\
&\le (1-\gamma_n-\delta_n)(1-\delta_n)\big\|x_n-\rho^*\big\|^2+\gamma_n(1-\delta_n)\big\|z_n-\rho^*\big\|^2.
\end{aligned} \tag{3.18}$$

Substituting (3.16) into (3.18), we obtain

$$\begin{aligned}
&\big\|(1-\gamma_n-\delta_n)(x_n-\rho^*)+\gamma_n(z_n-\rho^*)\big\|^2 \\
&\le (1-\gamma_n-\delta_n)(1-\delta_n)\big\|x_n-\rho^*\big\|^2+\gamma_n(1-\delta_n)\big\|x_n-\rho^*\big\|^2 \\
&= (1-\delta_n)^2\big\|x_n-\rho^*\big\|^2.
\end{aligned} \tag{3.19}$$

Therefore, we have

$$\big\|(1-\gamma_n-\delta_n)(x_n-\rho^*)+\gamma_n(z_n-\rho^*)\big\| \le (1-\delta_n)\big\|x_n-\rho^*\big\|. \tag{3.20}$$

Combining (3.17) and (3.20), we get

$$\begin{aligned}
\big\|x_{n+1}-\rho^*\big\| &\le (1-\delta_n)\big\|x_n-\rho^*\big\|+\delta_n\big\|\rho^*\big\| \\
&\le \max\Big\{\big\|x_n-\rho^*\big\|,\big\|\rho^*\big\|\Big\} \\
&\le \max\Big\{\big\|x_{n_0}-\rho^*\big\|,\big\|\rho^*\big\|\Big\}.
\end{aligned} \tag{3.21}$$

Thus, the above expression implies that $\{x_n\}$ is bounded sequence.

Next, our aim is to prove that the sequence $\{x_n\}$ is strongly convergent. Indeed, by the use of definition of $\{x_{n+1}\}$, we have

$$\begin{aligned}
\big\|x_{n+1}-\rho^*\big\|^2 &= \big\|(1-\gamma_n-\delta_n)x_n+\gamma_n z_n-\rho^*\big\|^2 \\
&= \big\|(1-\gamma_n-\delta_n)(x_n-\rho^*)+\gamma_n(z_n-\rho^*)-\delta_n\rho^*\big\|^2 \\
&= \big\|(1-\gamma_n-\delta_n)(x_n-\rho^*)+\gamma_n(z_n-\rho^*)\big\|^2+\delta_n^2\big\|\rho^*\big\|^2 \\
&\quad - 2\big\langle(1-\gamma_n-\delta_n)(x_n-\rho^*)+\gamma_n(z_n-\rho^*),\,\delta_n\rho^*\big\rangle.
\end{aligned} \tag{3.22}$$

By the use of (3.18), we have

$$\begin{aligned}&\big\|(1-\gamma_n-\delta_n)(x_n-\rho^*)+\gamma_n(z_n-\rho^*)\big\|^2\\&\le(1-\gamma_n-\delta_n)(1-\delta_n)\big\|x_n-\rho^*\big\|^2+\gamma_n(1-\delta_n)\big\|z_n-\rho^*\big\|^2.\end{aligned}\tag{3.23}$$

Combining (3.22) and (3.23) (for some $K_2>0$), we get

$$\begin{aligned}&\big\|x_{n+1}-\rho^*\big\|^2\\&\le(1-\gamma_n-\delta_n)(1-\delta_n)\big\|x_n-\rho^*\big\|^2+\gamma_n(1-\delta_n)\big\|z_n-\rho^*\big\|^2+\delta_nK_2\\&\le(1-\gamma_n-\delta_n)(1-\delta_n)\big\|x_n-\rho^*\big\|^2+\delta_nK_2\\&\quad+\gamma_n(1-\delta_n)\Big[\|x_n-\rho^*\|^2-\Big(1-\frac{\mu\chi_n}{\chi_{n+1}}\Big)\|x_n-y_n\|^2\\&\quad-\Big(1-\frac{\mu\chi_n}{\chi_{n+1}}\Big)\|z_n-y_n\|^2\Big]\\&=(1-\delta_n)^2\|x_n-\rho^*\|^2+\delta_nK_2\\&\quad-\gamma_n(1-\delta_n)\Big[\Big(1-\frac{\mu\chi_n}{\chi_{n+1}}\Big)\|x_n-y_n\|^2+\Big(1-\frac{\mu\chi_n}{\chi_{n+1}}\Big)\|z_n-y_n\|^2\Big]\\&\le\|x_n-\rho^*\|^2+\delta_nK_2\\&\quad-\gamma_n(1-\delta_n)\Big[\Big(1-\frac{\mu\chi_n}{\chi_{n+1}}\Big)\|x_n-y_n\|^2+\Big(1-\frac{\mu\chi_n}{\chi_{n+1}}\Big)\|z_n-y_n\|^2\Big].\end{aligned}\tag{3.24}$$

