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A Note on Yamabe Solitons and Gradient Yamabe Solitons

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Abstract. We set our target to investigate *Yamabe solitons, gradient Yamabe solitons* and *gradient Einstein solitons* within the structure of 3-dimensional non-cosymplectic normal almost contact metric manifolds. Also, we provide a nontrivial example and validate a result of our paper.

1. Introduction

In [8], Hamilton introduced the notion of Yamabe solitons. According to the author, a Riemannian metric g of a complete Riemannian manifold (M^n, g) is called a Yamabe soliton if it satisfies

$$\frac{1}{2}\pounds_W g = (\lambda - r)g, \tag{1.1}$$

where W, λ, r and $\pounds$ denotes a smooth vector field, a real number, the scalar curvature and Lie-derivative respectively. The vector field W is said to be the soliton field of the Yamabe solitons. If W is the gradient of a C^∞ function $f : M^n \to \mathbb{R}$, then the manifold will be called *gradient yamabe soliton*. In this occasion, the previous equation reduces to

$$\nabla^2 f = (\lambda - r)g, \tag{1.2}$$

where $\nabla^2 f$ indicates the Hessian of f. A Yamabe soliton is said to be *shrinking*, *steady* or *expanding* according to $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. Yamabe

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solitons have been investigated by many researchers in different context (see, [1],[4], [5], [6]).

The concept of the *Gradient Einstein solitons* was initiated by Catino and Mazzieri [3]. For some smooth function f and some constant $\lambda \in \mathbb{R}$, the *Gradient Einstein solitons* are Riemannian manifolds obeying

$$S - \frac{1}{2}rg + \nabla^2 f = \lambda g, \tag{1.3}$$

where S denotes the Ricci tensor.

Many years ago in [10], Olszak investigated the 3-dimensional normal almost contact metric(briefly, *acm*) manifolds mentioning several examples. After the citation of [10], in recent years normal *acm* manifolds have been studied by many researchers in different context (see, [7],[10] and references contained in those).

The present article is constructed as follows:

In section 2, we recall a few basic facts and formulas of 3-dimensional non-cosymplectic normal *acm* manifolds which will be needed throughout the article. In section 3, we investigate the Yamabe, gradient Yamabe and gradient Einstein solitons. Specifically, we establish the below stated Theorems:

Theorem 1.1. If a 3-dimensional non-cosymplectic normal *acm* manifold admits a Yamabe soliton of the type (M^3, g, ξ), then the scalar curvature is constant and the characteristic vector field ξ is Killing.

Theorem 1.2. If a 3-dimensional non-cosymplectic normal *acm* manifold M^3 admits a Yamabe soliton of the type (M^3, g, W), then either the manifold is quasi-Sasakian or the scalar curvature of the manifold is constant and the soliton vector field W is Killing provided α, β are constants and the scalar curvature r is invariant under the characteristic vector field ξ.

Theorem 1.3. Let the Riemannian metric of a 3-dimensional non-cosymplectic normal *acm* manifold with α, β =constant be the gradient Yamabe soliton. Then either the manifold is of constant sectional curvature $-(\alpha^2 - \beta^2)$ or the manifold is α-Kenmotsu, provided the gradient yamabe soliton is trivial.

Theorem 1.4. Let the Riemannian metric of a 3-dimensional non-cosymplectic normal *acm* manifold with α, β =constant and $\alpha \neq \pm\beta$ be the gradient Einstein metric. Then either the manifold is α-Kenmotsu or is a manifold of constant sectional curvature.

