J. Appl. & Pure Math. Vol. **4***(2022), No. 1 - 2, pp. 37 - 50*
https://doi.org/10.23091/japm.2022.037

TRIPLE SEQUENCES IN THE TOPOLOGY INDUCED BY RANDOM 2-NORMS

VERDA GÜRDAL

Abstract. In this article we define and study the notions of $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy of triple sequences in the topology induced by random 2-normed spaces and prove some theorems based on them.

AMS Mathematics Subject Classification : 40A35, 46A70, 54E70.
Key words and phrases : Triple sequence, t-norm, random 2-normed space, ideal convergence, F-topology.

1. Introduction

The theory of statistical convergence is an active area of research. Statistical convergence of a real number sequence was firstly originated by Fast [5] and since then several generalizations and application of this concept have been investigated by various authors, e.g. [4], [7], [13], [11], [17], [19], [20], [28]. In the wake of the study of ideal convergence defined by Kostyrko et al. [15], there has been comprehensive research to discover applications and summability studies of the classical theories. A lot of development have been seen in area about $\mathcal{I}$-convergence of sequences after the work of [12, 21, 24, 25, 27, 35]. Note that $\mathcal{I}$-convergence is an interesting generalization of statistical convergence.

Menger [16] introduced the notion of probabilistic metric spaces, which is an interesting and important generalization of metric spaces, the study of these spaces was under the name of statistical metric. In this theory, the notion of distance has a probabilistic nature. Namely, the distance between two points x and y is represented by a distribution function F_{xy}; and for $t > 0$, the value $F_{xy}(t)$ is interpreted as the probability that the distance from x to y is less than t. Infact the probabilistic theory has become an area of active research for the last forthy years. An important family of probabilistic metric spaces are probabilistic normed spaces. The notion of probabilistic normed spaces was

Received December 23, 2021. Revised February 10, 2022. Accepted February 24, 2022.

introduced in [30, 31] and further it was extended to random/probabilistic 2-normed spaces by Golet [9] using the concept of 2-norm of Gähler [8].

This paper consists of three sections with the new results in section 3. In Section 3 the concepts of $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy of triple sequences in a more general setting, i.e., in random 2-normed spaces are introduced and its fundamental properties are studied.

2. Preliminaries

In this section we recall some basic definitions and notations which form the background of the present work.

The notion of statistical convergence is based on the asymptotic density of the subsets of the set $\mathbb{N}$ of positive integers. In [6] an axiomatic approach is given for introducing the notion of density of sets $K \subseteq \mathbb{N}$.

Let us denote the set of natural numbers by $\mathbb{N}$ and let $A \subseteq \mathbb{N}$. Then the asymptotic density of A denoted by

$$\delta(A) = \lim_n n^{-1} \left|\{k \leq n : k \in A\}\right|,$$

where the vertical bars denote the cardinality of the enclosed set.

A number sequence $x = (x_n)$ is said to be statistically convergent ([5], [7]) to the number L if for each $\varepsilon > 0$, the set

$$A(\varepsilon) = \{k \leq n : |x_n - L| > \varepsilon\}$$

has asymptotic density zero, i.e.

$$\lim_n n^{-1} \left|\{k \leq n : |x_n - L| > \varepsilon\}\right| = 0.$$

In this case we write $st\text{-}\lim x_n = L$.

The notion of statistical convergence was further generalized in the paper [15] using the notion of an ideal of subsets of the set $\mathbb{N}$. An ideal $\mathcal{I}$ on $\mathbb{N}$ for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$ is called a proper ideal. A proper ideal $\mathcal{I}$ is called admissible if $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$.

A family of the sets $\mathcal{I} \subset 2^{\mathbb{N}}$ is called an ideal if and only if
(i) $\emptyset \in \mathcal{I}$.
(ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$.
(iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

A family of sets $\mathcal{F} \subset 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if
(i) $\emptyset \notin \mathcal{F}$.
(ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$.
(iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

If $\mathcal{I}$ is proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \backslash A\}$$

is a filter of $\mathcal{N}$, it is called the filter associated with the ideal.

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$. A real sequence (x_n) is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$,

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathcal{I}.$$

If (x_n) is $\mathcal{I}$-convergent to L, then we write $\mathcal{I}-\lim x = L$.

Take for $\mathcal{I}$ the class $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is a non-trivial admissible ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence. For more information about $\mathcal{I}$-convergence, see the references in [26].

Definition 2.1. ([8]) Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies
(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent.
(ii) $\|x, y\| = \|y, x\|$.
(iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$, $\alpha \in \mathbb{R}$.
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.
The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space.

As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|, \quad x = (x_1, x_2), \quad y = (y_1, y_2).$$

All the concepts listed below are studied in depth in the fundamental book by Schweizer and Sklar [31].

Definition 2.2. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $S = [0, 1]$ the closed unit interval. A mapping $f : \mathbb{R} \to S$ is called a distribution function if it is nondecreasing and left continuous with $\inf_{t \in \mathbb{R}} f(t) = 0$ and $\sup_{t \in \mathbb{R}} f(t) = 1$.

We denote the set of all distribution functions by D^+ such that $f(0) = 0$. If $a \in \mathbb{R}_+$, then $H_a \in D^+$, where

$$H_a(t) = \begin{cases} 1 & \text{if } t > a, \\ 0 & \text{if } t \leq a. \end{cases}$$

It is obvious that $H_0 \geq f$ for all $f \in D^+$.

Definition 2.3. A triangular norm (t-norm) is a continuous mapping $* : S \times S \to S$ such that $(S, *)$ is an abelian monoid with unit one and $c * d \leq a * b$ if $c \leq a$ and $d \leq b$ for all $a, b, c, d \in S$. A triangle function τ is a binary operation on D^+ which is commutive, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.