From the conditions (**C1**) and (**C2**), the solution set $EP(f,\mathbb{C})$ is a closed and convex set, see for example, [27]). Given that $\rho^*=P_{EP(f,\mathbb{C})}(0)$, and by Lemma 2.1 (ii), we have

$$\langle 0-\rho^*,y-\rho^*\rangle\le 0,\ \forall\, y\in EP(f,\mathbb{C}).\tag{3.25}$$

Now we divide the rest of the proof into the following two parts:

Case 1: Suppose that there is a fixed number $n_1\in\mathbb{N}$ such that

$$\|x_{n+1}-\rho^*\|\le\|x_n-\rho^*\|,\ \forall\, n\ge n_1.\tag{3.26}$$

Then $\lim_{n\to\infty}\|x_n-\rho^*\|$ exists. From (3.24), we have

$$\begin{aligned}&\gamma_n(1-\delta_n)\Big[\Big(1-\frac{\mu\chi_n}{\chi_{n+1}}\Big)\|x_n-y_n\|^2+\Big(1-\frac{\mu\chi_n}{\chi_{n+1}}\Big)\|z_n-y_n\|^2\Big]\\&\le\|x_n-\rho^*\|^2+\delta_nK_2-\|x_{n+1}-\rho^*\|^2.\end{aligned}\tag{3.27}$$

The existence of $\lim_{n\to\infty}\|x_n-\rho^*\|$ and $\delta_n\to 0$, we infer that

$$\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|z_n-y_n\|=0.\tag{3.28}$$

It follows that

$$\lim_{n\to\infty}\|x_n-z_n\|\le\lim_{n\to\infty}\|x_n-y_n\|+\lim_{n\to\infty}\|y_n-z_n\|=0.\tag{3.29}$$

It follows from (3.29) and $\delta_n \to 0$, that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(1 - \gamma_n - \delta_n)x_n + \gamma_n z_n - x_n\| \\ &= \|x_n - \delta_n x_n + \gamma_n z_n - \gamma_n x_n - x_n\| \\ &\leq \gamma_n\|z_n - x_n\| + \delta_n\|x_n\|, \end{aligned} \tag{3.30}$$

which gives that

$$\|x_{n+1} - x_n\| \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.31}$$

We can also deduce that $\{y_n\}$ and $\{z_n\}$ are bounded. The reflexivity of $\mathbb{H}$ and the boundedness of $\{x_n\}$ guarantee that there is a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in \mathbb{H}$ as $k \to +\infty$.

Next, we need to show that $\hat{x} \in EP(f, \mathbb{C})$. By the use of expression (3.4), the Lipschitz-type continuity of f and (3.10), we get

$$\begin{aligned}\chi_{n_k} f(y_{n_k}, y) &\geq \chi_{n_k} f(y_{n_k}, z_{n_k}) + \langle x_{n_k} - z_{n_k}, y - z_{n_k}\rangle \\ &\geq \chi_{n_k} f(x_{n_k}, x_{n_{k+1}}) - \chi_{n_k} f(x_{n_k}, y_{n_k}) - \frac{\mu\chi_{n_k}}{2\chi_{n_k+1}}\|x_{n_k} - y_{n_k}\|^2 \\ &\quad - \frac{\mu\chi_{n_k}}{2\chi_{n_k+1}}\|y_{n_k} - z_{n_k}\|^2 + \langle x_{n_k} - z_{n_k}, y - z_{n_k}\rangle \\ &\geq \langle x_{n_k} - y_{n_k}, z_{n_k} - y_{n_k}\rangle - \frac{\mu\chi_{n_k}}{2\chi_{n_k+1}}\|x_{n_k} - y_{n_k}\|^2 \\ &\quad - \frac{\mu\chi_{n_k}}{2\chi_{n_k+1}}\|y_{n_k} - z_{n_k}\|^2 + \langle x_{n_k} - z_{n_k}, y - z_{n_k}\rangle, \end{aligned} \tag{3.32}$$

where y is an arbitrary point in $\mathbb{H}_n$. The boundedness of $\{x_n\}$ and from (3.28), (3.29) right-hand side converge to zero. Since $\chi_n > 0$, condition (**C3**) and $y_{n_k} \rightharpoonup \hat{x}$, we have

$$0 \leq \limsup_{k\to\infty} f(y_{n_k}, y) \leq f(\hat{x}, y), \ \forall\, y \in \mathbb{H}_n. \tag{3.33}$$

Thus, above implies that $f(\hat{x}, y) \geq 0$, for all $y \in \mathbb{C}$, and hence $\hat{x} \in EP(f, \mathbb{C})$. Thus

$$\begin{aligned}&\limsup_{n\to\infty}\langle \rho^*, \rho^* - x_n\rangle \\ &= \limsup_{k\to\infty}\langle \rho^*, \rho^* - x_{n_k}\rangle = \langle \rho^*, \rho^* - \hat{x}\rangle \leq 0. \end{aligned} \tag{3.34}$$

By the use of $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. We might conclude that

$$\begin{aligned}&\limsup_{n\to\infty}\langle \rho^*, \rho^* - x_{n+1}\rangle \\ &\leq \limsup_{n\to\infty}\langle \rho^*, \rho^* - x_n\rangle + \limsup_{n\to\infty}\langle \rho^*, x_n - x_{n+1}\rangle \leq 0. \end{aligned} \tag{3.35}$$

