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CONVOLUTION SUM OF RAMANUJAN'S SUM

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Abstract. This article is the result of calculating the convolution of Ramanujan's sum and natural number multiplied. Among these results, special values are expressed by Euler and Bernoulli functions.

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1. Introduction

The theory of convolution sums of arithmetic functions is being studied a lot because it is helpful for the various number theory and special functions. Various results can be seen in the convolution sum for Ramanujan's sum. In fact,

Let n and r be positive integers. Define

$$c(n,r) := \sum_{\substack{x=1 \\ gcd(x,r)=1}}^{r} e^{\frac{2\pi n x}{r}}$$

and

$$c(m,n,r) := \sum_{d|gcd(m,n,r)} d^2 \mu(r/d), \tag{1}$$

where μ is the Möbius function. $c(n,r)$ is called Ramanujan's sum. In this article, we will use $c(m,n,r) = c^{(2)}(gcd(m,n),r)$ interchangeably. It is well known that $c(n,r) = \sum_{d|gcd(n,r)} \mu(r/d)d$ and $c(0,r) = \phi(r)$.

In 1918, Srinivasa Ramanujan considered Ramanujan's sums (see [6]). In fact, Srinivasa Ramanujan sum was usefully used when proving Vinogradov's theorem [5, Chapter 8]. In this paper we define the quasi-generalized Ramanujan's sum and give the relation to the Ramanujan's sum, Bernoulli polynomials and Euler

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polynomials. Bernoulli polynomials and Euler polynomials have been studied by many mathematicians (see [1, 2, 4, 7]).

For any rational numbers k_1, k_2, k_3, the quasi-generalized Ramanujan's sums are

$$C^{+}_{(k_1,k_2,k_3)}(l) := \sum_{n=1}^{l} \sum_{\substack{m=1 \\ r|n}}^{m\le \frac{n}{k_1}} c(k_1 m, n-k_1 m, r) m^{k_2} n^{k_3}$$

and

$$C^{-}_{(k_1,k_2,k_3)}(l) := \sum_{n=1}^{l} \sum_{\substack{m=1 \\ r|n}}^{m\le \frac{n}{k_1}} c(k_1 m, n-k_1 m, r)(-1)^n m^{k_2} n^{k_3},$$

where l is a positive integer.

In fact, we will prove the following result.

Theorem 1.1. *Let k_3 be an integer and l be a positive integer. Then we have*

$$C^{+}_{(1,0,k_3)}(l) = \begin{cases} \frac{B_{k_3+3}(l+1)-B_{k_3+3}}{k_3+3} & \text{if } k_3 > -2, \\ l & \text{if } k_3 = -2, \\ H_l^{(-k_3-2)} & \text{if } k_3 < -2, \end{cases}$$

$$C^{+}_{(1,1,k_3)}(l) = \begin{cases} \frac{B_{k_3+4}(l+1)-B_{k_3+4}}{2k_3+8} & \text{if } k_3 > -3, \\ \frac{l}{2} & \text{if } k_3 = -3, \\ \frac{1}{2}H_l^{(-k_3-3)} & \text{if } k_3 < -3, \end{cases}$$

$$C^{-}_{(1,0,k_3)}(l) = \begin{cases} \frac{E_{k_3+2}(0)+(-1)^l E_{k_3+2}(l+1)}{2} & \text{if } k_3 > -2, \\ \chi(l) & \text{if } k_3 = -2, \\ S_l^{(-k_3-2)} & \text{if } k_3 < -2 \end{cases}$$

and

$$C^{-}_{(1,1,k_3)}(l) = \begin{cases} \frac{E_{k_3+3}(0)+(-1)^l E_{k_3+3}(l+1)}{4} & \text{if } k_3 > -3, \\ \frac{\chi(l)}{2} & \text{if } k_3 = -3, \\ S_l^{(-k_3-3)} & \text{if } k_3 < -3. \end{cases}$$

Here, $H_l^{(t)} = 1 + \frac{1}{2^t} + ... + \frac{1}{l^t}$, $S_l^{(t)} = -1 + \frac{1}{2^t} - ... + (-1)^l \frac{1}{l^t}$, $B_n(z)$(resp., $E_n(z)$) is the nth Bernoulli polynomial (resp., Euler polynomial) and

$$\chi(l) = \begin{cases} 0, & \text{if } l \text{ is even,} \\ -1, & \text{otherwise.} \end{cases}$$