Golet [9] introduced the notion of a random 2-normed space as follows.

Definition 2.4. Let X be a linear space of dimension greater than one, τ is a triangle function, and $F : X \times X \to D^+$. Then F is called a probabilistic 2-norm and (X, F, τ) a probabilistic 2-normed space if the following conditions are satisfied:

(i) $F(x,y;t) = H_0(t)$ if x and y are linearly dependent, where $F(x,y;t)$ denotes the value of $F(x,y)$ at $t \in \mathbb{R}$,

(ii) $F(x,y;t) \neq H_0(t)$ if x and y are linearly independent,

(iii) $F(x,y;t) = F(y,x;t)$ for all $x,y \in X$,

(iv) $F(\alpha x,y;t) = F(x,y;\frac{t}{|\alpha|})$ for every $t > 0$, $\alpha \neq 0$ and $x,y \in X$,

(v) $F(x+y,z;t) \geq \tau(F(x,z;t),F(y,z;t))$ whenever $x,y,z \in X$, and $t > 0$.

If (v) is replaced by

(vi) $F(x+y,z;t_1+t_2) \geq F(x,z;t_1) * F(y,z;t_2)$ for all $x,y,z \in X$ and $t_1,t_2 \in \mathbb{R}_+$;

then $(X,F,*)$ is called a random 2-normed space (for short, RTN space).

Remark 2.1. Note that every 2-norm space $(X,\|.,.\|)$ can be made a random 2-normed space in a natural way, by setting

(a) $F(x,y;t) = H_0(t-\|x,y\|)$, for every $x,y \in X$, $t > 0$ and $a*b = \min\{a,b\}$, $a,b \in S$; or

(b) $F(x,y;t) = \frac{t}{t+\|x,y\|}$ for every $x,y \in X$, $t > 0$ and $a * b = ab$ for $a,b \in S$.

Let $(X,F,*)$ be a RTN space. Since $*$ is a continuous t-norm, the system of (ε,λ)-neighborhoods of θ (the null vector in X)

$$\{\mathcal{N}_\theta(\varepsilon,\lambda) : \varepsilon > 0,\ \lambda \in (0,1)\},$$

where

$$\mathcal{N}_\theta(\varepsilon,\lambda) = \{x \in X : F_x(\varepsilon) > 1-\lambda\}.$$

determines a first countable Hausdorff topology on X, called the F-topology. Thus, the F-topology can be completely specified by means of F-convergence of sequences. It is clear that $x - y \in \mathcal{N}_\theta$ means $y \in \mathcal{N}_x$ and vice-versa.

Recently, Mursaleen and Edely [23] presented the idea of statistical convergence for multiple sequences, and there are several papers dealing with double and triple statistical and ideal convergence (see literature [1, 3, 10, 14, 18, 29]). Also, the readers should refer to the monographs [2] and [22] for the background on the sequence spaces and related topics.

We now recall the following basic concepts from [33, 34, 36] which will be needed throughout the paper.

A function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$) is called a real (or complex) triple sequence. A triple sequence (x_{jkl}) is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $|x_{jkl} - L| < \varepsilon$ whenever $j,k,l \geq n_0$. A triple sequence (x_{jkl}) is said to be bounded if there exists $M > 0$ such that $|x_{jkl}| < M$ for all $j,k,l \in \mathbb{N}$. We denote the space of all bounded triple sequences by ℓ^3_∞.

Definition 2.5. A subset K of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ is said to have natural density $\delta_3(K)$ if

$$\delta_3(K) = P - \lim_{j,k,l\to\infty} \frac{|K_{jkl}|}{jkl}$$

exists, where the vertical bars denote the number of (j,k,l) in K such that $p \leq j$, $q \leq k$, $r \leq l$. Then, a real triple sequence $x = (x_{jkl})$ is said to be statistically convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\delta_3 \left(\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{jkl} - L| \geq \varepsilon\} \right) = 0.$$

As can be seen from the following example a st_3-convergent sequence does not need to be bounded.

Example 2.6. Let us define the triple sequence $x = (x_{jkl})$ by

$$(x_{jkl}) = \begin{cases} jkl & , \quad j,k,l \text{ are cubes} \\ 4 & , \quad \text{otherwise.} \end{cases}$$

Then st_3-$\lim x_{jkl} = 4$ but (x_{jkl}) is neither convergent in Pringsheim's sense nor bounded.

If a triple sequence $x = (x_{jkl})$ satisfies some property P for all j,k,l except a set of natural density zero, then we say that the triple sequence x satisfies P for almost all (j,k,l) and we abbreviate this by a.a. (j,k,l).

Definition 2.7. Let $\mathcal{I}_3$ be an admissible ideal on $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$, then a triple sequence (x_{jkl}) is said to be $\mathcal{I}_3$-convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{jkl} - L| \geq \varepsilon\} \in \mathcal{I}_3.$$

In this case, one writes $\mathcal{I}_3$-$\lim x_{jkl} = L$.

Remark 2.2. (i) Let $\mathcal{I}_3\left(f\right)$ be the family of all finite subsets of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Then $\mathcal{I}_3\left(f\right)$ convergence coincides with the convergence of the triple sequences given in [34].

(ii) Let $\mathcal{I}_3\left(\delta\right) = \{A \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \delta\left(A\right) = 0\}$. Then $\mathcal{I}_3\left(\delta\right)$ convergence coincides with the statistical convergence introduced in [34].