Next, assume that $t_n = (1 - \gamma_n)x_n + \gamma_n z_n$. Then, we obtain

$$x_{n+1} = t_n - \delta_n x_n = (1-\delta_n)t_n - \delta_n(x_n - t_n) = (1-\delta_n)t_n - \delta_n\gamma_n(x_n - z_n). \tag{3.36}$$

where $x_n - t_n = x_n - (1-\gamma_n)x_n - \gamma_n z_n = \gamma_n(x_n - z_n)$. Thus, we have

$$
\begin{aligned}
&\|x_{n+1} - \rho^*\|^2 \\
&= \|(1-\delta_n)t_n + \gamma_n\delta_n(z_n - x_n) - \rho^*\|^2 \\
&= \|(1-\delta_n)(t_n - \rho^*) + [\gamma_n\delta_n(z_n - x_n) - \delta_n\rho^*]\|^2 \\
&\le (1-\delta_n)^2\|t_n - \rho^*\|^2 \\
&\quad + 2\langle \gamma_n\delta_n(z_n - x_n) - \delta_n\rho^*,\ (1-\delta_n)(t_n - \rho^*) + \gamma_n\delta_n(z_n - x_n) - \delta_n\rho^* \rangle \\
&= (1-\delta_n)^2\|t_n - \rho^*\|^2 \\
&\quad + 2\langle \gamma_n\delta_n(z_n - x_n) - \delta_n\rho^*,\ t_n - \delta_n t_n - \delta_n(x_n - t_n) - \rho^* \rangle \\
&= (1-\delta_n)\|t_n - \rho^*\|^2 + 2\gamma_n\delta_n\langle z_n - x_n,\ x_{n+1} - \rho^*\rangle + 2\delta_n\langle \rho^*,\ \rho^* - x_{n+1}\rangle \\
&\le (1-\delta_n)\|t_n - \rho^*\|^2 + 2\gamma_n\delta_n\|z_n - x_n\|\|x_{n+1} - \rho^*\| + 2\delta_n\langle \rho^*,\ \rho^* - x_{n+1}\rangle.
\end{aligned} \tag{3.37}
$$

Next, we need to evaluate

$$
\begin{aligned}
&\|t_n - \rho^*\|^2 \\
&= \|(1-\gamma_n)x_n + \gamma_n z_n - \rho^*\|^2 \\
&= \|(1-\gamma_n)(x_n - \rho^*) + \gamma_n(z_n - \rho^*)\|^2 \\
&= (1-\gamma_n)^2\|x_n - \rho^*\|^2 + \gamma_n^2\|z_n - \rho^*\|^2 + 2\langle (1-\gamma_n)(x_n - \rho^*), \gamma_n(z_n - \rho^*)\rangle \\
&\le (1-\gamma_n)^2\|x_n - \rho^*\|^2 + \gamma_n^2\|z_n - \rho^*\|^2 + 2\gamma_n(1-\gamma_n)\|x_n - \rho^*\|\|z_n - \rho^*\| \\
&\le (1-\gamma_n)^2\|x_n - \rho^*\|^2 + \gamma_n^2\|z_n - \rho^*\|^2 + \gamma_n(1-\gamma_n)\|x_n - \rho^*\|^2 \\
&\quad + \gamma_n(1-\gamma_n)\|z_n - \rho^*\|^2 \\
&= (1-\gamma_n)\|x_n - \rho^*\|^2 + \gamma_n\|z_n - \rho^*\|^2 \\
&\le (1-\gamma_n)\|x_n - \rho^*\|^2 + \gamma_n\|x_n - \rho^*\|^2 \\
&= \|x_n - \rho^*\|^2.
\end{aligned} \tag{3.38}
$$

Combining expressions (3.37) and (3.38) gives that

$$
\begin{aligned}
&\|x_{n+1} - \rho^*\|^2 \\
&\le (1-\delta_n)\|x_n - \rho^*\|^2 + \delta_n\Big[2\gamma_n\|z_n - x_n\|\|x_{n+1} - \rho^*\| + 2\delta_n\langle \rho^*,\ \rho^* - x_{n+1}\rangle\Big].
\end{aligned} \tag{3.39}
$$

By the use of expressions (3.35), (3.39) and Lemma 2.4, we can derive that $\|x_n - \rho^*\| \to 0$ as $n \to +\infty$.

