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EXISTENCE AND EXPONENTIAL STABILITY OF NEUTRAL STOCHASTIC PARTIAL INTEGRODIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION WITH IMPULSIVE EFFECTS

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ABSTRACT. The purpose of this work is to study the existence and continuous dependence on neutral stochastic partial integrodifferential equations with impulsive effects, perturbed by a fractional Brownian motion with Hurst parameter $\mathtt{H} \in (\frac{1}{2}, 1)$. We use the theory of resolvent operators developed in Grimmer [19] to show the existence of mild solutions. Further, we establish a new impulsive-integral inequality to prove the exponential stability of mild solutions in the mean square moment. Finally, an example is presented to illustrate our obtained results.

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1. Introduction

Stochastic differential equations have been investigated as mathematical models to describe the dynamical behavior of real life phenomena. It is essential to take into account the environmental disturbances as well as the time delay while constructing realistic models in the area of engineering, biology, etc. The qualitative behavior of stochastic delay differential equations, regarding the stability, existence and uniqueness of solutions, has been studied by many investigators, (see [1, 2, 3, 4, 8, 9, 12] and references therein). In particular the existence of the exponential and asymptotic stability of mild solutions of stochastic integrodifferential equations with delays has been established [16, 17].

On the other hand, a fractional Brownian motion (fBm) is a Gaussian stochastic process, which varies pointedly from semi-martingales and a standard

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Brownian motion to other processes usually utilized in the theory of stochastic processes. An fBm depends on the Hurst index $\mathtt{H} \in (0, 1)$, the parameter introduced by Kolmogorov [21]. We refer to [24] for more detail on fBm. As a centered Gaussian process, it is examined by its stationary increments and the medium- or long-memory property. The fBm reduces to standard Brownian motion when $\mathtt{H} = \frac{1}{2}$. However, when $\mathtt{H} \neq \frac{1}{2}$, the fBm acts in a entirely different way, that is, it is neither a Markov process nor a semi-martingale. Recently, the theory of differential equations driven by a fBm has been investigated intensively by many researchers (see [5]-[18] and references therein).

This model gives arbitrage opportunities. In 2003, Cheridito [13] proved that even the market allows for arbitrage strategies, these strategies cannot be constructed in practice. In fact, he proved that if there is a minimum amount of time between transactions, the arbitrage opportunities disappear. In 2006, Guasoni [20] proved that the arbitrage opportunities also disappear under transaction costs. To achieve an arbitrage, at some point t_0 we have to start trading. This decision generates a transaction cost that must be recovered at a later time, and this is possible only if the asset price moves enough in the future. Hence, if at all times there is a remote possibility of arbitrary small price changes, then downside risk cannot be eliminated, and arbitrage is impossible. The above results by Cheridito and Guasoni open a new scenario, where the fBm can be appropriate for stock price modeling if we assume that the non-existence of arbitrage strategies is not due to the market, but to the existence of restrictions on the trading strategies. Nevertheless, the dependence of the implied volatility on time to maturity (term structure) is not well explained by classical stochastic volatility models. In practice, the decreasing of the small amplitude when the time to maturity increases, turns out to be much slower than it goes according to stochastic volatility models. With this aim, in 2012, Comte, Coutin, and Renault [14] have proposed stochastic volatility models based on the fBm. This model allows us to explain the observed long-time behavior of the implied volatility. Thus the fractional stochastic volatility models allow us to explain the long-time behavior of the implied volatility.

Motivated by this works, we will generalize the existence and stability of neutral impulsive stochastic integrodifferential equations under Lipschitz condition. Moreover, we use the theory of resolvent operator and through the continuous dependence on the initial values by means of Corollary of Bihari's inequality.
In order to fill this gap, this paper studies the existence and exponential stability of mild solutions of the following neutral stochastic partial integrodifferential equations driven by fractional Brownian motion with impulsive effects of the form:

$$
\begin{aligned}
& d\left[u(t) + p(t, u_t)\right] \\
& = A\left[u(t) + p(t, u_t)\right] dt + \left[\int_0^t B(t-s)\left[u(s) + p(s, u_s)\right] ds\right.
\end{aligned}
$$

$$+ f(t,u_t)\Big] dt + \sigma(t) d\mathtt{B}^{\mathtt{H}}_{\mathtt{Q}}(t),\ t \in [0,T],\ t \neq t_k, \tag{1}$$

$$\Delta u(t_k) = u(t_k^+) - u(t_k^-) = I_k(u(t_k)),\ t = t_j,\ k = 1,2,..., \tag{2}$$

$$u(t) = \varphi(t),\ -\tau \leq t \leq 0, \tag{3}$$

where A is the infinitesimal generator of a strongly continuous semigroup $(T(t))$, $t \geq 0$ of bounded linear operators in a Hilbert space $\mathtt{Y}$; $\mathtt{B}^H_Q$ is a fractional Brownian motion with Hurst parameter $\mathtt{H} \in (\frac{1}{2}, 1)$ on a real and separable Hilbert space $\mathtt{Y}$. $f, p : [0,+\infty) \times \mathtt{X} \to \mathtt{X}$, $\sigma : [0,+\infty) \to \mathcal{L}^0_Q(\mathtt{Y},\mathtt{X})$, $I_k : \mathtt{X} \to \mathtt{X}$ are appropriate functions specified later. The impulsive moments t_k satisfies $0 = t_0 < t_1 < \cdots, \lim_{k\to\infty} t_k = \infty$, $u(t_k^+)$ and $u(t_k^+)$ denote the right and left limits of $u(t)$ at time t_k. And $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ represents the jump in the state u at time t_k, where I_k determines the size of the jump.

2. Preliminaries

Let us start with some basic facts on fractional Brownian motion (fBm) and Wiener integral with respect to fBm. Additionally, we introduce the resolvent operator of infinitesimal generator which is the basis of our study.