Example 2.8. Let $\mathcal{I} = \mathcal{I}_3\left(\delta\right)$. Define the triple sequence (x_{jkl}) by

$$(x_{jkl}) = \begin{cases} 1 & , \quad j,k,l \text{ are cubes} \\ 4 & , \quad \text{otherwise.} \end{cases}$$

Then for every $\varepsilon > 0$

$$\delta\left(\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{jkl} - 4| \geq \varepsilon\}\right) \leq \lim_{p,q,r} \frac{\sqrt{p}\sqrt{q}\sqrt{r}}{pqr} = 0.$$

This implies that $\mathcal{I}$-$\lim x_{jkl} = 4$. But, the triple sequence (x_{jkl}) is not convergent to 4.

Throughout the chapter we consider the ideals of $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ by $\mathcal{I}_3$.

3. Main results

A triple sequences $x=(x_{jkl})$ in X is said to be F-convergence to $L\in X$ if for every $\varepsilon>0$, $\lambda\in(0,1)$ and for each nonzero $z\in X$ there exists a positive integer N such that

$$x_{jkl}, z-L\in\mathcal{N}_\theta(\varepsilon,\lambda) \text{ for each } j,k,l\geq N$$

or equivalently,

$$x_{jkl}, z\in\mathcal{N}_L(\varepsilon,\lambda) \text{ for each } j,k,l\geq N.$$

In this case we write $F\text{-}\lim x_{jkl}, z=L$.

Lemma 3.1. *Let $(X,\|.,.\|)$ be a real 2-normed space and $(X,F,*)$ be a RTN space induced by the random norm $F_{x,y}(t)=\frac{t}{t+\|x,y\|}$, where $x,y\in X$ and $t>0$. Then for every triple sequence $x=(x_{jkl})$ and nonzero y in X*

$$\lim\|x-L,y\|=0\Rightarrow F\text{-}\lim(x-L),y=0.$$

Proof. Let suppose that $\lim\|x-L,y\|=0$. Then for every $t>0$ and for every $y\in X$ there exists a positive integer $N=N(t)$ such that

$$\|x_{jkl}-L,y\|<t \text{ for each } j,k,l\geq N.$$

We observe that for any given $\varepsilon>0$,

$$\frac{\varepsilon+\|x_{jkl}-L,y\|}{\varepsilon}<\frac{\varepsilon+t}{\varepsilon}$$

which is equivalent to

$$\frac{\varepsilon}{\varepsilon+\|x_{jkl}-L,y\|}>\frac{\varepsilon}{\varepsilon+t}=1-\frac{t}{\varepsilon+t}.$$

Therefore, by letting $\lambda=\frac{t}{\varepsilon+t}\in(0,1)$ we have

$$F_{x_{jkl}-L,y}(\varepsilon)>1-\lambda \text{ for each } j,k,l\geq N.$$

This implies that $x_{jkl}, y\in\mathcal{N}_L(\varepsilon,\lambda)$ for each $j,k,l\geq N$ as desired. □

3.1. $\mathcal{I}_3^F$ and $\mathcal{I}_3^{F*}$-convergence for triple sequences in RTN spaces. In this subsection we study the concept $\mathcal{I}$ and $\mathcal{I}^*$-convergence of a triple sequence in $(X,F,*)$ and prove some important results.

Definition 3.2. Let $(X,F,*)$ be a RTN space and $\mathcal{I}$ be a proper ideal in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. The triple sequence $x=(x_{jkl})$ in X is said to be $\mathcal{I}_3^F$-convergent to $L\in X$ ($\mathcal{I}_3^F$-convergent to $L\in X$ with respect to F-topology) if for each $\varepsilon>0$, $\lambda\in(0,1)$ and each nonzero $z\in X$,

$$\{(j,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: x_{jkl}, z\notin\mathcal{N}_L(\varepsilon,\lambda)\}\in\mathcal{I}_3.$$

In this case the vector L is called the $\mathcal{I}_3^F$-limit of the triple sequence $x=(x_{jkl})$ and we write $\mathcal{I}_3^F\text{-}\lim x, z=L$.

Lemma 3.3. *Let $(X,F,*)$ be a RTN space. If a triple sequence $x=(x_{jkl})$ is $\mathcal{I}_3^F$-convergent with respect to the random 2-norm F, then $\mathcal{I}_3^F$-limit is unique.*

Proof. Let us assume that $\mathcal{I}_3^F$-$\lim x, z = L_1$ and $\mathcal{I}_3^F$-$\lim x, z = L_2$ where $L_1 \neq L_2$. Since $L_1 \neq L_2$, select $\varepsilon > 0$, $\lambda \in (0,1)$ and each nonzero $z \in X$ such that $\mathcal{N}_{L_1}(\varepsilon, \lambda)$ and $\mathcal{N}_{L_2}(\varepsilon, \lambda)$ are disjoint neighborhoods of L_1 and L_2. Since L_1 and L_2 both are $\mathcal{I}_3^F$-limit of the sequence (x_{jkl}), we have

$$A = \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl}, z \notin \mathcal{N}_{L_1}(\varepsilon, \lambda)\}$$

and

$$B = \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl}, z \notin \mathcal{N}_{L_2}(\varepsilon, \lambda)\}$$

both belongs to $\mathcal{I}_3^F$. This implies that the sets

$$A^c = \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl}, z \in \mathcal{N}_{L_1}(\varepsilon, \lambda)\}$$

and

$$B^c = \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl}, z \in \mathcal{N}_{L_2}(\varepsilon, \lambda)\}$$

belongs to $\mathcal{F}(\mathcal{I}_3)$. In this way we obtain a contradiction to the fact that the neighborhoods $\mathcal{N}_{L_1}(\varepsilon, \lambda)$ and $\mathcal{N}_{L_2}(\varepsilon, \lambda)$ of L_1 and L_2 are disjoints. Hence we have $L_1 = L_2$. This completes the proof. □

Lemma 3.4. *Let $(X, F, *)$ be a RTN space. Then we have*

(a) F-$\lim x_{jkl}, z = L$, then $\mathcal{I}_3^F$-$\lim x_{jkl}, z = L$.