Case 2: Assume that there is a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - \rho^*\| \leq \|x_{n_{i+1}} - \rho^*\|, \ \forall\, i \in \mathbb{N}.$$

Then, by Lemma 2.5, there exists a sequence $\{m_k\} \subset \mathbb{N}$ ($\{m_k\} \to \infty$), such that

$$\|x_{m_k} - \rho^*\| \leq \|x_{m_{k+1}} - \rho^*\| \quad \text{and} \quad \|x_k - \rho^*\| \leq \|x_{m_{k+1}} - \rho^*\|, \ \forall\, k \in \mathbb{N}. \tag{3.40}$$

By the use of expression (3.27), we have

$$\begin{aligned}
&\gamma_{m_k}(1-\delta_{m_k})\Big[\Big(1 - \frac{\mu\chi_{m_k}}{\chi_{m_k+1}}\Big)\|x_{m_k} - y_{m_k}\|^2 + \Big(1 - \frac{\mu\chi_{m_k}}{\chi_{m_k+1}}\Big)\|z_{m_k} - y_{m_k}\|^2\Big] \\
&\leq \|x_{m_k} - \rho^*\|^2 + \delta_{m_k}K_2 - \|x_{m_k+1} - \rho^*\|^2.
\end{aligned} \tag{3.41}$$

Due to $\delta_{m_k} \to 0$, we can deduce the following:

$$\lim_{n\to\infty} \|x_{m_k} - y_{m_k}\| = \lim_{n\to\infty} \|z_{m_k} - y_{m_k}\| = 0. \tag{3.42}$$

It continues from that

$$\begin{aligned}
\big\|x_{m_k+1} - x_{m_k}\big\| &= \big\|(1-\gamma_{m_k}-\delta_{m_k})x_{m_k} + \gamma_{m_k}z_{m_k} - x_{m_k}\big\| \\
&= \big\|x_{m_k} - \delta_{m_k}x_{m_k} + \gamma_{m_k}z_{m_k} - \gamma_{m_k}x_{m_k} - x_{m_k}\big\| \\
&\leq \gamma_{m_k}\big\|z_{m_k} - x_{m_k}\big\| + \delta_{m_k}\big\|x_{m_k}\big\| \longrightarrow 0.
\end{aligned} \tag{3.43}$$

By using similar argument as in Case 1, we get

$$\limsup_{k\to\infty}\langle \rho^*, x_{m_k+1} - \rho^*\rangle \leq 0. \tag{3.44}$$

By the use of expressions (3.39) and (3.40), we have

$$\begin{aligned}
&\big\|x_{m_k+1} - \rho^*\big\|^2 \\
&\leq (1-\delta_{m_k})\big\|x_{m_k} - \rho^*\big\|^2 \\
&\quad + \delta_{m_k}\Big[2\gamma_{m_k}\big\|z_{m_k} - x_{m_k}\big\|\big\|x_{m_k+1} - \rho^*\big\| + 2\delta_{m_k}\big\langle \rho^*,\, \rho^* - x_{m_k+1}\big\rangle\Big] \\
&\leq (1-\delta_{m_k})\big\|x_{m_{k+1}} - \rho^*\big\|^2 \\
&\quad + \delta_{m_k}\Big[2\gamma_{m_k}\big\|z_{m_k} - x_{m_k}\big\|\big\|x_{m_k+1} - \rho^*\big\| + 2\delta_{m_k}\big\langle \rho^*,\, \rho^* - x_{m_k+1}\big\rangle\Big].
\end{aligned} \tag{3.45}$$

It follows that

$$\big\|x_{m_k+1} - \rho^*\big\|^2 \leq 2\gamma_{m_k}\big\|z_{m_k} - x_{m_k}\big\|\big\|x_{m_k+1} - \rho^*\big\| + 2\delta_{m_k}\big\langle \rho^*,\, \rho^* - x_{m_k+1}\big\rangle. \tag{3.46}$$

Since $\delta_{m_k} \to 0$ and $\big\|x_{m_k} - \rho^*\big\|$ is bounded, (3.44) and (3.46), yield

$$\|x_{m_k+1} - \rho^*\|^2 \to 0, \ \text{as } k \to \infty. \tag{3.47}$$

This implies that

$$\lim_{n\to\infty} \|x_k - \rho^*\|^2 \le \lim_{n\to\infty} \|x_{m_k+1} - \rho^*\|^2 \le 0. \tag{3.48}$$

As a result, $x_n \to \rho^*$ and the desired result will be obtained. □

4. Applications

In this section, we extracted the results from our main proposed results to figure out variational inequalities. In the last few years, variational inequalities have drawn a considerable amount of attention from both researchers and readers. It is well established that variational inequalities deal with a broad variety of topics in partial differential equations, optimal control, optimization techniques, applied mathematics, engineering, finance, operational science.

The variational inequality problem for an operator $\mathcal{A} : \mathbb{H} \to \mathbb{H}$ is described as follows:

$$\text{Find } \rho^* \in \mathbb{C} \ \text{ such that } \ \big\langle \mathcal{A}(\rho^*), y - \rho^* \big\rangle \ge 0, \ \forall\, y \in \mathbb{C}. \tag{VIP}$$

We consider the following conditions to study variational inequalities.

($\mathcal{A}1$) A solution set of the problem (VIP) indicate by $VI(\mathcal{A}, \mathbb{C})$ is nonempty.