A two-sided one-dimensional fBm $\beta^{\mathtt{H}} = \beta^{\mathtt{H}}(t)$ with Hurst parameter $\mathtt{H} \in (0,1)$ is a centered Gaussian process with the covariance function

$$\mathtt{R}_{\mathtt{H}}(s,t) \quad = \quad \mathbf{E}\big[\beta^{\mathtt{H}}(t)\beta^{\mathtt{H}}(s)\big] = \frac{1}{2}(t^{2\mathtt{H}} + s^{2\mathtt{H}} - |t-s|^{2\mathtt{H}})$$

In this paper, we consider, $\mathtt{H} > \frac{1}{2}$, then $\beta^{\mathtt{H}}(t)$ has the following representation

$$\beta^{\mathtt{H}}(t) \quad = \quad \int_0^t K_{\mathtt{H}}(t,s) d\beta(s),$$

where $\beta(s)$ is a standard Brownian motion and $\mathtt{K}(t,s)$ is the kernel given by

$$\mathtt{K}_{\mathtt{H}}(t,s) \quad = \quad c_{\mathtt{H}} s^{\frac{1}{2}-\mathtt{H}} \int_s^t (u-s)^{\mathtt{H}-\frac{3}{2}} u^{\mathtt{H}-\frac{1}{2}} du,\ t \geq s,$$

which $c_{\mathtt{H}}$ is a non-negative constant with respect to $\mathtt{H}$.

For the deterministic function $\varphi \in \mathcal{L}^2([0,T])$, the fractional Wiener integral of φ with respect to $\beta^{\mathtt{H}}$ is defined by

$$\int_0^T \varphi(s) d\beta^{\mathtt{H}}(s) \quad = \quad \int_0^T \mathtt{K}^*_{\mathtt{H}} \varphi(s) d\beta(s),$$

where $(\mathtt{K}^*_{\mathtt{H}}\varphi)(s) = \displaystyle\int_0^t \varphi(t) \frac{\partial \mathtt{K}(t,s)}{\partial t} dt.$

Let $\mathtt{X}$ and $\mathtt{Y}$ be two real separable Hilbert spaces. $\mathcal{L}(\mathtt{Y},\mathtt{X})$ denotes the space of all bounded linear operators from $\mathtt{Y}$ into $\mathtt{X}$. Let $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space satisfying the usual conditions. $\mathbf{E}(\cdot)$ denotes the mathematical expectation with respect to $\mathbb{P}$, $\mathtt{Q}$ is the non-negative self-adjoint operator in $\mathcal{L}(\mathtt{Y},\mathtt{Y})$, $\mathcal{L}^0_{\mathtt{Q}}$ is symbolized the space of all $\gamma \in \mathcal{L}(\mathtt{Y},\mathtt{X})$ such that $\gamma \mathtt{Q}^{\frac{1}{2}}$ is a Hilbert-Schmidt operator and the norm is given by

$$\|\gamma\|^2_{\mathcal{L}^0_{\mathtt{Q}}(\mathtt{Y},\mathtt{X})} \quad = \quad \operatorname{tr}(\gamma \mathtt{Q} \gamma^*).$$

Then γ is called $\mathtt{Q}$-Hilbert-Schmidt operator mapping from $\mathtt{Y}$ into $\mathtt{X}$. e_n $(n = 1,2,...)$ denote a complete orthonormal basis in $\mathtt{Y}$ and $\mathtt{Q} \in \mathcal{L}(\mathtt{Y},\mathtt{X})$ is an operator defined by $\mathtt{Q}e_n = \lambda_n e_n$ are non-negative real numbers. We define the infinite dimensional fBm on $\mathtt{Y}$ with covariance $\mathtt{Q}$ as

$$\mathtt{B}^{\mathtt{H}}_{\mathtt{Q}}(t) = \sum_{n=1}^{\infty} \beta^{\mathtt{H}}_n(t)\mathtt{Q}^{\frac{1}{2}} e_n = \sum_{n=1}^{\infty} \sqrt{\sigma_n}\beta^{\mathtt{H}}_n(t) e_n, \ t \geq 0.$$

where $\beta^{\mathtt{H}}_n(t)$ $(n = 1,2...)$ are real, independent fBms. The process $\mathtt{B}^{\mathtt{H}}_{\mathtt{Q}}(t)$ is called by $\mathtt{Y}$-valued $\mathtt{Q}$-fBm.

Now, we introduce the definition of the fractional Wiener integral of the function $\varphi : [0,T] \to \mathcal{L}^0_{\mathtt{Q}}$ with respect to $\mathtt{Q}$-fBm as follows

$$\begin{aligned} \int_0^t \varphi(s) d\mathtt{B}^{\mathtt{H}}_{\mathtt{Q}}(s) &= \sum_{n=1}^{\infty} \int_0^t \varphi(s)\mathtt{Q}^{\frac{1}{2}} e_n d\beta^{\mathtt{H}}_n(s) \\ &= \sum_{n=1}^{\infty} \int_0^t \left(\mathtt{K}^*_{\mathtt{H}}(\varphi \mathtt{Q}^{\frac{1}{2}} e_n)\right)(s) d\beta_n(s), \end{aligned}$$

where β_n is the standard Brownian motion with respect to $\beta^{\mathtt{H}}_n$.

Lemma 2.1. *If $\varphi : [0,T] \to \mathcal{L}^0_{\mathtt{Q}}(\mathtt{Y},\mathtt{X})$ satisfies $\int_0^t \|\varphi(s)\|^2_{\mathcal{L}^0_{\mathtt{Q}}} ds < \infty$. Then the integral $\int_0^t \varphi(s) d\mathtt{B}^{\mathtt{H}}_{\mathtt{Q}}(s)$ is well defined as an $\mathtt{X}$-valued random variable and we have*

$$\mathbf{E}\left\|\int_0^t \varphi(s) d\mathtt{B}^{\mathtt{H}}_{\mathtt{Q}}(s)\right\|^2 \quad \leq \quad 2\mathtt{H}t^{2\mathtt{H}-1} \int_0^t \|\varphi(s)\|^2_{\mathcal{L}^0_{\mathtt{Q}}} ds.$$

For more details for fractional Brownian motion and fractional Wiener integral, one can see [25, 26].