2. Preliminaries

Let M^3 be an *acm* manifold endowed with a triplet of almost contact structure(η, ξ, ϕ). In details, M^3 is an odd-dimensional differentiable manifold equipped with a global 1-form η, a unique characteristic vector field ξ and a $(1,1)$-type tensor field ϕ, respectively, such that

$$\phi^2 E = -E + \eta(E)\xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0. \tag{2.1}$$

An *almost complex structure* J on $M \times \mathbb{R}$ is defined by

$$J(E, \lambda\frac{d}{dt}) = (\phi E - \lambda\xi, \eta(X)\frac{d}{dt}), \tag{2.2}$$

where $(E, \lambda\frac{d}{dt})$ denotes a tangent vector on $M \times \mathbb{R}$, E and $\lambda\frac{d}{dt}$ being tangent to M and $\mathbb{R}$ respectively. After fulfilling the condition, the structure J is integrable, M and (ϕ, ξ, η) are called normal (see, [2]).
The *Nijenhuis* torsion is defined by

$$[\phi, \phi](E, F) = \phi^2[E, F] + [\phi E, \phi F] - \phi[\phi E, F] - \phi[E, \phi F].$$

The structure (η, ξ, ϕ) is said to be normal if and only if

$$[\phi, \phi] + 2d\eta \otimes \xi = 0. \tag{2.3}$$

A Riemannian metric g on M^3 is called compatible with the structure (η, ξ, ϕ) if the condition

$$g(\phi E, \phi F) = g(E, F) - \eta(E)\eta(F), \tag{2.4}$$

holds for any $E, F \in \chi(M)$. In such case, the quadruple (η, ξ, ϕ, g) is termed as an *acm structure* on M^3 and M^3 is an *acm manifold*. The equation

$$\eta(E) = g(E, \xi), \tag{2.5}$$

is also valid on such a manifold.

Certainly, we can define the fundamental 2-form Φ by

$$\Phi(F, Z) = g(F, \phi Z), \tag{2.6}$$

where $F, Z \in \chi(M)$.

For a normal *acm*, we can write [10]:

$$(\nabla_E \phi)(F) = g(\phi\nabla_E \xi, F) - \eta(F)\phi\nabla_E \xi, \tag{2.7}$$

$$\nabla_E \xi = \alpha[E - \eta(E)\xi] - \beta\phi E, \tag{2.8}$$

$$(\nabla_E \eta)(F) = \alpha g(\phi E, \phi F) - \beta g(\phi E, F), \tag{2.9}$$

where $2\alpha = div\xi$ and $2\beta = tr(\phi\nabla\xi)$, $div\xi$ is the divergent of ξ defined by $div\xi = trace\{E \longrightarrow \nabla_E \xi\}$ and $tr(\phi\nabla\xi) = trace\{E \longrightarrow \phi\nabla_E \xi\}$. Utilizing (2.8) in (2.7) we lead

$$(\nabla_E \phi)(F) = \alpha[g(\phi E, F)\xi - \eta(F)\phi E] + \beta[g(E, F)\xi - \eta(F)E]. \tag{2.10}$$

Also in this manifold the subsequent relations hold [10]:

$$\begin{aligned} R(E,F)\xi &= [F\alpha + (\alpha^2-\beta^2)\eta(F)]\phi^2 E \\ &\quad -[E\alpha + (\alpha^2-\beta^2)\eta(E)]\phi^2 F \\ &\quad +[F\beta + 2\alpha\beta\eta(F)]\phi E \\ &\quad -[E\beta + 2\alpha\beta\eta(E)]\phi F, \end{aligned} \tag{2.11}$$

$$\begin{aligned} S(E,\xi) &= -E\alpha - (\phi E)\beta \\ &\quad -[\xi\alpha + 2(\alpha^2-\beta^2)]\eta(E), \end{aligned} \tag{2.12}$$

$$\xi\beta + 2\alpha\beta = 0. \tag{2.13}$$

$$(\nabla_E \eta)(F) = \alpha g(\phi E, \phi F) - \beta g(\phi E, F). \tag{2.14}$$

It is well admitted that the Riemann curvature tensor always satisfies

$$\begin{aligned} R(E,F)Z &= S(F,Z)E - S(E,Z)F + g(F,Z)QE - g(E,Z)QF \\ &\quad -\frac{r}{2}[g(F,Z)E - g(E,Z)F]. \end{aligned} \tag{2.15}$$