($\mathcal{A}2$) $\mathcal{A} : \mathbb{H} \to \mathbb{H}$ is pseudomonotone, that is, if $\big\langle \mathcal{A}(x), y - x \big\rangle \ge 0$, then

$$\big\langle \mathcal{A}(y), x - y \big\rangle \le 0, \ \forall\, x, y \in \mathbb{C}.$$

($\mathcal{A}3$) $\mathcal{A} : \mathbb{H} \to \mathbb{H}$ is *Lipschitz continuous*, that is, if there exits a constant $L > 0$ such that

$$\|\mathcal{A}(x) - \mathcal{A}(y)\| \le L\|x - y\|, \ \forall\, x, y \in \mathbb{C}.$$

($\mathcal{A}4$) $\mathcal{A} : \mathbb{H} \to \mathbb{H}$ is *sequentially weakly continuous*, that is, $\{\mathcal{A}(x_n)\}$ converges weakly to $\mathcal{A}(x)$ for every weakly convergent sequence $\{x_n\}$ to x.

On the other hand, we have also developed the results to deal with fixed point problems from our main results. The existence of a solution to a theoretical or real-world problem should be analogous to the existence of a fixed point for an appropriate map or operator. There is, therefore, a great deal of importance to fixed point theorems in several fields of mathematics, engineering and science. In many cases, it is not difficult to find an exact solution; therefore, it is crucial to create effective techniques to approximate the desired result.

The fixed point problem for an operator $\mathcal{B} : \mathbb{H} \to \mathbb{H}$ is defined as follows:

$$\text{Find } \rho^* \in \mathbb{C} \text{ such that } \mathcal{B}(\rho^*) = \rho^*. \tag{FPP}$$

The following conditions are considered to solve the fixed point problems:

($\mathcal{B}1$) The solution set of the problem (FPP) denoted by $Fix(\mathcal{B}, \mathbb{C})$ is nonempty.
($\mathcal{B}2$) $\mathcal{B} : \mathbb{C} \to \mathbb{C}$ is a *κ-strict pseudo-contraction* [7] on $\mathbb{C}$, that is,

$$\|\mathcal{B}x - \mathcal{B}y\|^2 \leq \|x - y\|^2 + \kappa\|(x - \mathcal{B}x) - (y - \mathcal{B}y)\|^2, \ \forall\, x, y \in \mathbb{C}.$$

($\mathcal{B}3$) $\mathcal{B} : \mathbb{H} \to \mathbb{H}$ is *weakly sequentially continuous.*

Corollary 4.1. *Assume that an operator $\mathcal{A} : \mathbb{C} \to \mathbb{H}$ satisfies the conditions ($\mathcal{A}1$)-($\mathcal{A}4$). Let $x_0 \in \mathbb{C}$, $\chi_0 > 0$, $\{\gamma_n\} \subset (a, b) \subset (0, 1 - \delta_n)$ and $\{\delta_n\} \subset (0, 1)$ such that*

$$\lim_{n\to\infty} \delta_n = 0 \quad \textit{and} \quad \sum_{n=1}^{+\infty} \delta_n = +\infty.$$

Consider the iterative sequence as follows:

$$\begin{cases} y_n = P_{\mathbb{C}}(x_n - \chi_n \mathcal{A}(x_n)), \\ z_n = P_{\mathbb{H}_n}(x_n - \chi_n \mathcal{A}(y_n)), \\ x_{n+1} = (1 - \gamma_n - \delta_n)x_n + \gamma_n z_n, \end{cases}$$

where $\mathbb{H}_n = \{z \in \mathbb{H} : \langle x_n - \chi_n \mathcal{A}(x_n) - y_n, z - y_n \rangle \leq 0\}$.
Compute

$$\chi_{n+1} = \begin{cases} \min\left\{\chi_n, \ \frac{\mu\|x_n - y_n\|^2 + \mu\|z_n - y_n\|^2}{2\left[\left\langle \mathcal{A}(x_n) - \mathcal{A}(y_n), z_n - y_n \right\rangle\right]}\right\} \\ \qquad \textit{if} \quad \left\langle \mathcal{A}(x_n) - \mathcal{A}(y_n), z_n - y_n \right\rangle > 0, \\ \chi_n, \quad \textit{others.} \end{cases}$$

Then, $\{x_n\}$ strongly converges to $\rho^* \in VI(\mathcal{A}, \mathbb{C})$.

Corollary 4.2. *Assume that $\mathcal{B} : \mathbb{C} \to \mathbb{C}$ is a mapping satisfying the conditions ($\mathcal{B}1$)-($\mathcal{B}3$) and $Fix(\mathcal{B}, \mathbb{C}) \neq \emptyset$. Let $x_0 \in \mathbb{C}$, $\chi_0 > 0$, $\{\gamma_n\} \subset (a, b) \subset (0, 1 - \delta_n)$ and $\{\delta_n\} \subset (0, 1)$ such that*

$$\lim_{n\to\infty} \delta_n = 0 \quad \textit{and} \quad \sum_{n=1}^{+\infty} \delta_n = +\infty.$$

Consider the iterative sequence update as follows:

$$\begin{cases} y_n = P_{\mathbb{C}}\left[x_n - \chi_n(x_n - \mathcal{B}(x_n))\right], \\ z_n = P_{\mathbb{H}_n}\left[x_n - \chi_n(y_n - \mathcal{B}(y_n))\right], \\ x_{n+1} = (1 - \gamma_n - \delta_n)x_n + \gamma_n z_n, \end{cases}$$

where $\mathbb{H}_n = \{z \in \mathbb{H} : \langle (1 - \chi_n)x_n + \chi_n \mathcal{B}(x_n) - y_n, z - y_n \rangle \leq 0\}$.