In the present section, we recall some definitions, notations and properties needed in the sequel. In what follows, $\mathtt{X}$ will denote a Banach space, A and $B(t)$ are closed linear operators on $\mathtt{X}$. $\mathtt{Y}$ represents the Banach space $D(A)$, the domain of operator A, equipped with the graph norm

$$|y|_{\mathtt{Y}} := |Ay| + |y| \quad \text{for } \ y \in \mathtt{Y}.$$

The notation $C([0,+\infty);\mathtt{Y})$ stands for the space of all continuous functions from $[0,+\infty)$ into $\mathtt{Y}$. We then consider the following Cauchy problem

$$\begin{cases} v'(t) &= Av(t) + \int_0^t B(t-s)v(s)ds \ \text{ for } \ t \geq 0, \\ v(0) &= v_0 \in \mathtt{X}. \end{cases} \tag{4}$$

Definition 2.2. ([19]) A resolvent operator for Equation (4) is a bounded linear operator valued function $R(t) \in \mathcal{L}(\mathtt{X})$ for $t \geq 0$, satisfying the following properties:

(i) $R(0) = I$ and $\|R(t)\| \leq Me^{\beta t}$ for some constants M and β.
(ii) For each $x \in \mathtt{X}$, $R(t)x$ is strongly continuous for $t \geq 0$.
(iii) For $x \in \mathtt{Y}$, $R(\cdot)x \in C^1([0,+\infty);\mathtt{X}) \cap C([0,+\infty);\mathtt{Y})$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds, \text{for } t \geq 0. \end{aligned} \tag{5}$$

The resolvent operator plays an important role to study the existence of solutions and to establish a variation of constants formula for nonlinear systems. For this reason, we need to know when the linear system (4) has a resolvent operator. For more details on resolvent operator, we refer to [19].
In what follows we suppose the following assumptions:
(H1) A is the infinitesimal generator of a C_0-semigroup on $\mathtt{X}$.
(H2) For all $t \geq 0$, $B(t)$ is a continuous linear operator from $(\mathtt{Y}, |\cdot|_{\mathtt{Y}})$ into $(\mathtt{X}, |\cdot|_{\mathtt{X}})$. Moreover, there exists an integrable function $c : [0,+\infty) \to \mathbb{R}^+$ such that for any $y \in \mathtt{Y}$, $t \mapsto B(t)k$ belongs to $W^{1,1}([0,+\infty),\mathtt{X})$ and

$$\left|\frac{d}{dt}B(t)(t)y\right|_{\mathtt{X}} \leq c(t)|y|_{\mathtt{Y}} \text{ for } y \in \mathtt{Y} \text{ and } t \geq 0.$$

Theorem 2.3. *([19]) Assume that hypotheses* **(H1)** *and* **(H2)** *hold. Then Equation (4) admits a resolvent operator* $(R(t))_{t\geq 0}$.

Theorem 2.4. *([23]) Assume that hypotheses* **(H1)** *and* **(H2)** *hold. Let* $R(t)$ *be a compact operator for* $t > 0$. *Then, the corresponding resolvent operator* $R(t)$ *of Equation (4) is continuous for* $t > 0$ *in the operator norm, namely, for all* $t_0 > 0$, *it holds that* $\lim_{h\to 0} \|R(t_0 + h) - R(t_0)\| = 0$.

In the sequel, we recall some results on the existence of solutions for the following integrodifferential equation

$$\begin{cases} v'(t) &= Av(t) + \displaystyle\int_0^t B(t-s)v(s)ds + q(t) \text{ for } t \geq 0, \\ v(0) &= v_0 \in \mathtt{X}, \end{cases} \tag{6}$$

where $q : [0,+\infty[\to \mathtt{X}$ is a continuous function.

Definition 2.5. ([19]) A continuous function $v : [0,+\infty) \to \mathtt{X}$ is said to be a strict solution of Equation (6) if

(i) $v \in C^1([0,+\infty);\mathtt{X}) \cap C([0,+\infty);\mathtt{Y})$,
(ii) v satisfies Equation (6) for $t \geq 0$.

Remark 2.1. From this definition we deduce that $v(t) \in D(A)$, and the function $B(t-s)v(s)$ is integrable, for all $t > 0$ and $s \in [0,+\infty)$.

Theorem 2.6. *([19]) Assume that **(H1)-(H2)** hold. If v is a strict solution of Equation (6), then the following variation of constants formula holds*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds \quad for \quad t \geq 0. \tag{7}$$

3. Main results

Definition 3.1. If $u : [-\tau, T] \to \mathtt{X}$ is a stochastic process and
(i) $u(t)$ is measurable, $\mathscr{F}_t$-adapted, and has cadlag paths almost surely for all $-\tau \leq t \leq T$.
(ii) $u(t) = \varphi(t),\ -\tau \leq t \leq 0$.
(iii) u satisfies the following integral equation:

$$\begin{aligned} u(t) &= R(t)\left[\varphi(0) + p(0,\varphi)\right] - p(t,u_t) + \int_0^t R(t-s)f(s,u_s)ds \\ &+ \int_0^t R(t-s)\sigma(s)d\mathtt{B}^{\mathtt{H}}_{\mathtt{Q}}(s) + \sum_{0<t_k<t} R(t-t_k)I_k(u(t_k)) \end{aligned} \tag{8}$$

To guarantee the existence and uniqueness of the solution, we impose some hypotheses:

(H3) There exist constants $\lambda > 0$ and $M \geq 1$ such that $\|R(t)\| \leq Me^{-\lambda t}$.

(H4) $f(t,\cdot)$ satisfy the following Lipschitz conditions for all $t \in [0,T]$ and $u, v \in \mathtt{X}$

$$\|f(t,u_t) - f(t,v_t)\|^2 \leq K_f^2 \|u-v\|_t^2$$

for some positive constants K_f^2. We further assume that, for $t \geq 0$, $f(t,0) = k_0$, where $k_0 > 0$ is a constant.

(H5) The function p satisfies

$$\|p(t,u_t) - p(t,v_t)\|^2 \leq K_p^2 \|u-v\|_t^2$$

and $p(t,0) = 0$, for all $t \in [0,T]$, $u, v \in \mathtt{X}$.

(H6) The function p is continuous in the quadratic mean sense:

$$\lim_{t\to s} \|p(t,u_t) - p(t,u_s)\|^2 = 0,\ u \in \mathtt{X}.$$

(H7) A function $\sigma : [0,+\infty) \to \mathcal{L}^0_{\mathtt{Q}}(\mathtt{Y},\mathtt{X})$ satisfies

(i) $\int_0^t \|\sigma(s)\|^2_{\mathcal{L}^0_{\mathtt{Q}}} ds < \infty,\ \ \forall t \in [0,T]$,

(ii) $\sum_{n=1}^{\infty} \left\|\sigma \mathtt{Q}^{1/2} e_n\right\|_{\mathcal{L}([0,T];\mathtt{X})} < \infty$,

(iii) $\sum_{n=1}^{\infty} \left\|\sigma \mathtt{Q}^{1/2} e_n\right\|_{\mathtt{Y}}$ is uniformly convergent for $t \in [0,T]$.