By (2.11), (2.12) and (2.15) we infer

$$\begin{aligned} S(F,Z) &= (\frac{r}{2} + \xi\alpha + \alpha^2 - \beta^2) g(\phi F, \phi Z) \\ &\quad -\eta(F)(Z\alpha + (\phi Z)\beta) - \eta(Z)(F\alpha + (\phi F)\beta) \\ &\quad -2(\alpha^2-\beta^2)\eta(F)\eta(Z). \end{aligned} \tag{2.16}$$

From (2.10) it follows that if α, β =constant, then the manifold is either α-Kenmotsu [9] or cosymplectic [2]or β-Sasakian. Also, it is well known that a 3-dimensional normal almost contact manifold reduces to a quasi-Sasakian manifold if and only if $\alpha = 0$ (see, [10]).

Now before producing the detailed proof of our main theorems, we first prove the following results:

Lemma 2.1. *For a 3-dimensional non-cosymplectic normal acm manifold with α, β =constant, we have*

$$(\nabla_E Q)\xi = -\{\frac{r}{2} + 3(\alpha^2-\beta^2)\}[\alpha\{E - \eta(E)\xi\} - \beta\phi E]. \tag{2.17}$$

Proof. For α, β =constants, we get from (2.16)

$$QF = \{\frac{r}{2} + (\alpha^2 - \beta^2)\}F - \{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}\eta(F)\xi. \tag{2.18}$$

Differentiating (2.18) covariantly in the direction of E and using (2.8) and (2.14), we get

$$\begin{aligned}(\nabla_E Q)F &= \frac{dr(E)}{2}(F - \eta(F)\xi) \\ &-\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}[\alpha g(E, F)\xi - 2\alpha\eta(E)\eta(F)\xi \\ &+\alpha\eta(F)E - \beta g(\phi E, F)\xi - \beta\eta(F)\phi E].\end{aligned} \tag{2.19}$$

Replacing F by ξ in (2.19) and using (2.8), we obtain

$$(\nabla_E Q)\xi = -\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}[\alpha\{E - \eta(E)\xi\} - \beta\phi E].$$

□

Lemma 2.2. *Let $M^3(\eta, \xi, \phi, g)$ be a non-cosymplectic normal acm manifold with α, β =constant. Then we have*

$$\xi r = -4\alpha\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\} \tag{2.20}$$

Proof. Recalling (2.19), we can write

$$\begin{aligned}g((\nabla_E Q)F, Z) &= \frac{dr(E)}{2}[g(F, Z) - \eta(F)\eta(Z)] \\ &-\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}[\alpha g(E, F)\eta(Z) - 2\alpha\eta(E)\eta(F)\eta(Z) \\ &+\alpha\eta(F)g(E, Z) - \beta g(\phi E, F)\eta(Z) - \beta\eta(F)g(\phi E, Z)].\end{aligned} \tag{2.21}$$

Putting $E = Z = e_i$ (where $\{e_i\}$ is an orthonormal basis for the tangent space of M and taking $\sum i$, $1 \le i \le 3$) in the foregoing equation and using the so called formula of Riemannian manifolds $divQ = \frac{1}{2}grad\ r$, we obtain

$$(\xi r)\eta(F) = -4\alpha\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}\eta(F). \tag{2.22}$$

Superseding $F = \xi$ in the previous equation we have the required result. □

Definition 2.1. A vector field W on an n dimensional Riemannian manifold (M, g) is said to be conformal if

$$\pounds_W g = 2\rho g, \tag{2.23}$$

ρ being the conformal coefficient. If the conformal coefficient is zero then the conformal vector field is a Killing vector field.

Lemma 2.3. [12] *On an n-dimensional Riemannian or, Pseudo-Riemannian manifold (M^n, g) endowed with a conformal vector field W, the following relations are satisfied:*

$$\begin{aligned}(\pounds_W S)(E,F) &= -(n-2)g(\nabla_E D\rho, F) + (\Delta\rho)g(E,F),\\ \pounds_W r &= -2\rho r + 2(n-1)\Delta\rho\end{aligned}$$

for $E, F \in \chi(M)$, D being the gradient operator and $\Delta = -divD$ being the Laplacian operator of g.