Compute

$$\chi_{n+1} = \begin{cases} \min\left\{\chi_n,\ \frac{\mu\|x_n-y_n\|^2+\mu\|z_n-y_n\|^2}{2\left[\left\langle (x_n-y_n)-[\mathcal{B}(x_n)-\mathcal{B}(y_n)], z_n-y_n\right\rangle\right]}\right\} \\ \quad \text{if} \quad \left\langle (x_n-y_n)-[\mathcal{B}(x_n)-\mathcal{B}(y_n)], z_n-y_n\right\rangle > 0, \\ \chi_n, \quad \text{others.} \end{cases}$$

Then, $\{x_n\}$ *converges strongly to* $Fix(\mathcal{B}, \mathbb{C})$.

5. Numerical illustrations

In this section, we include 2 numerical test problems and explain the numerical behaviour of designed method in comparisons to some related works in the literature.

Example 5.1. Suppose that the set $\mathbb{C}$ is defined by

$$\mathbb{C} := \{x \in \mathbb{R}^m : -10 \le x_i \le 10\}$$

and $f : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ is considered as follows

$$f(x, y) = \langle Mx + Ny + r, y - x\rangle, \ \forall\, x, y \in \mathbb{C},$$

where $r \in \mathbb{R}^m$ and M, N are matrices of order m and $c_1 = c_2 = \frac{1}{2}\|M - N\|$ (see [27] for details). Two matrices M, N are taken as follows:

$$M = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad N = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad r = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}.$$

Numerical results are presented in Figures 1-5 and Table 1 by letting $y_{-1} = (1, 1, 1, 1, 1)^T$ and $x_0 = y_0$ and $TOL = 10^{-6}$. The control parameters are taken in the following way:

(i) $\chi = \frac{1}{5c_1}$, $\gamma_n = \frac{1}{40(n+2)}$ and $D_n = \|x_n - y_n\|^2$ for Algorithm 3.2 (Alg3.2) in [10];

(ii) Algorithm 4.1 in [11] (Alg4.1): $\chi_0 = 0.55$, $\mu = 0.45$, $\gamma_n = \frac{1}{(n+1)^{0.5}}$ and $D_n = \max\left\{\|x_{n+1} - y_n\|^2, \|x_n - y_n\|^2\right\}$;

(iii) $\chi_0 = 0.55$, $\mu = 0.45$, $\delta_n = \frac{1}{60(n+2)}$, $\gamma_n = \frac{6}{10}(1-\delta_n)$ and $D_n = \|x_n - y_n\|^2$ for Algorithm 1 (Alg1).

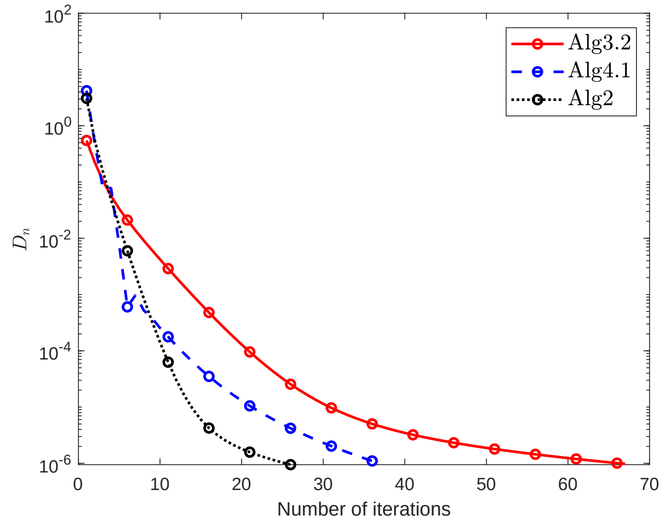

FIGURE 1. Numerical comparison between Algorithm 1 with Algorithm 4.1 in [11] and Algorithm 3.2 in [10] with $(1, 0, 1, 0, 1)^T$.

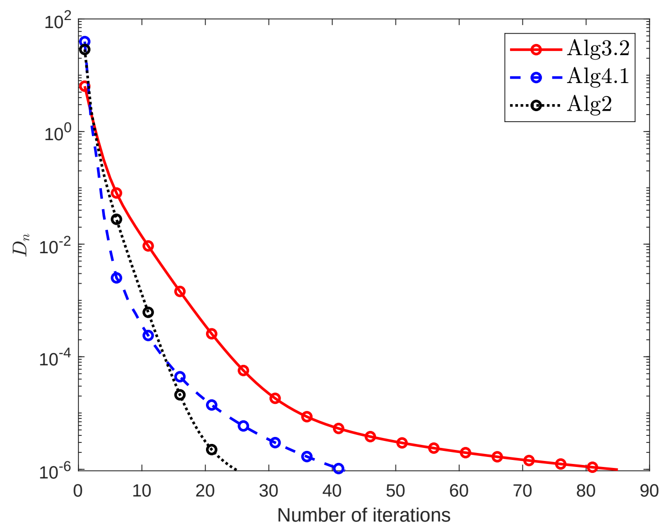

FIGURE 2. Numerical comparison between Algorithm 1 with Algorithm 4.1 in [11] and Algorithm 3.2 in [10] with $(2, 3, 0, 3, 2)^T$.