(H8) The function $I_k \in \mathcal{C}(\mathtt{X},\mathtt{X})$ for all $u, v \in \mathtt{X}$,

$\|I_k(u(t_k)) - I_k(v(t_k))\|^2 \leq q_k^2 \|u-v\|_t^2$, where q_j is a constant and $k = 1, 2, ...$

Theorem 3.2. *Suppose that* **(H1)**-**(H7)** *hold. Then for all* $T > 0$*, system (1)-(3) has a unique mild solution on* $[-\tau, T]$ *provided that*

$$\frac{2M^2\sum_{k=1}^{\infty} q_k^2}{(1-k)^2} < 1 \tag{9}$$

where $k = K_p$*.*

Proof. Define the operator $\Theta : \mathscr{B}_T \to \mathscr{B}_T$ by

$$\begin{aligned}
\Theta(u(t)) &= \varphi(t), \text{ for } t \in [-\tau, T],\\
\Theta(u(t)) &= R(t)\left[\varphi(0) + p(0,\varphi)\right] - p(t,u_t) + \int_0^t R(t-s)f(s,u_s)ds\\
&+ \int_0^t R(t-s)\sigma(s)d\mathtt{B}_{\mathtt{Q}}^{\mathtt{H}}(s) + \sum_{0<t_k<t} R(t-t_k)I_k(u(t_k)), \text{ for } t \in [-\tau, T].\\
&= \sum_{i=1}^{5} \Delta_i(t).
\end{aligned}$$

Now, to prove the existence of mild solutions of (1)-(3), it is sufficient to show that Θ has a fixed point.

Step 1: First, we verify that $t \to \Theta(u(t))$ is Cadillac on $[0,T]$. Let $|\gamma|$ be small enough, for $u \in \mathscr{B}_T$, $0 < t < T$, Then

$$\mathbf{E}\left\|(\Theta u)(t+\gamma) - (\Theta u)(t)\right\|_{\mathtt{X}}^2 \leq 5\sum_{i=1}^{5}\mathbf{E}\left\|\Delta_i(t+\gamma) - \Delta_i(t)\right\|_{\mathtt{X}}^2.$$

We can easily see that $\mathbf{E}\left\|\Delta_i(t+\gamma) - \Delta_i(t)\right\|_{\mathtt{X}}^2 \to 0$, $i = 1,2,3$ as $\gamma \to 0$. For the case $i = 5$, Then we have

$$\begin{aligned}
&\mathbf{E}\left\|\Delta_5(t+\gamma) - \Delta_5(t)\right\|^2\\
&\leq 2\mathbf{E}\left\|\sum_{0<t_k<t}\left[R(t+\gamma-t_k) - R(t-t_k)\right]I_k(u(t_k))\right\|^2\\
&+ 2\mathbf{E}\left\|\sum_{t<t_k<t+\gamma}\left[R(t+\gamma-t_k)\right]I_k(u(t_k))\right\|^2\\
&\leq 2\sum_{0<t_k<t}\mathbf{E}\left\|\left[R(t+\gamma-t_k) - R(t-t_k)\right]\right\|^2\left[q_k^2\mathbf{E}\left\|u(t_k)\right\|^2\right]\\
&+ 2\sum_{t<t_k<t+\gamma}\mathbf{E}\left\|\left[R(t+\gamma-t_k)\right]\right\|^2\left[q_k^2\mathbf{E}\left\|u(t_k)\right\|^2\right]\\
&\to 0 \text{ as } |\gamma| \to 0.
\end{aligned}$$

Further, using Lemma 2.1, we get

$$\mathbf{E}\left\|\Delta_4(t+\gamma) - \Delta_4(t)\right\|^2 \leq 2\mathbf{E}\left\|\int_0^t\left[R(t+\gamma-s) - R(t-s)\right]\sigma(s)d\mathtt{B}_{\mathtt{Q}}^{\mathtt{H}}(s)\right\|^2$$

$$\begin{aligned} &+ \quad 2\mathbf{E}\left\|\int_t^{t+\gamma}[R(t+\gamma-s)]\,\sigma(s)d\mathtt{B}^{\mathtt{H}}_{\mathtt{Q}}(s)\right\|^2 \\ &= \quad J_1+J_2. \\ &+ \quad 2\mathtt{H}t^{2\mathtt{H}-1}\int_0^{t+\gamma}\|R(t+\gamma-s)\|^2_{\mathcal{L}_{\mathtt{Q}}}\,ds \end{aligned}$$

By **(H3)** and **(H7)**, we have

$$\begin{aligned} J_1 &\le 2\mathtt{H}t^{2\mathtt{H}-1}\int_0^t\|R(t+\gamma-s)-R(t-s)\sigma(s)\|^2_{\mathcal{L}^0_{\mathtt{Q}}}\,ds \\ &\le 4M^2\mathtt{H}T^{2\mathtt{H}-1}\int_0^t e^{2\lambda s}\left(e^{-2\lambda(t+\gamma)}+e^{-2\lambda t}\right)\|\sigma(s)\|^2_{\mathcal{L}^0_{\mathtt{Q}}}\,ds \\ &\le 4M^2\mathtt{H}T^{2\mathtt{H}-1}\int_0^\infty e^{2\lambda s}\,\|\sigma(s)\|^2_{\mathcal{L}^0_{\mathtt{Q}}}\,ds. \end{aligned}$$

and

$$\begin{aligned} J_2 &\le 2\mathtt{H}t^{2\mathtt{H}-1}\int_0^t\|R(t+\gamma-s)\sigma(s)\|^2_{\mathcal{L}^0_{\mathtt{Q}}}\,ds \\ &\le 2M^2\mathtt{H}T^{2\mathtt{H}-1}\int_0^\infty e^{2\lambda s}\,\|\sigma(s)\|^2_{\mathcal{L}^0_{\mathtt{Q}}}\,ds. \end{aligned}$$

Hence, we have

$$\mathbf{E}\,\|\Delta_4(t+\gamma)-\Delta_4(t)\|^2\to 0 \text{ as } |\gamma|\to 0.$$

Hence, the above argument imply that $t\to\Theta(u(t))$ is Cadillac on $[0,T]$ a.s.
Step 2: Next, we will verify that Θ is a contraction mapping in $\mathscr{B}_{T_1}$ with some $T_1<T$ to be specified later. Let $u,v\in\mathscr{B}_T$ and $t\in[0,T]$, we have

$$\begin{aligned} \mathbf{E}\,\|\Theta(u(t))-\Theta(v(t))\|^2 &\le \frac{1}{k}\mathbf{E}\,\|p(t,u_t)-p(t,v_t)\|^2 \\ &+ \frac{2}{1-k}\mathbf{E}\left\|\int_0^t R(t-s)\,[f(s,u_s)-f(s,v_s)]\,ds\right\|^2 \\ &+ \frac{2}{1-k}\mathbf{E}\left\|\sum_{0<t_k<t}R(t-t_k)\,[I_k(u(t_k))-I_k(v(t_k))]\right\|^2. \end{aligned}$$