3. Proof of The Main Theorems

Proof of Theorem 1.1. Let a normal *acm* manifold M^3 admits a Yamabe soliton of the type (g, ξ). Then superseding $W = \xi$ in (1.1) yields

$$(\pounds_\xi g)(E,F) = (\lambda - r)g(E,F). \tag{3.1}$$

In view of (2.8), (3.1) becomes

$$(2\alpha - \lambda + r)g(E,F) - 2\alpha\eta(E)\eta(F) = 0. \tag{3.2}$$

Superseding $E = F = \xi$ in (3.2) and using (2.1), we have

$$r = \lambda. \tag{3.3}$$

Therefore the scalar curvature r is constant. Putting $\lambda = r$ in (3.1) yields $\pounds_\xi g = 0$ i.e., ξ is Killing vector field. Hence the theorem. □

Proof of Theorem 1.2. Let M^3 be a normal *acm* manifold with the structure which endowed the Yamabe soliton (g, W).

Taking Lie differential of $g(\xi, \xi) = 1$ along the soliton vector field V and making use of (1.1) yields

$$\eta(\pounds_V \xi) = -(\pounds_V \eta)(\xi) = \frac{r-\lambda}{2}. \tag{3.4}$$

Again, in view of (1.1) and (2.23) it is obvious that the soliton vector field W is conformal with the conformal coefficient $\rho = \frac{\lambda - r}{2}$. As we consider the metric g of the 3-dimensional normal *acm* manifold M is a Yamabe soliton, using $\rho = \frac{\lambda - r}{2}$ and $n = 3$ in Lemma 2.3, we have

$$(\pounds_W S)(E,F) = \frac{1}{2}g(\nabla_E Dr, F) - \frac{1}{2}(\Delta r)g(E,F), \tag{3.5}$$

and

$$\pounds_W r = r(r-\lambda) - 2\Delta r. \tag{3.6}$$

Here we consider α, β are constants. So from (2.16), we have

$$S(E,F) = [\frac{r}{2} + \alpha^2 - \beta^2]g(E,F) - [\frac{r}{2} + 3(\alpha^2 - \beta^2)]\eta(E)\eta(F). \tag{3.7}$$

Taking Lie differentiation of (3.7) along W and making use of (1.1) and (2.10), we obtain

$$\begin{aligned}(\pounds_W S)(E,F) &= -(\Delta r)g(E,F) + (\lambda - r)(\alpha^2-\beta^2)g(E,F) \\ &\quad -\{\frac{r(r-\lambda)}{2} - \Delta r\}\eta(E)\eta(F) \\ &\quad -\{\frac{r}{2} + 3(\alpha^2-\beta^2)\}\{(\pounds_W\eta)(E)\eta(F) \\ &\quad +\eta(E)(\pounds_W\eta)(F)\}.\end{aligned} \tag{3.8}$$

Making use of (3.5) in (3.8), we acquire

$$\begin{aligned}g(\nabla_E Dr, F) &= \{2(\alpha^2-\beta^2)(\lambda-r) - \Delta r\}g(E,F) \\ &\quad +\{r(\lambda - r) - 2\Delta r\}\eta(E)\eta(F) \\ &\quad -\{r + 6(\alpha^2-\beta^2)\}\{(\pounds_W\eta)(E)\eta(F) \\ &\quad +\eta(E)(\pounds_W\eta)(F)\}.\end{aligned} \tag{3.9}$$