Example 1.2. Let l be a positive integer. Then

$$C_{+}(l) := C^{+}_{(1,0,0)}(l) = \sum_{n=1}^{l} \sum_{\substack{m=1 \\ r|n}}^{n} c(m, n-m, r) = \frac{B_3(l+1)}{3} = \frac{l(l+1)(2l+1)}{6}$$

and

$$C_{-}(l) := C^{-}_{(1,0,0)}(l) = \sum_{n=1}^{l} \sum_{\substack{m=1 \\ r|n}}^{n} (-1)^n c(m, n-m, r) = \frac{(-1)^l E_2(l+1)}{2} = \frac{(-1)^l l(l+1)}{2}.$$

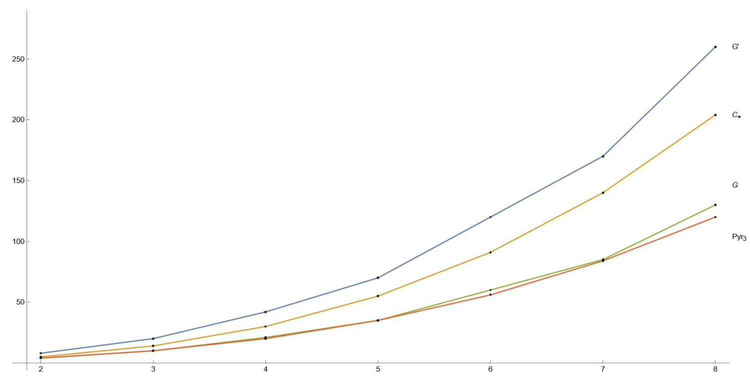

FIGURE 1. Values of $G'(l), C_{+}(l), G(l), Pyr_3(l)$ $(2 \le l \le 8)$

Remark 1.1. Let $G(l) := \sum_{(a+b+c)x=l+2} a$ and $G'(l) := \sum_{(a+b+c)x=l+2} 2a$ with $a, b, c, l, x \in \mathbb{N}$. The result of studying the values of functions such as $G(l)$ is in [8].

Using [3, Theorem 1.1 and Remark 2.2] and Theorem 1.1, we obtain

$$G'(l) > C_{+}(l) > G(l) \ge Pyr_3(l)$$

with $l \ge 1$. Here, $Pyr_a(z) = \frac{1}{6}(z(z+1)((a-2)z+5-a))$ be the ath order pyramid number. In Figure 1, we plot the graphs for the values of $G'(l)$, $C_{+}(l)$, $G(l)$ and $Pyr_3(l)$ when $l = 2, 3, 4, 5, 6, 7, 8$.

It is easly checked that if $l = 2$ or $l+2$ is a prime number for $l \ge 1$, then we obtain

$$G(l) = Pyr_3(l) = \frac{l(l+1)(l+2)}{6}.$$

2. Proof of Theorem 1.1

We need several lemmas to prove Theorem 1.1, so we introduce them first.

Lemma 2.1. *Let k, n and r be positive integers. Then*

$$\sum_{m=1}^{n} c(m, n-m, r)^k = (\varphi * c^{(2)}(\cdot, r)^k)(n) \tag{2}$$

and

$$\sum_{m=1}^{n-1} c(m,n-m,r)^k = (\varphi * c^{(2)}(\cdot,r)^k)(n) - c^{(2)}(n,r)^k.$$

Here, $*$ is a Dirichlet convolution, that is, $(f*g)(n) = \sum_{d|n} f(d)g(\frac{n}{d})$ and $c^{(2)}(\cdot,r)$ means that one of the two variables, r, is given as a fixed number, and the other is regarded as a one varible function.

Proof. Let

$$A := \sum_{m=1}^{n} c(m,n-m,r)^k = \sum_{d|n} \sum_{\substack{x=1 \\ gcd(x,n/d)=1}}^{n/d} c(xd,n-xd,r)^k.$$

Since $gcd(x,n/d) = 1$ implies $gcd(x,n/d-x) = 1$, $gcd(xd,n-xd) = d$ and $c(xd,n-xd,r) = c^{(2)}(d,r)$ by (1). Thus, we deduce that

$$A = \sum_{d|n} c^{(2)}(d,r)^k \sum_{\substack{x=1 \\ gcd(x,n/d)=1}}^{n/d} 1 = \sum_{d|n} \varphi(n/d)c^{(2)}(d,r)^k.$$