By using Holder's inequality, together with **(H4)**, **(H5)** and **(H8)**, we get

$$\begin{aligned} \mathbf{E}\,\|\Theta(u(t))-\Theta(v(t))\|^2 &\le k\mathbf{E}\,\|u-v\|_t^2+\frac{2}{1-k}M^2tK_f^2\int_0^t\mathbf{E}\,\|u-v\|_s^2\,ds \\ &+ \frac{2}{1-k}M^2\sum_{k=1}^\infty q_k^2\mathbf{E}\,\|u-v\|_t^2 \\ &\le k\mathbf{E}\,\|u-v\|_t^2+\frac{2}{1-k}\left[M^2tK_f^2\right]\int_0^t\mathbf{E}\,\|u-v\|_s^2\,ds \end{aligned}$$

$$+ \quad \frac{2}{1-k} M^2 \sum_{k=1}^{\infty} q_k^2 \mathbf{E} \|u-v\|_t^2 .$$

Hence,

$$\sup_{s\in[-\tau,T]} \mathbf{E} \|\Theta(u(t)) - \Theta(v(t))\|^2 \leq \Gamma(t) \sup_{s\in[-\tau,T]} \mathbf{E} \|u-v\|_s^2 ,$$

where $\Gamma(t) = k + \frac{2}{1-k} \left[M^2 t K_f^2 \right] + \frac{2}{1-k} M^2 \sum_{k=1}^{\infty} q_k^2$.

By Equation (7), we have $\Gamma(0) = k + \frac{2}{1-k} M^2 \sum_{k=1}^{\infty} q_k^2 = \frac{2M^2 \sum_{k=1}^{\infty} q_k^2}{(1-k)^2} < 1$. Hence, there exists $0 < T_1 < T$ such that $0 < \Gamma(T_1) < 1$ and Θ is a contraction mapping on $\mathscr{B}_{T_1}$. Therefore it is clear that it has a unique fixed point, which is a mild solution of (1)-(3). By repeating a similar process the solution can be extended to the entire interval $[-\tau, T]$ in infinitely many steps. Hence the proof. □

4. Stability

Definition 4.1. Let u, v be different mild solutions of (1)-(3) with initial values φ_1 and φ_2 respectively. If for all $\epsilon > 0$, there exists $\delta > 0$ such that $\mathbf{E} \|u(t) - v(t)\|^2 \leq \epsilon$ when $\mathbf{E} \|\varphi_1 - \varphi_2\|^2 \leq \delta$ for all $t \in [0, T]$, then $u(t)$ is said to be stable in mean square.

Theorem 4.2. *Assume that any two mild solutions of (1)-(3) are $u(t)$ and $v(t)$ with initial values φ_1 and φ_2, respectively. Suppose that* **(H3)**-**(H6)** *are satisfied, then the mild solution of (1)-(3) is stable in the quadratic mean.*

Proof.

$$\begin{aligned} \mathbf{E} \|u(t) - v(t)\|^2 \leq\ & 4\mathbf{E} \Big\| R(t) \left[(\varphi_1(0) - \varphi_2(0)) + (p(0,\varphi_1) - p(0,\varphi_2)) \right] \Big\|^2 \\ +\ & 4\mathbf{E} \Big\| p(t,u_t) - p(t,v_t) \Big\|^2 \\ +\ & 4\mathbf{E} \Big\| \int_0^t R(t-s) \left[f(s,u_s) - f(s,v_s) \right] ds \Big\|^2 \\ +\ & 4\mathbf{E} \Big\| \sum_{0<t_k<t} R(t-t_k) \left[I_k(u(t_k)) - I_k(v(t_k)) \right] \Big\|^2 . \end{aligned}$$

By using Holder's inequality and **(H3)**, **(H4)** and **(H6)**, we get

$$\begin{aligned} \mathbf{E} \|u(t) - v(t)\|^2 \leq\ & 4M^2 \left[1 + K_p^2 \right] \mathbf{E} \|\varphi_1 - \varphi_2\|^2 \\ +\ & 4 \left[K_p^2 + M^2 \sum_{k=1}^{\infty} q_k^2 \right] \mathbf{E} \|u-v\|_t^2 \\ +\ & 4M^2 \left[t K_f^2 \right] \int_0^t \mathbf{E} \|u-v\|_s^2 \, ds. \end{aligned}$$

It follows that

$$\begin{aligned}\sup_{t\in[\tau,T]}\mathbf{E}\left\|u-v\right\|_t^2 &\leq \frac{4M^2\left[1+K_p^2\right]}{1-Q}\mathbf{E}\left\|\varphi_1-\varphi_2\right\|^2\\ &+ \frac{4M^2\left[tK_f^2\right]}{1-Q}\int_0^t\sup_{t\in[\tau,T]}\mathbf{E}\left\|u-v\right\|_s^2 ds.\end{aligned}$$

where $4\left[K_p^2+M^2\sum_{k=1}^{\infty}q_k^2\right]$.
By applying Gronwall's inequality, we have

$$\begin{aligned}\sup_{t\in[\tau,T]}\mathbf{E}\left\|u-v\right\|_t^2 &\leq \frac{4M^2\left[1+K_p^2\right]}{1-Q}\mathbf{E}\left\|\varphi_1-\varphi_2\right\|^2\times\exp\frac{4M^2\left[tK_f^2\right]}{1-Q}\\ &\leq \Lambda\mathbf{E}\left\|\varphi_1-\varphi_2\right\|^2,\end{aligned}$$

where $\Lambda=\frac{4M^2\left[1+K_p^2\right]}{1-Q}\mathbf{E}\left\|\varphi_1-\varphi_2\right\|^2\exp\frac{4M^2\left[tK_f^2\right]}{1-Q}$.
Now, given $\epsilon>0$, choose $\delta=\frac{\epsilon}{\Lambda}$ such that

$$\mathbf{E}\left\|\varphi_1-\varphi_2\right\|^2<\delta.$$

Then

$$\sup_{t\in[\tau,T]}\mathbf{E}\left\|u-v\right\|^2<\epsilon.$$

Hence the proof. □

5. Exponential stability

Definition 5.1. System (1)-(3) is said to be exponentially stable in the quadratic mean if there exist positive constant M_1 and $\lambda>0$ such that

$$\mathbf{E}\left\|u(t)\right\|^2 \leq M_1\mathbf{E}\left\|\varphi\right\|^2 e^{-\lambda(t-t_0)},\ t\geq t_0.$$

We assume that $f(t,0)=0$ for all $t\geq 0$. So that system (1)-(3) admits a trivial solution. We further need the following assumptions

(H9) $\|R(t)\|\leq Me^{-\lambda(t-t_0)}$, $t\geq t_0$, where $M\geq 1$, $\lambda>0$.