(b) If $\mathcal{I}_3^F$-$\lim x_{jkl}, z = L_1$ and $\mathcal{I}_3^F$-$\lim y_{jkl}, z = L_2$, then $\mathcal{I}_3^F$-$\lim (x_{jkl} + y_{jkl}), z = L_1 + L_2$.

(c) If $\mathcal{I}_3^F$-$\lim x_{jkl}, z = L$ and $\alpha \in \mathbb{R}$, then $\mathcal{I}_3^F$-$\lim \alpha x_{jkl}, z = \alpha L$.

(d) If $\mathcal{I}_3^F$-$\lim x_{jkl}, z = L_1$ and $\mathcal{I}_3^F$-$\lim y_{jkl}, z = L_2$, then $\mathcal{I}_3^F$-$\lim (x_{jkl} - y_{jkl}), z = L_1 - L_2$

Proof. (a) Suppose that F-$\lim x_{jkl}, z = L$. Let $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$. Then there exists positive integer N such that $x_{jkl}, z \in \mathcal{N}_L(\varepsilon, \lambda)$ for each $j, k, l > N$. Since the set

$$\begin{aligned} A &= \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl}, z \notin \mathcal{N}_L(\varepsilon, \lambda)\} \\ &\subseteq \{1,2,3,...,N-1\} \times \{1,2,3,...,N-1\} \times \{1,2,3,...,N-1\} \end{aligned}$$

and the ideal a $\mathcal{I}_3^F$ is admissible, we have $A \in \mathcal{I}_3^F$. This shows that $\mathcal{I}_3^F$-$\lim x_{jk}, z = L$.

(b) Let $\varepsilon > 0, \lambda \in (0,1)$ and nonzero $z \in X$. Choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > (1-\lambda)$. Since $\mathcal{I}_3^F$-$\lim x_{jk}, z = L_1$ and $\mathcal{I}_3^F$-$\lim y_{jk}, z = L_2$, the sets

$$A = \left\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl}, z \notin \mathcal{N}_{L_1}\left(\frac{\varepsilon}{2}, \lambda\right)\right\}$$

and

$$B = \left\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : y_{jkl}, z \notin \mathcal{N}_{L_2}\left(\frac{\varepsilon}{2}, \lambda\right)\right\}$$

are belongs to $\mathcal{I}_3^F$. Let $C = \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : (x_{jkl} + y_{jkl}), z \notin \mathcal{N}_{L_1+L_2}(\varepsilon, \lambda)\}$. Since $\mathcal{I}_3^F$ is an ideal, it is sufficient to show that $C \subset A \cup B$. This is equivalent to show that $C^c \supset A^c \cap B^c$ where A^c and B^c are belongs to $\mathcal{F}(\mathcal{I}_3)$. Let

$(j,k,l) \in A^c \cap B^c$, i.e., $(j,k,l) \in A^c$ and $(j,k,l) \in B^c$, and we have

$$\begin{aligned} F_{(x_{jkl}+y_{jkl})-(L_1+L_2),z}(\varepsilon) &\geq F_{x_{jkl}-L_1,z}\left(\frac{\varepsilon}{2}\right) * F_{y_{jkl}-L_2,z}\left(\frac{\varepsilon}{2}\right) \\ &> (1-\eta) * (1-\eta) \\ &> (1-\lambda). \end{aligned}$$

Since $(j,k,l) \in C^c \supset A^c \cap B^c \in \mathcal{F}(\mathcal{I}_3)$, we have $C \subset A \cup B \in \mathcal{I}_3^F$.

(c) It is trivial for $\alpha = 0$. Now let $\alpha \neq 0$, $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$. Since $\mathcal{I}_3^F$-$\lim x_{jkl}, z = L$, we have

$$A = \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : x_{jkl}, z \notin \mathcal{N}_L(\varepsilon,\lambda)\} \in \mathcal{I}_3$$

This implies that

$$A^c = \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : x_{jkl}, z \in \mathcal{N}_L(\varepsilon,\lambda)\} \in \mathcal{F}(\mathcal{I}_3)$$

Let $(j,k,l) \in A^c$. Then we have

$$\begin{aligned} F_{\alpha x_{jkl}-\alpha L,z}(\varepsilon) &= F_{x_{jkl}-L,z}\left(\frac{\varepsilon}{|\alpha|}\right) \\ &\geq F_{x_{jkl}-L,z}(\varepsilon) * F_0\left(\frac{\varepsilon}{|\alpha|} - \varepsilon\right) \\ &> (1-\lambda) * 1 = (1-\lambda). \end{aligned}$$

So $\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \alpha x_{jkl}, z \notin \mathcal{N}_{\alpha L}(\varepsilon,\lambda)\} \in \mathcal{I}_3$. Hence $\mathcal{I}_3^F$-$\lim \alpha x_{jkl}, z = \alpha L$.

(d) The result follows from (b) and (c). $\square$

We introduce the concept of $\mathcal{I}_3^{F_*}$-convergence closely related to $\mathcal{I}_3^F$-convergence of triple sequences in random 2-normed space and show that $\mathcal{I}_3^{F*}$-convergence implies $\mathcal{I}_3^F$-convergence but not conversely.

Definition 3.5. Let $(X,F,*)$ be a RTN space. We say that a sequence $x = (x_{jkl})$ in X is said to be $\mathcal{I}_3^{F*}$-convergent to $L \in X$ with respect to the random 2-norm F if there exists a subset

$$K = \{(j_m,k_m,l_m) : j_1 < j_2 < ...;\ k_1 < k_2 < ...;\ l_1 < l_2 < ...\} \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$$

such that $K \in \mathcal{F}(\mathcal{I}_3)$ (i.e., $\mathbb{N}\times\mathbb{N}\times\mathbb{N}\backslash K \in \mathcal{I}_3$) and F-$\lim_m x_{j_m,k_m,l_m}, z = L$ for each nonzero $z \in X$.