Replacing $E = F = \xi$ in (3.9) and using of (2.1) and (3.4), we obtain

$$\xi(\xi r) = \Delta r - 4(\alpha^2-\beta^2)(\lambda - r). \tag{3.10}$$

Let r is invariant under the characteristic vector field ξ. Then either $\alpha = 0$ or r =constant. Thus we conclude that either the manifold is quasi-Sasakian or using r =constant in (3.10), we have $\lambda - r = 0$. Hence (1.1) immediately yields $\pounds_W g = 0$, i.e., the soliton vector field W is Killing. This completes the proof. □

If $\alpha = 0$ and $\beta = 1$, then the manifold reduces to a 3-dimensional Sasakian manifold. Since the characteristic vector field ξ is Killing in a Sasakian manifold, therefore $\xi r = 0$. Hence from the above theorem we can state the following:

Corollary 3.1. *If a three dimensional Sasakian manifold admits a Yamabe soliton, then the scalar curvature of the manifold is constant and the soliton vector field W is Killing.*

The foregoing Corollary was established by Sharma in [11].

Proof of Theorem 1.3. Let us consider a gradient Yamabe soliton on a 3-dimensional non-cosymplectic normal *acm* manifold with α, β =constant. Then from (1.2) we obtain

$$\nabla_E Df = (\lambda - r)E, \tag{3.11}$$

from which we acquire

$$R(E,F)Df = dr(E)F - dr(F)E. \tag{3.12}$$

Contraction of previous equation along F yields

$$S(E, Df) = 2dr(E). \tag{3.13}$$

Now, the equation (2.16) gives

$$S(E, Df) = \{\frac{r}{2} + (\alpha^2 - \beta^2)\}(Ef) - \{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}\eta(E)(\xi f). \tag{3.14}$$

Equation (3.13) and (3.14) together reveal that

$$2dr(E) = \{\frac{r}{2} + (\alpha^2 - \beta^2)\}(Ef) - \{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}\eta(E)(\xi f). \tag{3.15}$$

Putting $E = \xi$ and using (2.20), we get

$$(\xi f) = \frac{4\alpha}{\alpha^2 - \beta^2}\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}. \tag{3.16}$$

Hence, using (3.16) in (3.15), we have

$$2dr(E) = \{\frac{r}{2} + (\alpha^2 - \beta^2)\}(Ef) - \frac{4\alpha}{\alpha^2 - \beta^2}\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}\eta(E). \tag{3.17}$$

Now, from (3.12) we infer that

$$g(R(E,F)\xi, Df) = dr(E)\eta(F) - dr(F)\eta(E). \tag{3.18}$$

Again (2.11) implies that

$$g(R(E,F)\xi, Df) = (\alpha^2 - \beta^2)[\eta(E)(Ff) - \eta(F)(Ef)]. \tag{3.19}$$

Combining equation (3.18) and (3.19), we acquire

$$dr(E)\eta(F) - dr(F)\eta(E) = (\alpha^2 - \beta^2)[\eta(E)(Ff) - \eta(F)(Ef)]. \tag{3.20}$$

Setting $F = \xi$ in the above equation gives

$$Er = -(\alpha^2 - \beta^2)(Ef). \tag{3.21}$$

Using (3.21) in (3.17) we infer that

$$\{\frac{r}{2}+3(\alpha^2-\beta^2)\}[(Ef)-\frac{4\alpha}{\alpha^2-\beta^2}\{\frac{r}{2}+3(\alpha^2-\beta^2)\}\eta(E)]=0. \tag{3.22}$$

This shows that either $r=-6(\alpha^2-\beta^2)$ or $Df=(\xi f)\xi$. Next, we consider the above two cases as follows.

Case i: If $r=-6(\alpha^2-\beta^2)$, then from (2.16) we get $S=-2(\alpha^2-\beta^2)g$, that is the manifold is an Einstein manifold and hence from (2.15) it follows that the manifold is of constant sectional curvature $-(\alpha^2-\beta^2)$.