(H10) There exist non-negative real numbers $G_1,G_2\geq 0$ and continuous functions $\nu_1,\nu_2,\nu_3:[0,+\infty)\to\mathbb{R}_+$ such that, for all $t\geq 0$ and $u,v\in\mathtt{X}$,

$$\begin{aligned}\left\|p(t,u_t)\right\|^2 &\leq G_1\left\|u\right\|_t^2+\nu_1(t),\\ \left\|f(t,u_t)\right\|^2 &\leq G_2\left\|u\right\|_t^2+\nu_2(t),\end{aligned}$$

(H11) There exist non-negative real numbers $s_i\geq 0$, $i=1,2,3$ such that

$$\nu_i(t) \leq s_ie^{-\lambda t},\ \text{for all } t\geq 0,\ i=1,2,3.$$

(H12) The function $\sigma : [0, +\infty) \to \mathcal{L}^0_{\mathtt{Q}}(\mathtt{Y}, \mathtt{X})$ satisfies the following condition in addition to assumptions (ii) and (iii):

$$\int_0^t e^{\lambda s} \left\|\sigma(s)\right\|^2_{\mathcal{L}^0_{\mathtt{Q}}} ds < \infty, \quad \forall t \in [0, T],$$

Lemma 5.2. *([27]) Let $N : [-\tau, +\infty) \longrightarrow [0, +\infty)$ be a function and suppose that there exist some constants $\gamma > 0$, $\lambda_i > 0$ $(i = 1, 2, 3)$ such that*

$$N(t) \quad \leq \quad \lambda_1 e^{-\gamma t} + \lambda_2 \sup_{\theta \in [-\tau, 0]} N(t+\theta) + \lambda_3 \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} N(s+\theta) ds, t \geq 0$$

and

$$N(t) \quad \leq \quad \lambda_1 e^{-\gamma t}, \ t \in [-\tau, 0].$$

If $\lambda_2 + \frac{\lambda_3}{\gamma} < 1$. Then , we have $N(t) \leq M e^{-\mu t}$, $(t \geq -\tau)$, where μ is a positive root of the algebra equation $\lambda_2 + \frac{\lambda_3}{\gamma} e^{\mu\tau} = 1$ and $M = \max\left\{\frac{\lambda_t(\gamma-\mu)}{\lambda_3 e^{\mu\tau}}, \lambda_1\right\}$.

Theorem 5.3. *Assume that* **(H7)**-**(H8)** *are fulfilled and that*

$$\frac{4\left\{M^2 G_1/\lambda^2 + M^2 \sum_{=1}^{\infty} q_k^2\right\}}{(1-k)^2} < 1. \tag{10}$$

where $k = \sqrt{G_1}$. Then the mild solution of system (1)-(3) is exponentially stable in mean square moment.

Proof. From inequality (8), we have a smal number $\epsilon > 0$ such that $\eta = \lambda - \epsilon$ satisfies the following inequality

$$\frac{4M^2 G_1}{\lambda(\lambda-\epsilon)(1-k)} + \frac{4M^2 \sum_{k=1}^{\infty} q_k^2}{1-k} < 1. \tag{11}$$

From (8), we have

$$\begin{aligned}
\mathbf{E}\left\|u(t)\right\|^2 \quad &\leq \quad \frac{1}{k}\mathbf{E}\left\|p(t,u_t)\right\|^2 + \frac{4}{1-k}\mathbf{E}\Bigg\{ \left\|R(t\left[\varphi(0) + p(0,\varphi)\right])\right\|^2 \\
&+ \quad \left\|\int_0^t R(t-s) f(s,u_s) ds\right\|^2 \\
&+ \quad \left\|\int_0^t R(t-s)\sigma(s) d\mathtt{B}^{\mathtt{H}}_{\mathtt{Q}}(s)\right\|^2 + \left\|\sum_{0<t_k<t} R(t-t_k) I_k(u_{t_k})\right\|^2 \Bigg\} \\
&= \quad \sum_{i=1}^4 \Delta_i.
\end{aligned}$$

By **(H7)**-**(H9)**, we have

$$\Delta_1 \quad = \quad \frac{1}{k}\mathbf{E}\left\|p(t,u_t)\right\|^2$$

$$\begin{aligned} &\leq \frac{1}{k}\left\{G_1\mathbf{E}\left\|u_t\right\|^2+\nu_1(t)\right\} \\ &\leq k\mathbf{E}\left\|u_t\right\|^2+Q_1e^{-\eta t}, \end{aligned}$$

where $Q_1=\frac{s_1}{k}$.

$$\begin{aligned} \Delta_2 &= \frac{8}{1-k}\left[\mathbf{E}\left\|R(t)\varphi(0)\right\|^2+\mathbf{E}\left\|R(t)p(0,\varphi)\right\|^2\right] \\ &\leq \frac{8M^2}{1-k}e^{-2\lambda t}\mathbf{E}\left\|\varphi(0)\right\|^2+\frac{8M^2}{1-k}e^{-2\lambda t}\left\{G_1\mathbf{E}\left\|\varphi\right\|^2+\nu_1(t)\right\} \\ &\leq Q_2e^{-\eta t}. \end{aligned}$$

where $Q_2=\frac{8M^2}{1-k}\left[\mathbf{E}\left\|\varphi(0)\right\|^2+G_1\left\|\varphi\right\|^2+s_1\right]$.
Combining **(H7)-(H9)** and Holder's inequality, we obtain that

$$\begin{aligned} \Delta_3 &= \frac{4}{1-k}\mathbf{E}\left\|\int_0^t R(t-s)f(s,u_s)ds\right\|^2 \\ &\leq \frac{4}{1-k}\int_0^t M^2e^{-\lambda(t-s)}ds\int_0^t e^{-\lambda(t-s)}\mathbf{E}\left\|f(s,u_s)\right\|^2ds \\ &\leq \frac{4M^2G_2}{\lambda(1-k)}\int_0^t e^{-\lambda(t-s)}\mathbf{E}\left\|u_s\right\|^2ds+Q_3e^{-\eta t}. \end{aligned}$$