In this case we write $\mathcal{I}_3^{F*}$-$\lim x, z = L$ and L is called the $\mathcal{I}_3^{F*}$-limit of the triple sequence $x = (x_{jkl})$.

Theorem 3.6. *Let $(X,F,*)$ be a RTN space and $\mathcal{I}_3$ be an admissible ideal. If $\mathcal{I}_3^{F*}$-$\lim x, z = L$, then $\mathcal{I}_3^F$-$\lim x, z = L$.*

Proof. Suppose that $\mathcal{I}_3^{F*}$-$\lim x, z = L$. Then by definition, there exists

$$K = \{(j_m,k_m,l_m) : j_1 < j_2 < ...;\ k_1 < k_2 < ...;\ l_1 < l_2 < ...\} \in \mathcal{F}(\mathcal{I}_3)$$

such that $F\text{-}\lim_m x_{j_m,k_m,l_m}, z = L$. Let $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$ be given. Since $F\text{-}\lim_m x_{j_m k_m, l_m}, z = L$, there exists $N \in \mathbb{N}$ such that $x_{j_m k_m, l_m}, z \in \mathcal{N}_L(\varepsilon, \lambda)$ for every $m \geq N$. Since

$$A = \{(j_m, k_m, l_m) \in K : x_{j_m k_m, l_m}, z \notin \mathcal{N}_L(\varepsilon, \lambda)\}$$

is contained in

$$B = \{j_1, j_2, ..., j_{N-1};\ k_1, k_2, ..., k_{N-1};\ l_1, l_2, ..., l_{N-1}\}$$

and the ideal $\mathcal{I}_3$ is admissible, we have $A \in \mathcal{I}_3$. Hence

$$\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl}, z \notin \mathcal{N}_L(\varepsilon, \lambda)\} \subseteq K \cup B \in \mathcal{I}_3$$

for $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$. Therefore, we conclude that $\mathcal{I}_3^F\text{-}\lim x, z = L$. $\square$

The following example shows that the converse of Theorem 3.6 need not be true.

Example 3.7. Consider $X = \mathbb{R}^2$ with $\|x, y\| := |x_1 y_2 - x_2 y_1|$ where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$ and let $a * b = ab$ for all $a, b \in S$. For all $(x, y) \in \mathbb{R}^2$ and $t > 0$, consider

$$F_{x,y}(t) = \frac{t}{t + \|x, y\|}.$$

Then $(\mathbb{R}^2, F, *)$ is an RTN space. Consider a decomposition of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ as $\mathbb{N} \times \mathbb{N} \times \mathbb{N} = \cup_{i,j,k} \Delta_{ijk}$ such that for any $(m,n,o) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ each Δ_{ijk} contains infinitely many (i,j,k)'s where $i \geq m$, $j \geq n$, $k \geq o$ and $\Delta_{ijk} \cap \Delta_{mno} = \emptyset$ for $(i,j,k) \neq (m,n,o)$. $\mathcal{I}_3$ be the class of all subsets of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ which intersect almost a finite number of Δ_{ijk}'s. Then $\mathcal{I}_3$ is an admissible ideal. We define a triple sequence (x_{mno}) as follows: $x_{mn} = \left(\frac{1}{ijk}, 0\right) \in \mathbb{R}^2$ if $(m,n,o) \in \Delta_{ij}$. Then for nonzero $z \in X$, we have

$$F_{x_{mno}, z}(t) = \frac{t}{t + \|x_{mno}, z\|} \to 1$$

as $m, n, o \to \infty$. Hence $\mathcal{I}_3^F\text{-}\lim_{m,n,o} x_{mno}, z = 0$.

Now, we show that $\mathcal{I}_3^{F_*}\text{-}\lim_{m,n,o} x_{mno}, z \neq 0$. Suppose that $\mathcal{I}_3^{F_*}\text{-}\lim_{m,n,o} x_{mno}, z = 0$. Then by definition, there exists a subset

$$K = \{(m_j, n_j, o_j) : m_1 < m_2 < ...;\ n_1 < n_2 < ...;\ o_1 < o_2 < ...\} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$$

such that $K \in \mathcal{F}(\mathcal{I}_3)$ and $F\text{-}\lim_j x_{m_j n_j o_j}, z = 0$. Since $K \in \mathcal{F}(\mathcal{I}_3)$, there exists $H \in \mathcal{I}_3$ such that $K = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \backslash H$. Then there exists positive integers p, q and r such that

$$H \subset \left(\bigcup_{o=1}^{r} \left(\bigcup_{m=1}^{p} \left(\bigcup_{n=1}^{\infty} \Delta_{mno}\right)\right)\right) \cup \left(\bigcup_{o=1}^{r} \left(\bigcup_{n=1}^{q} \left(\bigcup_{m=1}^{\infty} \Delta_{mno}\right)\right)\right).$$

Thus $\Delta_{p+1,q+1,r+1} \subset K$ and so $x_{m_j n_j o_j} = \frac{1}{(p+1)(q+1)(r+1)} > 0$ for infinitely many values (m_j, n_j, o_j)'s in K. This contradicts the assumption that F-$\lim_j x_{m_j n_j o_j}, z = 0$. Hence $\mathcal{I}_3^{F_*}$-$\lim_{m,n,o} x_{mno}, z \neq 0$.