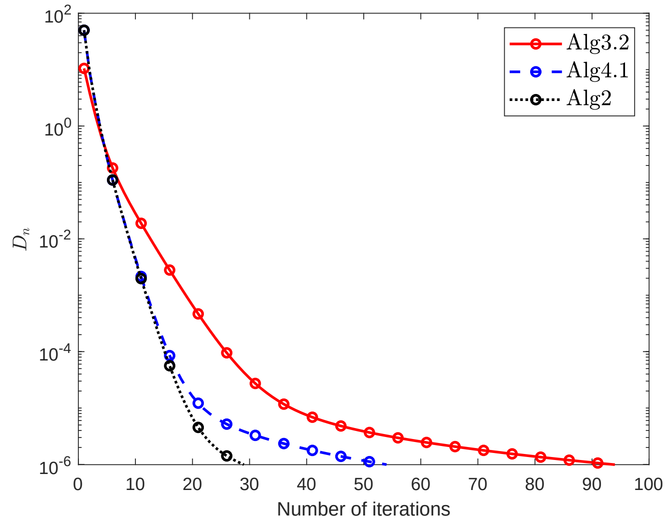

FIGURE 3. Numerical comparison between Algorithm 1 with Algorithm 4.1 in [11] and Algorithm 3.2 in [10] with $(4, 2, -1, 3, 5)^T$.

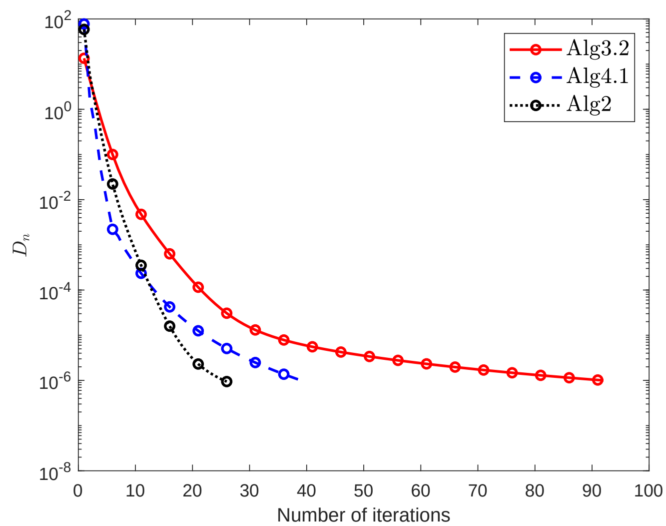

FIGURE 4. Numerical comparison between Algorithm 1 with Algorithm 4.1 in [11] and Algorithm 3.2 in [10] with $(-1, -1, 3, 4, -5)^T$.

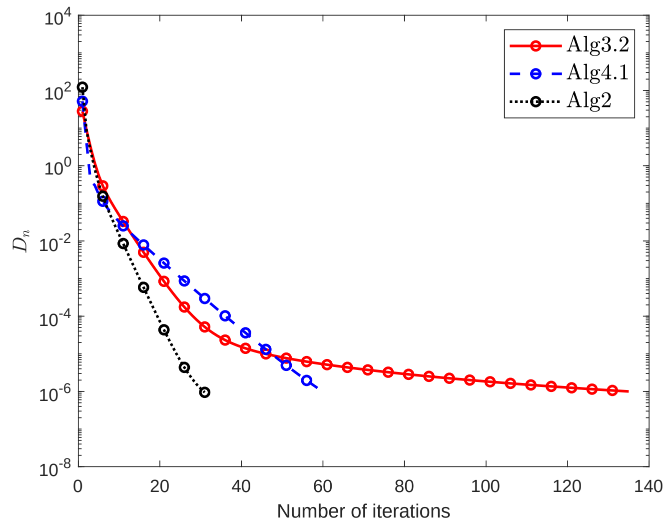

FIGURE 5. Numerical comparison between Algorithm 1 with Algorithm 4.1 in [11] and Algorithm 3.2 in [10] with $(-5,1,3,9,-1)^T$.

TABLE 1. Numerical values for Figures 1-5.