where $Q_3=\frac{4M^2}{\lambda(1-k)}\frac{s_2}{\lambda-\eta}$.
By applying Lemma 5.1, and **(H9)**, **(H12)**, we have

$$\begin{aligned} \Delta_4 &= \frac{4}{1-k}\mathbf{E}\left\|\int_0^t R(t-s)\sigma(s)d\mathtt{B}^{\mathtt{H}}_{\mathtt{Q}}(s)\right\|^2 \\ &\leq \frac{4M^2c\mathtt{H}(2\mathtt{H}-1)t^{2\mathtt{H}-1}}{1-k}\int_0^t e^{-2\lambda(t-s)}\mathbf{E}\left\|\sigma(s)\right\|^2ds \\ &\leq e^{-\eta t}\frac{4M^2c\mathtt{H}(2\mathtt{H}-1)t^{2\mathtt{H}-1}e^{-\epsilon t}}{1-k}\int_0^t e^{-\lambda(t-s)}\mathbf{E}\left\|u_s\right\|^2ds. \end{aligned}$$

Therefore, **(H12)** ensures the existence of a positive constant $Q_4>0$, for all $t\geq 0$ such that $\frac{4M^2c\mathtt{H}(2\mathtt{H}-1)t^{2\mathtt{H}-1}e^{-\epsilon t}}{1-k}\int_0^t e^{-\lambda(t-s)}\mathbf{E}\left\|u_s\right\|^2ds\leq Q_4$, Then $\Delta_4\leq Q_4e^{-\eta t}$.
By **(H6)**, we have

$$\begin{aligned} \Delta_4 &= \frac{4}{1-k}\mathbf{E}\Big\|\sum_{0<t_k<t}R(t-t_k)I_k(u_{t_k})\Big\|^2 \\ &\leq \frac{4M^2}{1-k}\sum_{k=1}^{\infty}q_k^2e^{-2\lambda(t-t_k)}\mathbf{E}\left\|u(t_k)\right\|^2 \\ &\leq \frac{4M^2}{1-k}\sum_{k=1}^{\infty}q_k^2e^{-\eta(t-t_k)}\mathbf{E}\left\|u(t_k)\right\|^2 \end{aligned}$$

The above inequality together with Lemma 2.1, imply that

$$\mathbf{E}\left\|u(t)\right\|^2 \quad \leq \quad \delta e^{-\eta t},\ \text{for } t\in[-r,0].$$

and

$$\begin{aligned}\mathbf{E}\left\|u(t)\right\|^2 \quad &\leq \quad \delta e^{-\gamma t}+k\sup_{-\tau\leq u\leq 0}\mathbf{E}\left\|u(t+\theta)\right\|^2 \\ &+ \quad k^{'}\int_0^t e^{-\gamma(t-s)}\sup_{-\tau\leq u\leq 0}\mathbf{E}\left\|u(s+\theta)\right\|^2 ds \\ &+ \quad \sum_{k=1}^{+\infty} e^{-\eta(t-t_k)}\mathbf{E}\left\|u(t_k^-)\right\|^2,\ t\geq 0,\end{aligned}$$

where $\delta = \max\left(\sum_{k=1}^{4} Q_k, \sup_{-\tau\leq u\leq 0}\mathbf{E}\left\|\varphi(\theta)\right\|^2\right)$ and $k^{'} = \frac{4M^2G_2}{\lambda(1-k)}$, since $k + \frac{k^{'}}{\gamma} + \sum_{k=1}^{+\infty} d_k < 1$, then it follows from Lemma 2.1 that there exists positive constant $\theta > 0$ and $k > 0$ such that

$$\mathbf{E}\left\|u(t)\right\|^2 \quad \leq \quad Qe^{-\theta t},\ \text{forall } t\geq -\tau$$

which is our crave inequality. Then the proof is completed. □

Remark 5.1. If the impulsive term $\Delta(u(t_k)) = I_k(\cdot) = 0,\ k = 1,2,...,$ then (1)-(3) takes the following form:

$$\begin{aligned}&d\left[u(t)+p(t,u_t)\right] \\ &= A\left[u(t)+p(t,u_t)\right]dt+\left[\int_0^t B(t-s)\left[u(s)+p(s,u_s)\right]ds\right. \\ &\left.+ f(t,u_t)\right]dt+\sigma(t)d\mathtt{B}_{\mathtt{Q}}^{\mathtt{H}}(t),\ t\in[0,T],\ t\neq t_k, \qquad (12)\end{aligned}$$

$$u(t)=\varphi(t),\ -\tau\leq t\leq 0, \qquad (13)$$

where $\mathcal{C}=\mathcal{C}([-\tau,0];\mathtt{X})$ denotes the family of almost surely bounded and continuous functions φ from $[-\tau,0]$ into $\mathtt{X}$ and, as usual, with $\|\varphi\|_{\mathcal{C}}=\sup_{\theta\in[-\tau,0]}\|\varphi(\theta)\|$. Also, if we assume that all the functions are defined the same as earlier, then by the same procedure as in Theorem 5.1, we may deduce the next corollary.

Corollary 5.4. *Suppose that* **(H3)**-**(H9)** *are satisfied, then the mild solution of (1)-(3) is exponentially stable in the mean square moment if the following inequality holds:*

$$\frac{3M^2\left\{G_2/\lambda^2\right\}}{(1-k)^2} < 1.$$

6. Illustration

Consider the following neutral impulsive stochastic partial integrodifferential equations with Poisson jumps :

$$\begin{cases} \frac{\partial}{\partial t}[x(t,\xi)-p(t,x(t-r,\xi))] = \frac{\partial^2}{\partial \xi^2}[x(t,\xi)-p(t,x(t-r,\xi))] \\ \qquad + \int_0^t b(t-s)\frac{\partial^2}{\partial \xi^2}[x(t,\xi)-p(t,x(t-r,\xi))]ds \\ \qquad + f(t,x(t-r_2,\xi)) + \sigma(s)d\mathtt{B}^{\mathtt{H}}_{\mathtt{Q}}(s), 0 \le x \le \pi, t \in I, t \neq t_k, \\ \Delta x(t_k,x) = I_k(x(t_k-h,x)), \quad t = t_k, k = 1,2,\cdots, \\ x(t,0) + p(t,x(t-r,0)) = 0, \quad t \ge 0, \\ x(t,\pi) + p(t,x(t-r,\pi)) = 0, \quad t \ge 0, \\ x(\theta,\xi) = x_0(\theta,\xi), \quad \theta \in]-\infty,0] \text{ and } 0 \le \xi \le \pi, \end{cases} \tag{14}$$