Definition 3.8. An admissible ideal $\mathcal{I}_3 \subset 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ is said to satisfy the condition (AP) if for every sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{I}_3$ there are sets $B_n \subset \mathbb{N}$, $n \in \mathbb{N}$, such that the symmetric difference $A_n \Delta B_n$ is a finite set for every n and $\cup_{n\in\mathbb{N}} B_n \in \mathcal{I}_3$.

The following theorem shows that the converse holds if the ideal $\mathcal{I}_3$ satisfies condition (AP).

Theorem 3.9. *Let $(X, F, *)$ be a RTN space and the ideal $\mathcal{I}_3$ satisfy the condition (AP). If $x = (x_{jkl})$ is a triple sequence in X such that $\mathcal{I}_3^F$-$\lim x, z = L$, then $\mathcal{I}_3^{F_*}$-$\lim x, z = L$.*

Proof. Since $\mathcal{I}_3^F$-$\lim x, z = L$, so for every $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$, the set

$$\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : x_{jkl}, z \notin \mathcal{N}_L(\varepsilon, \lambda)\} \in \mathcal{I}_3.$$

We define the set A_p for $p \in \mathbb{N}$ as

$$A_p = \left\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : 1 - \frac{1}{p} \leq F_{x_{jkl}, z-L} < 1 - \frac{1}{p+1}\right\}.$$

Then it is clear that $\{A_1, A_2, ...\}$ is a countable family of mutually disjoint sets belonging to $\mathcal{I}_3$ and so by the condition (AP) there is a countable family of sets $\{B_1, B_2, ...\} \in \mathcal{I}_3$ such that the symmetric difference $A_i \Delta B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \cup_{i=1}^{\infty} B_i \in \mathcal{I}_3$. Since $B \in \mathcal{I}_3$, there is a set $K \in F(\mathcal{I}_3)$ such that $K = \mathbb{N}\times\mathbb{N}\times\mathbb{N} \backslash B$. Now we prove that the subsequence $(x_{jkl})_{(j,k,l)\in K}$ is convergent to L with respect to the random 2-norm F. Let $\eta \in (0,1)$, $\varepsilon > 0$ and nonzero $z \in X$. Choose a positive q such that $q^{-1} < \eta$. Then

$$\begin{aligned} &\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : x_{jkl}, z \notin \mathcal{N}_L(\varepsilon, \eta)\} \\ &\subset \left\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : x_{jkl}, z \notin \mathcal{N}_L\left(\varepsilon, \frac{1}{q}\right)\right\} \subset \cup_{i=1}^{q-1} A_i. \end{aligned}$$

Since $A_i \Delta B_i$ is a finite set for each $i = 1, 2, ..., q-1$, there exists $(j_0, k_0, l_0) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that

$$\begin{aligned} &\left(\cup_{i=1}^{q-1} B_i\right) \cap \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : j \geq j_0,\ k \geq k_0 \text{ and } l \geq l_0\} \\ &= \left(\cup_{i=1}^{q-1} A_i\right) \cap \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : j \geq j_0,\ k \geq k_0 \text{ and } l \geq l_0\}. \end{aligned}$$

If $j \geq j_0$, $k \geq k_0$, $l \geq l_0$ and $(j,k,l) \in K$, then $(j,k,l) \notin \cup_{i=1}^{q-1} B_i$ and $(j,k,l) \notin \cup_{i=1}^{q-1} A_i$. Hence for every $j \geq j_0$, $k \geq k_0$, $l \geq l_0$ and $(j,k,l) \in K$ we have

$$x_{jkl}, z \notin \mathcal{N}_L(\varepsilon, \eta).$$

Since this holds for every $\varepsilon > 0$, $\eta \in (0,1)$ and nonzero $z \in X$, we have $\mathcal{I}_3^{F_*}$-$\lim x, z = L$. Then, the desired result is obtained. $\square$

3.2. $\mathcal{I}_3^F$ and $\mathcal{I}_3^{F_*}$-triple Cauchy sequences in RTN spaces. In this subsection we study the concepts $\mathcal{I}_3$-Cauchy and $\mathcal{I}_3^*$-Cauchy of a triple sequence in $(X, F, *)$. Also, we will study the relations between these concepts.

Definition 3.10. Let $(X, F, *)$ be a RTN space and $\mathcal{I}_3$ be an admissible ideal of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. If for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero $z \in X$, there exists $s = s(\varepsilon)$, $t = t(\varepsilon)$, $u = u(\varepsilon)$ such that

$$\{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl} - x_{stu}, z \notin \mathcal{N}_\theta(\varepsilon, \lambda)\} \in \mathcal{I},$$

then a triple sequence $x = (x_{jkl})$ of elements in X is called to be $\mathcal{I}_3^F$-Cauchy sequence in X

Definition 3.11. Let $(X, F, *)$ be a RTN space and $\mathcal{I}_3$ be an admissible ideal of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. If for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero $z \in X$, there exists a set

$$K = \{(j_m, k_m, l_m) : j_1 < j_2 < ...;\ k_1 < k_2 < ...;\ l_1 < l_2 < ...\} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$$

such that $K \in F(\mathcal{I}_3)$ and (x_{j_m, k_m, l_m}) is an ordinary F-Cauchy in X, we say that a triple sequence $x = (x_{jkl})$ of elements in X is called to be $\mathcal{I}_3^{F_*}$-Cauchy sequence in X.