	Number of Iterations			Execution Time in Seconds		
x_0	**Alg2**	**Alg4.1**	**Alg1**	**Alg2**	**Alg4.1**	**Alg1**
$(1,0,1,0,1)^T$	67	37	26	0.5769519	0.3086849	0.229005400
$(2,3,0,3,2)^T$	85	42	25	0.7270275	0.3868875	0.2305259
$(4,2,-1,3,5)^T$	94	54	29	0.8391188	0.463802	0.2518623
$(-1,-1,3,4,-5)^T$	92	40	26	0.8199833	0.3376767	0.24075900
$(-5,1,3,9,-1)^T$	135	60	31	1.2119099	0.4957612	0.2694812

Example 5.2. Let a bifunction $f : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ is defined by

$$f(x,y) = \sum_{i=2}^{5} (y_i - x_i)\|x\|, \ \forall\, x, y \in \mathbb{R}^5,$$

where $\mathbb{C} \subset \mathbb{R}^5$ is taken as follows:

$$\mathbb{C} = \big\{(x_1, \cdots, x_5) : x_1 \geq -1, x_i \geq 1, i = 2, \cdots, 5\big\}.$$

Then, f is Lipschitz-like continuous with $c_1 = c_2 = 2$, and satisfies the conditions (C1)-(C4). All numerical results are reported in Table 2-4 by letting different initial points and $TOL = 10^{-3}$. The control parameters are taken in the following way:

(i) $\chi = \frac{1}{4c_1}$, $\gamma_n = \frac{1}{10(n+2)}$ and $D_n = \|x_n - y_n\|^2$ for Algorithm 3.2 (Alg3.2) in [10];

(ii) Algorithm 4.1 in [11] (Alg4.1): $\chi_0 = 0.45$, $\mu = 0.75$, $\gamma_n = \frac{1}{(n+1)^{0.6}}$ and $D_n = \max\left\{\|x_{n+1} - y_n\|^2, \|x_n - y_n\|^2\right\}$;

(iii) $\chi_0 = 0.45$, $\mu = 0.75$, $\delta_n = \frac{1}{10(n+2)}$, $\gamma_n = \frac{7}{10}(1-\delta_n)$ and $D_n = \|x_n - y_n\|^2$ for Algorithm 1 (Alg1).

Table 2. Example 5.2: Numerical results of Algorithm 3.2 in [10] while $x_0 = (5, 2, 1, 3, 4)^T$.

It.(n)	x_1	x_2	x_3	x_4	x_5
1	4.99999	1.49676	1.00000	2.49676	3.49676
2	4.99999	1.06668	1.00000	2.04283	3.04283
3	4.99999	1.05000	1.00000	1.61835	2.61835
4	4.99999	1.04000	1.00000	1.21606	2.21606
5	5.00000	1.03333	1.00000	1.06666	1.82696
6	4.99999	1.02857	1.00000	1.05714	1.44679
7	5.00000	1.02500	1.00000	1.05000	1.07534
8	4.99999	1.02222	1.00000	1.04444	1.06666
⋮	⋮	⋮	⋮	⋮	⋮
71	4.99999	1.00277	1.00000	1.00555	1.00833
72	4.99999	1.00273	1.00000	1.00547	1.00821
73	4.99999	1.00270	1.00000	1.00540	1.00810
74	4.99999	1.00266	1.00000	1.00533	1.00800
75	4.99999	1.00263	1.00000	1.00526	1.00789
CPU time is seconds	2.206290				

Table 3. Example 5.2: Numerical results of Algorithm 4.1 in [11] while $x_0 = (5, 2, 1, 3, 4)^T$.

Iter (n)	x_1	x_2	x_3	x_4	x_5
1	4.99999	1.46880	1.00000	2.46880	3.46880
2	4.99999	1.03572	1.00000	1.98570	2.98570
3	4.99999	1.02777	1.00000	1.53336	2.53336
4	4.99999	1.02272	1.00000	1.10451	2.10449
5	4.99999	1.01923	1.00000	1.03846	1.68846
6	4.99999	1.01666	1.00000	1.03333	1.28191
7	5.00000	1.01470	1.00000	1.02941	1.04412
8	4.99999	1.01315	1.00000	1.02631	1.03947
⋮	⋮	⋮	⋮	⋮	⋮
43	4.99999	1.00280	1.00000	1.00561	1.00842
44	4.99999	1.00274	1.00000	1.00549	1.00824
45	4.99999	1.00268	1.00000	1.00537	1.00806
46	4.99999	1.00263	1.00000	1.00526	1.00789
47	4.99999	1.00257	1.00000	1.00515	1.00773
CPU time is seconds	1.395494				

TABLE 4. Example 5.2: Numerical results of Algorithm 1 while $x_0 = (5, 2, 1, 3, 4)^T$.

It.(n)	x_1	x_2	x_3	x_4	x_5
1	4.97500	1.19400	0.99500	1.39300	1.6223
2	4.95841	1.03533	0.99567	1.07500	1.1207
3	4.94602	1.00455	0.99663	1.01246	1.0215
4	4.93612	0.99890	0.99732	1.00048	1.0023
5	4.92790	0.99811	0.99780	0.99843	0.9987
6	4.92086	0.99819	0.99813	0.99825	0.9983
7	4.91471	0.99839	0.99837	0.99840	0.9984
8	4.90925	0.99856	0.99856	0.99857	0.9985
⋮	⋮	⋮	⋮	⋮	⋮
22	4.86499	0.99945	0.999450	0.99945	0.9994
23	4.86296	0.99947	0.999474	0.99947	0.9994
24	4.86102	0.99949	0.999495	0.99949	0.9994
25	4.85915	0.99951	0.999515	0.99951	0.9995
26	4.85735	0.99953	0.999533	0.99953	0.9995
CPU time is seconds	0.697144				

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