where $I=[0,T]$ and $\mathtt{B}^{\mathtt{H}}$ denotes a fractional Brownian motion, $p,f : I\times\mathbb{R}\to\mathbb{R}$, and $b: I\to\mathbb{R}$ are continuous fuctions. Let $Y = L^2([0,\pi])$, and let

$$e_n := \sqrt{\frac{2}{\pi}}\sin nx, \quad n = 1,2,\cdots$$

Then $(e_n)_{n\in\mathbb{N}}$ is a complete orthonormal basis in $\mathtt{Y}$. Let $\mathtt{X} = \mathcal{L}^2([0,\pi])$, and let $A = \frac{\partial^2}{\partial z^2}$, with domain

$$D(A) = H^2([0,\pi]) \cap H^1_0([0,\pi]).$$

Then, it is well known that

$$Az = -\sum_{n=1}^{\infty} n^2\langle\langle z, e_n\rangle e_n, \quad \forall z \in \mathtt{X},$$

and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{(T(t))_{t\ge 0}\}$ on $\mathtt{X}$, which is given by

$$T(t)\xi = \sum_{n=1}^{\infty} e^{-n^2 t}\langle \xi, e_n\rangle e_n, \quad \xi \in D(A).$$

Let $B : D(A)\subset \mathtt{X}\to\mathtt{X}$ be the operator defined by $B(t)(y) = b(t)Ay$ for $t\ge 0$ and $y \in D(A)$.
In order to define the operator $\mathtt{Q} : \mathtt{Y}\to\mathtt{T}$, we choose a sequence $\{e_n\}_{n\ge 1}\subset\mathbb{R}^+$, set $\mathtt{Q}e_n = \sigma_n e_n$, and assume that

$$\mathrm{tr}(\mathtt{Q}) \quad = \quad \sum_{n=1}^{\infty}\sqrt{\sigma_n} < \infty.$$

Define the process $\mathtt{B}^H$ by

$$\mathtt{B}^{\mathtt{H}} \quad = \quad \sum_{n=1}^{\infty}\sqrt{\lambda_n\gamma_n(t)}e_n. \text{ where } \mathtt{H}\in(\frac{1}{2},1), \text{ and } \{\gamma_n^{\mathtt{H}}\}_{n\in\mathbb{N}}.$$

We suppose that
(1) For $t\ge 0$, $p(t,0) = f(t,0) = k_0$.

(2) there exists a positive constant l_1, such that

$$\|p(t,\xi_1)-p(t,\xi_2)\|\leq l_1\|\xi_1-\xi_2\|,\ \ t\geq 0,\ \xi_1,\xi_2\in\mathbb{R};$$

(3) there exists a positive constant k_1, such that

$$\|f(t,\xi_1)-f(t,\xi_2)\|^2\leq k_1(\|\xi_1-\xi_2\|^2),\ \ t\geq 0,\ \xi_1,\xi_2\in\mathbb{R};$$

(4) there exists a positive constant q_k, $k=1,2,...$, such that

$$\|I_k(\xi_1)-I_k(\xi_2)\|^2\leq q_k(\|\xi_1-\xi_2\|^2),\ \ k=1,2,...,\ \xi_1,\xi_2\in\mathbb{R};$$

(5) The function $\sigma:[0,+\infty)\to\mathcal{L}_2^0(\mathcal{L}^2[0,\pi],\mathcal{L}^2[0,\pi])$ satisfies

$$\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0}<\infty,\ \forall T>0.$$

For $t\geq 0$ and $\xi\in\mathtt{X}$, define the operators $F,P:[0,T]\times\mathtt{X}\to\mathtt{X}$ for $\xi\in[0,\pi]$ by

$$P(t,\xi)(\xi)=p(t,\xi(-r_1)(\xi_1)),$$

$$F(t,\xi)(\xi)=f(t,\xi(-r_1)(\xi_2)),$$

If we put

$$\begin{cases}u(t)(\xi)=x(t,\xi),\ \ t\geq 0,\\ \varphi(\theta)(\xi)=x_0(\theta,\xi),\ \ \theta\in[-r,0]\end{cases}\quad \xi\in[0,\pi],$$

then (14) takes the following abstract form:

$$\begin{aligned}&d\left[u(t)+p(t,u_t)\right]\\&=A\left[u(t)+p(t,u_t)\right]dt+\left[\int_0^t B(t-s)\left[u(s)+p(s,u_s)\right]ds\right.\\&\left.+f(t,u_t)\right]dt+\sigma(t)d\mathtt{B}^{\mathtt{H}}_{\mathtt{Q}}(t),\ t\in[0,T],\ t\neq t_k,\end{aligned}\tag{15}$$

$$\Delta u(t_k)=u(t_k^+)-u(t_k^-)=I_k(u(t_k)),\ t=t_j,\ k=1,2,...,\tag{16}$$

$$u(t)=\varphi(t),\ -\tau\leq t\leq 0,\tag{17}$$

Moreover, if b is a bounded and C^1 function such that b' is bounded and uniformly continuous, then (**H1**) and (**H2**) are satisfied, and hence, by Theorem 2.1, (14) has a resolvent operator $(R(t))_{t\geq 0}$ on $\mathtt{X}$. As a consequence of the continuity of f and assumption (1), it follows that F and P are continuous on $[0,T]\times\mathtt{X}$ with values in $\mathtt{X}$. By assumption (3), one can see that

$$\|F(t,\xi_1)-F(t,\xi_2)\|^2\leq k(\|\xi_1-\xi_2\|^2),\ \ t\geq 0,\ \xi_1,\xi_2\in\mathbb{R};$$

Similarly, the same property for $P(t,u)$ can also be verified.
The remaining conditions can be verified similarly. Therefore, the existence of a unique mild solution of (14) follows.

7. Conclusion

In this article we have studied the existence and exponential stability of neutral stochastic partial integrodifferential equations driven by fractional Brownian motion with impulsive effects with Hurst parameter $\mathtt{H} \in (\frac{1}{2}, 1)$ using theory of resolvent operators. Further, we established a new impulsive-integral inequality to prove the exponential stability of mild solutions in the mean square moment. One can extend the same system to second order non-autonomous with infinite delay. Also one can study the existence and exponential stability of neutral stochastic fractional order partial integrodifferential equations driven by Rossenblatt process with impulsive effects.

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