The next theorem gives that $\mathcal{I}_3^{F_*}$-triple Cauchy sequence implies $\mathcal{I}_3^F$-triple Cauchy sequence:

Theorem 3.12. *Let $(X, F, *)$ be a RTN space and $\mathcal{I}_3$ be a nontrivial ideal of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. If $x = (x_{jkl})$ is a $\mathcal{I}_3^{F*}$-triple Cauchy sequence, then $x = (x_{jkl})$ is a $\mathcal{I}_3^F$-triple Cauchy sequence.*

Proof. Let (x_{jkl}) be a $\mathcal{I}_3^{F*}$-Cauchy sequence, that is, for $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero $z \in X$, there exists

$$K = \{(j_m, k_m, l_m) : j_1 < j_2 < ...;\ k_1 < k_2 < ...;\ l_1 < l_2 < ...\} \in \mathcal{F}(\mathcal{I}_3)$$

and a number $N \in \mathbb{N}$ such that

$$x_{j_m k_m, l_m} - x_{j_p k_p l_p}, z \in \mathcal{N}_\theta(\varepsilon, \lambda)$$

for every $m, p \geq N$. Now, fix $p = j_{N+1}$, $r = k_{N+1}$, $s = l_{N+1}$. Moreover, for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero $z \in X$, we have

$$x_{j_m k_m l_m} - x_{prs}, z \in \mathcal{N}_\theta(\varepsilon, \lambda) \text{ for every } m \geq N.$$

Let $H = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \backslash K$. It is obvious that $H \in \mathcal{I}_3$ and

$$\begin{aligned} A(\varepsilon, \lambda) &= \{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl} - x_{prs}, z \notin \mathcal{N}_\theta(\varepsilon, \lambda)\} \\ &\subset H \cup \{j_1 < ... < j_N;\ k_1 < ... < k_N;\ l_1 < ... < l_N\} \in \mathcal{I}_3. \end{aligned}$$

Therefore, for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and nonzero $z \in X$, we can find $(p, r, s) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $A(\varepsilon, \lambda) \in \mathcal{I}_3$, i.e., (x_{jkl}) is a $\mathcal{I}_3^F$-triple Cauchy sequence. The proof of this theorem is complete. □

Now we will prove that $\mathcal{I}_3^{F_*}$-convergence implies $\mathcal{I}_3^F$-Cauchy condition in 2-normed space.

Theorem 3.13. *Let $(X, F, *)$ be a RTN space and $\mathcal{I}_3$ be an admissible ideal of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. If a sequence $x = (x_{jkl})$ is $\mathcal{I}_3^{F*}$-convergent, then it is a $\mathcal{I}_3^F$-triple Cauchy sequence.*

Proof. By assumption there exists a set

$$K = \{(j_m, k_m, l_m) : j_1 < j_2 < ...;\ k_1 < k_2 < ...;\ l_1 < l_2 < ...\} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$$

such that $K \in \mathcal{F}(\mathcal{I}_3)$ and $F\text{-}\lim_m x_{j_m,k_m,l_m}, z = L$ for each nonzero z in X, i.e., there exists $N \in \mathbb{N}$ such that $x_{j_m k_m l_m}, z \in \mathcal{N}_L(\varepsilon, \lambda)$ for every $\varepsilon > 0$, $\lambda \in (0,1)$, each nonzero z in X and $m > N$. Choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > (1-\lambda)$. Since

$$\begin{aligned} F_{x_{j_m k_m l_m} - x_{j_p k_p l_p}, z}(\varepsilon) &\geq F_{x_{j_m k_m l_m} - L, z}\left(\frac{\varepsilon}{2}\right) * F_{x_{j_p k_p l_p} - L, z}\left(\frac{\varepsilon}{2}\right) \\ &> (1-\eta) * (1-\eta) \\ &> 1 - \lambda \end{aligned}$$

for every $\varepsilon > 0$, $\lambda \in (0,1)$, each nonzero z in X and $m > N$, $p > N$, we have $x_{j_m k_m l_m} - x_{j_p k_p l_p}, z \notin \mathcal{N}_L(\varepsilon, \lambda)$ for every $m, p > N$ and each nonzero $z \in X$, i.e., (x_{jkl}) in X is an $\mathcal{I}_3^{F*}$-triple Cauchy sequence in X. Then by Theorem 3.12 (x_{jkl}) is a $\mathcal{I}_3^F$-triple Cauchy sequence in RTN space, as desired. □

Theorem 3.14. *Let $(X, F, *)$ be a RTN space and $\mathcal{I}_3$ be an admissible ideal of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. If a sequence $x = (x_{jkl})$ of elements in X is $\mathcal{I}_3^F$-convergent, then it is $\mathcal{I}_3^F$-triple Cauchy sequence.*

Proof. Suppose that (x_{jkl}) is $\mathcal{I}_3^F$-convergent to $L \in X$. Let $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$ be given. Then we have

$$A = \left\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl}, z \notin \mathcal{N}_L\left(\frac{\varepsilon}{2}, \lambda\right)\right\} \in \mathcal{I}_3$$

This implies that

$$A^c = \left\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl}, z \in \mathcal{N}_L\left(\frac{\varepsilon}{2}, \lambda\right)\right\} \in \mathcal{F}(\mathcal{I}_3)$$

Choose $\eta \in (0,1)$ such that $(1-\eta)*(1-\eta) > (1-\lambda)$. Then for every $(j,k,l), (s,t,u) \in A^c$,

$$\begin{aligned} F_{x_{jkl} - x_{stu}, z}(\varepsilon) &\geq F_{x_{jkl} - L, z}\left(\frac{\varepsilon}{2}\right) * F_{x_{stu} - L, z}\left(\frac{\varepsilon}{2}\right) \\ &> (1-\eta) * (1-\eta) \\ &> (1-\lambda). \end{aligned}$$

Hence $\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl} - x_{stu}, z \in \mathcal{N}_\theta(\varepsilon, \lambda)\} \in \mathcal{F}(\mathcal{I}_3)$ for nonzero $z \in X$. This implies that

$$\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl} - x_{stu}, z \notin \mathcal{N}_\theta(\varepsilon, \lambda)\} \in \mathcal{I}_3,$$

i.e., (x_{jkl}) is a $\mathcal{I}_3^F$-triple Cauchy sequence. This completes the proof. □