Case ii: If

$$Df=(\xi f)\xi. \tag{3.23}$$

Taking the covariant differentiation of (3.23) along any vector field $E\in\chi(M)$ we get

$$\nabla_E Df=E(\xi f)\xi+(\xi f)\nabla_E\xi. \tag{3.24}$$

Replacing E by ϕE and taking inner product with ϕF yields

$$g(\nabla_{\phi E}Df,\phi F)=(\xi f)\{\alpha g(E,F)-\alpha\eta(E)\eta(F)+\beta g(E,\phi F)\}. \tag{3.25}$$

Interchanging E and F in the foregoing equation, we infer

$$g(\nabla_{\phi F}Df,\phi E)=(\xi f)\{\alpha g(E,F)-\alpha\eta(E)\eta(F)+\beta g(F,\phi E)\}. \tag{3.26}$$

Applying Poincare's lemma, we have $d^2f(E,F)=0$ and hence by a straightforward calculation we lead

$$\nabla_E g(gradf,F)-\nabla_F g(gradf,E)-g(gradf,\nabla_E F)+g(gradf,\nabla_F,E)=0. \tag{3.27}$$

Since $\nabla g=0$, the above equation yields

$$g(\nabla_E gradf,F)-g(\nabla_F gradf,E)=0. \tag{3.28}$$

Replacing E by ϕE and F by ϕF in (3.28) and utilizing (3.25) and (3.26), we obtain

$$2(\xi f)\beta g(E,\phi F)=0,$$

which implies that

$$(\xi f)\beta d\eta(E,F)=0. \tag{3.29}$$

Since $d\eta\neq 0$, either $\beta=0$ or $(\xi f)=0$. Hence we conclude that either the manifold is α-Kenmotsu or ($f=$ constant) the gradient yamabe soliton is trivial. This completes the proof. □

Proof of Theorem 1.4. Let us suppose that the Riemannian metric of a 3-dimensional non-cosymplectic normal *acm* manifold with α, β =constant is a gradient Einstein metric. Then from (1.3) we obtain

$$\nabla_E Df = (\lambda + \frac{r}{2})E - QE. \tag{3.30}$$

From which we get

$$R(E,F)Df = (\nabla_F Q)E - (\nabla_E Q)F. \tag{3.31}$$

Now, from (2.19) we infer that

$$\begin{aligned} R(E,F)Df &= \frac{(Fr)}{2}[E - \eta(E)\xi] - \frac{(Er)}{2}[F - \eta(F)\xi] \\ &\quad -\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}[\alpha F\eta(E) - \alpha E\eta(F) \\ &\quad -2\beta g(E, \phi F)\xi - \beta\phi F\eta(E) + \beta\phi E\eta(F)]. \end{aligned} \tag{3.32}$$

The contraction of above equation along E and using (2.20), gives

$$S(F, Df) = \frac{(Fr)}{2}. \tag{3.33}$$

Equation (3.14) and (3.33) together reveal that

$$\frac{(Er)}{2} = \{\frac{r}{2} + (\alpha^2 - \beta^2)\}(Ef) - \{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}\eta(E)(\xi f). \tag{3.34}$$

Putting $E = \xi$ and utilizing (2.20), we have

$$(\xi f) = \frac{\alpha}{\alpha^2 - \beta^2}\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}. \tag{3.35}$$

Hence, using (3.35) in (3.34), we get

$$\frac{(Er)}{2} = \{\frac{r}{2} + (\alpha^2 - \beta^2)\}(Ef) - \frac{\alpha}{\alpha^2 - \beta^2}\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}^2\eta(E). \tag{3.36}$$

Now, from (3.32) we infer that

$$g(R(E,F)Df, \xi) = \{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}2\beta g(E, \phi F). \tag{3.37}$$

Combining equation (3.19) and (3.37) reveal that

$$\{\frac{r}{2} + 3(\alpha^2 - \beta^2)\}2\beta g(E, \phi F) = (\alpha^2 - \beta^2)[\eta(E)(Ff) - \eta(F)(Ef)]. \tag{3.38}$$