From (2), we obtain that

$$\sum_{m=1}^{n-1} c(m,n-m,r)^k = (\varphi * c^{(2)}(\cdot,r)^k)(n) - c^{(2)}(n,r)^k.$$

This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let n and r be positive integers. Then*

$$(\varphi * c^{(2)}(\cdot,r))(n) = nc(n,r). \tag{3}$$

Proof. To prove Lemma 2.2, let us define $g(n,r)$ as follows

$$g(n,r) := \begin{cases} r & \text{if } r|n \\ 0 & \text{otherwise.} \end{cases}$$

From the left hand side of (3), we get the following process by rearranging terms. That is,

$$\begin{aligned}(\varphi * c^{(2)}(\cdot,r))(n) &= \sum_{d|n} \varphi(n/d) \sum_{e|gcd(d,r)} e^2\mu(r/e) = \sum_{d|n} \varphi(n/d) \sum_{d|r} e\mu(r/e)g(d,e) \\ &= \sum_{e|r} e^2\mu(r/e) \sum_{\substack{d|n \\ e|d}} \varphi(n/d) = \sum_{e|gcd(n,r)} e^2\mu(r/e) \sum_{\substack{D|n/e \\ d=De}} \varphi(n/De) \\ &= \sum_{e|gcd(n,r)} e^2\mu(r/e)\frac{n}{e} = n \sum_{e|gcd(n,r)} e\mu(r/e) = nc(n,r).\end{aligned}$$

□

By Lemma 2.2, we have Corollary 2.3.

Corollary 2.3. *Let n and r be positive integers. Then*

$$\sum_{m=1}^{n-1} c(m, n-m, r) = nc(n,r) - c^{(2)}(n,r) = \sum_{d|gcd(n,r)} d(n-d)\mu(r/d).$$

Furthermore, by using the Möbius inversion formula, we have

$$c^{(2)}(n,r) = (\zeta_1\mu * \zeta_0 * \zeta_1 c(\cdot, r))(n).$$

Here, $\zeta_i(t) = t^i$ and $fg(t) = f(t)g(t)$, where i is a non-negatvie integer and t is a positive integer.

Lemma 2.4. *Let n and r be positive integers. If $r|n$, then*

$$\sum_{m=1}^{n} c(m,r)c(n-m,r) = nc(n,r). \tag{4}$$

Proof. Let B be the left hand side of (4) and let $r|n$. By the definition of the Ramanujan' sum, we derive that

$$\begin{aligned} B &= \sum_{m=1}^{n} \sum_{\substack{x=1 \\ gcd(x,r)=1}}^{r} \sum_{\substack{y=1 \\ gcd(y,r)=1}}^{r} e^{2\pi mx/r} \cdot e^{2\pi(n-m)y/r} \\ &= \sum_{\substack{y=1 \\ gcd(y,r)=1}}^{r} e^{2\pi ny/r} \sum_{\substack{x=1 \\ gcd(x,r)=1}}^{r} \left[\sum_{m=1}^{n} e^{2\pi m(x-y)/r}\right]. \end{aligned}$$

Since $1 \leq x, y \leq r$ and $r|(x-y)$ implies $x = y$, the sum in brackets is equal to n if $x = y$ and 0 otherwise. Therefore,

$$B = n \sum_{\substack{x=1 \\ gcd(x,r)=1}}^{r} e^{2\pi nx/r} = nc(n,r).$$

□

By Corollary 2.3 and Lemma 2.4, we have that

Corollary 2.5. *Let n and r be positive integers. If $r|n$, then*

$$\sum_{m=1}^{n} c(m, n-m, r) = \sum_{m=1}^{n} c(m,r)c(n-m,r) = nc(n,r) = n\varphi(r).$$

Proof of Theorem 1.1

It is well-known that

$$\sum_{r|n} \varphi(r) = n \tag{5}$$

by Lemma 2.5 and (5), we obtain that

$$\sum_{r|n}\sum_{m=1}^{n} c(m,n-m,r) = \sum_{r|n} n\varphi(r) = n^2. \tag{6}$$

It is easily checked that

$$\sum_{m=1}^{n} c(m,n-m,r)m^2 = \sum_{m=1}^{n} c(m,n-m,r)(n-m)^2$$

and

$$\sum_{m=1}^{n} c(m,n-m,r)m = \sum_{m=1}^{n} c(m,n-m,r)\frac{n}{2}. \tag{7}$$

Using (6), (7) and two results((2.2), (2.3)) in Sun's article (see [7]), the proof of Theorem 1.1 is completed. □

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