References

1. B. Altay and F. Başar, *Some new spaces of double sequences*, J. Math. Anal. Appl. **309** (2005), 70-90.
2. F. Başar, *Summability Theory and its Applications*, Bentham Science Publishers, Istanbul, 2012.
3. I.A. Demirci and M. Gürdal, *On lacunary generalized statistical convergent complex uncertain triple sequence*, J. Intell. Fuzzy Systems **41** (2021), 1021-1029.
4. I.A. Demirci and M. Gürdal, *On lacunary statistical ϕ-convergence for triple sequences of sets via ideals*, J. Appl. Math. Inform. in press, 2021.
5. H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241-244.
6. A.R. Freedman and J.J. Sember, *Densities and summability*, Pasific J. Math. **95** (1981), 293-305.
7. J.A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301-313.
8. S. Gähler, 2*-metrische räume und ihre topologische struktur*, Math. Nachr. **26** (1963), 115-148.
9. I. Golet, *On probabilistic* 2*-normed spaces*, Novi Sad. J. Math. **35** (2005), 95-102.
10. M. Gürdal and M.B. Huban, *On I-convergence of double sequences in the topology induced by random* 2*-norms*, Mat. Vesnik **66** (2014), 73-83.
11. M. Gürdal and S. Pehlivan, *The statistical convergence in* 2*-Banach spaces*, Thai. J. Math. **2** (2004), 107-113.
12. M. Gürdal and I. Açık, *On I-cauchy sequences in* 2*-normed spaces*, Math. Inequal. Appl. **11** (2008), 349-354.
13. M.B. Huban and M. Gürdal, *Wijsman lacunary invariant statistical convergence for triple sequences via Orlicz function*, J. Class. Anal. **17** (2021), 119-128.
14. M.B. Huban, M. Gürdal and H. Baytürk, *On asymptotically lacunary statistical equivalent triple sequences via ideals and Orlicz function*, Honam Math. J. **43** (2021), 343-357.
15. P. Kostyrko, M. Macaj and T. Salat, *I-convergence*, Real Anal. Exchange **26** (2000), 669-686.
16. K. Menger, *Statistical metrics*, Proc. Nat. Acad. Sci. USA **28** (1942), 535-537.
17. S.A. Mohiuddine and M. Aiyub, *Lacunary statistical convergence in random 2-normed spaces*, Appl. Math. Inf. Sci. **6** (2012), 581-585.
18. S.A. Mohiuddine, A. Alotaibi and S.M. Alsulami, *Ideal convergence of double sequences in random 2-normed spaces*, Adv. Differential Equations **2012:149** (2012), 1-8.
19. S.A. Mohiuddine and E. Savaş, *Lacunary statistically convergent double sequences in probabilistic normed spaces*, Ann. Univ. Ferrara Sez. VII Sci. Mat. **58** (2012), 331-339.
20. M. Mursaleen, *On statistically convergence in random* 2*-normed spaces*, Acta Sci. Math. (Sezeged) **76** (2010), 101-109.
21. M. Mursaleen and A. Alotaibi, *On I-convergence in random* 2*-normed spaces*, Math. Slovaca **61** (2011), 933-940.
22. M. Mursaleen and F. Başar, *Sequence Spaces: Topics in Modern Summability Theory*, CRC Press, Taylor & Francis Group, Series: Mathematics and Its Applications, Boca Raton, London, New York, 2020.
23. M. Mursaleen and O.H.H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl. **288** (2003), 223-231.
24. M. Mursaleen and S.A. Mohiuddine, *On ideal convergence of double sequences in probabilistic normed spaces*, Math. Rep. (Bucur.) **12** (2010), 359-371.
25. M. Mursaleen and S.A. Mohiuddine, *On ideal convergence in probabilistic normed spaces*, Math. Slovaca **62** (2012). 49-62.
26. A. Nabiev, S. Pehlivan and M. Gürdal, *On I-Cauchy sequences*, Taiwanese J. Math. **11** (2007), 569-576.
27. M.R.S. Rahmat and K.K. Harikrishnan, *On I-convergence in the topology induced by probabilistic norms*, European J. Pure Appl. Math. **2** (2009), 195-212.

28. T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca **30** (1980), 139-150.
29. E. Savaş and S.A. Mohiuddine, $\overline{\lambda}$*-statistically convergent double sequences in probabilistic normed spaces*, Math. Slovaca **62** (2012), 99-108.
30. B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math. **10** (1960), 313-334.
31. B. Schweizer and A. Sklar, *Probabilistic metric spaces*, North Holland, New York-Amsterdam-Oxford, 1983.
32. H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math. **2** (1951), 73-74.
33. N. Subramanian and A. Esi, *Wijsman rough lacunary statistical convergence on I Cesaro triple sequences*, Int. J. Anal. Appl. **16** (2018), 643-653.
34. A. Şahiner, M. Gürdal and F.K. Düden, *Triple sequences and their statistical convergence*, Selcuk J. Appl. Math. **8** (2007), 49-55.
35. A. Şahiner, M. Gürdal and S. Saltan, H. Gunawan, *Ideal convergence in* 2*-normed spaces*, Taiwanese J. Math. **11** (2007), 1477-1484.
36. A. Şahiner and B.C. Tripathy, *Some I-related properties of triple sequences*, Selcuk J. Appl. Math. **9** (2008), 9-18.

Verda Gürdal is a graduate student in Department of Mathematics, Suleyman Demirel University. Her research interests include summability theory and sequences spaces through functional analysis.

Department of Mathematics, Suleyman Demirel University, Isparta 32260, Turkey.
e-mail: verdagurdal@icloud.com