Putting $F = \xi$ in the above equation gives

$$(\alpha^2 - \beta^2)[\eta(E)(\xi f) - (Ef)]. \tag{3.39}$$

This shows that $Df = (\xi f)\xi$, provided $\alpha \neq \pm\beta$. Hence from the previous theorem, we conclude that either the manifold is α-Kenmotsu or $f =$ constant. If $f = constant$, then we get from (3.30) that the manifold is an Einstein manifold. Since the manifold is under consideration of dimension 3, hence the manifold is of constant sectional curvature.
This finishes the proof. □

4. Example

We consider a 3-dimensional Riemannian manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$, (x, y, z) being the standard coordinate in $\mathbb{R}^3$. Here we take the vector fields v_1, v_2 and v_3 given by

$$v_1 = \frac{\partial}{\partial x}, v_2 = \frac{\partial}{\partial y}, v_3 = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}.$$

We define the Riemannian metric g on M by $g(v_i, v_j) = \delta_{ij}$, $i, j = 1, 2, 3$ and η, a 1-form on M by $\eta(E) = g(E, v_1)$, for $E \in \chi(M)$. Let ϕ be a second order mixed tensor field defined by $\phi(v_1) = 0, \phi(v_2) = -v_3, \phi(v_3) = v_2$.

In this setting (ϕ, ξ, η, g) becomes an almost contact structure on M with $\xi = v_1$.

The setting of the vector fields v_1, v_2, v_3 gives

$$[v_1, v_2] = [v_3, v_1] = 0, [v_2, v_3] = -v_1.$$

Using Koszul's formula, we calculate the following:

$$\begin{aligned}
\nabla_{v_1} v_1 = 0, \nabla_{v_1} v_2 = \frac{1}{2}v_3, \nabla_{v_1} v_3 = -\frac{1}{2}v_2 \\
\nabla_{v_2} v_1 = \frac{1}{2}v_3, \nabla_{v_2} v_2 = 0, \nabla_{v_2} v_3 = -\frac{1}{2}v_1 \\
\nabla_{v_3} v_1 = -\frac{1}{2}v_2, \nabla_{v_3} v_2 = \frac{1}{2}v_1, \nabla_{v_3} v_3 = 0
\end{aligned}$$

It is easy to verify that the manifold M is a 3-dimensional non-cosymplictic normal almost contact metric manifold with $\alpha = 0$ and $\beta = \frac{1}{2}$.

By using the well-known formula $R(E, F)W = \nabla_E\nabla_F W - \nabla_F\nabla_E W - \nabla_{[E,F]}W$, we calculate the non-zero independent components of the curvature tensor as follows:

$$\begin{aligned}
R(v_1, v_2)v_2 = \frac{1}{4}v_1, R(v_1, v_3)v_3 = \frac{1}{4}v_1, R(v_2, v_1)v_1 = \frac{1}{4}v_2, \\
R(v_2, v_3)v_3 = -\frac{3}{4}v_2, R(v_3, v_1)v_1 = \frac{1}{4}v_3, R(v_3, v_2)v_2 = -\frac{3}{4}v_3.
\end{aligned}$$

Therefore we get the non-zero components of Ricci tensor as $S(v_1, v_1) = \frac{1}{2}, S(v_2, v_2) = S(v_3, v_3) = -\frac{1}{2}$. Hence the scalar curvature $r = -\frac{1}{2}$ =a constant.

Let $E = a_1v_1 + a_2v_2 + a_3v_3$ and $F = b_1v_1 + b_2v_2 + b_3v_3$. Then

$$\begin{aligned}(\pounds_{v_1} g)(E, F) &= g(\nabla_E v_1, F) + g(E, \nabla_F v_1) \\ &= a_2b_3 - a_3b_2 + a_3b_2 - a_2b_3 \\ &= 0.\end{aligned}$$

Therefore if we set $\lambda = -\frac{1}{2}$ the $(g, v_1 = \xi)$ becomes a Yamabe Soliton on M and also v_1 is a Killing vector field with the scalar curvature r =constant.
Hence the Theorem 1.1 is verified.

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