KYUNGPOOK Math. J. 62(2022), 89-106
https://doi.org/10.5666/KMJ.2022.62.1.89
pISSN 1225-6951 eISSN 0454-8124

An Iterative Method for Equilibrium and Constrained Convex Minimization Problems

Maryam Yazdi
Young Researchers and Elite Club, Malard Branch, Islamic Azad University, Malard, Iran
e-mail : msh_yazdi@yahoo.com

Mohammad Mehdi Shabani
Faculty of sciences, Imam Ali University, Tehran, Iran
e-mail : mohammadmehdishabani@yahoo.com

Saeed Hashemi Sababe*
Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Canada and
Young Researchers and Elite Club, Malard Branch, Islamic Azad University, Malard, Iran
e-mail : s.hashemi@ualberta.ca, hashemi_1365@yahoo.com

Abstract. We are concerned with finding a common solution to an equilibrium problem associated with a bifunction, and a constrained convex minimization problem. We propose an iterative fixed point algorithm and prove that the algorithm generates a sequence strongly convergent to a common solution. The common solution is identified as the unique solution of a certain variational inequality.

1. Introduction

Consider two problems in a Hilbert space: A constrained convex optimization (CCO) and an equilibrium problem (EP) associated with a bifunction satisfying certain properties. It is known that CCO can be solved by the gradient-projection algorithm (GPA). It is also known that EP is equivalent to a fixed point problem. Therefore, both problems can be solved by fixed point algorithms.

Let H be a real Hilbert space and C a nonempty closed convex subset of H. A

* Corresponding Author.
Received February 1, 2021; accepted November 8, 2021.
2020 Mathematics Subject Classification: 47H10, 47H09, 47J25, 65J15, 46T20.
Key words and phrases: Nonexpansive mapping, equilibrium problem, fixed point, convex minimization, averaged mapping, iterative method, variational inequality.

self-mapping T of C is said to be nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. A self-mapping f of C is said to be a κ-contraction if $\|f(x) - f(y)\| \leq \kappa\|x - y\|$ for all $x, y \in C$, where $\kappa \in [0, 1)$ is a constant. We use $Fix(T)$ to denote the set of fixed points of T; i.e., $Fix(T) = \{x \in C : Tx = x\}$.

Recall that the equilibrium problem (EP) associated to a bifunction $\phi : C \times C \to \mathbb{R}$ is to find a point $u \in C$ with the property

$$\phi(u, v) \geq 0, \quad v \in C. \tag{1.1}$$

The set of solutions of EP (1.1) is denoted by $EP(\phi)$.

If ϕ is of the form $\phi(u, v) = \langle Au, v - u\rangle$ for all $u, v \in C$, where $A : C \to H$ is a mapping, then $u \in EP(\phi)$ if and only if $u \in C$ is a solution to the variational inequality (VI):

$$\langle Au, v - u\rangle \geq 0, \quad v \in C.$$

Consequently, EP (1.1) includes, as special cases, numerous problems from various areas such as in physics, optimization and economics, and has been received a lot of attention; see, e.g., [7, 8, 9, 12, 13, 14, 16, 17, 18, 19, 22, 28] and the references therein.

In 2010, Tian [20] considered the following iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\lambda_n F)Tx_n, \ n \geq 0, \tag{1.2}$$

where F is Lipschitz and strongly monotone and $\gamma, \mu > 0$ are some constants. He proved that if the parameters $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy certain appropriate conditions, the sequence $\{x_n\}$ generated by (1.2) converges strongly to the unique solution $x^* \in Fix(T)$ of the variational inequality

$$\langle(\gamma f - \mu F)x^*, x - x^*\rangle \leq 0, \quad \text{for all} \quad x \in Fix(T).$$

Consider the constrained convex minimization problem:

$$\text{Minimize } \{g(x) : x \in C\}, \tag{1.3}$$

where $g : C \to \mathbb{R}$ is a real-valued convex function. We denote by U the set of solutions of (1.3). It is well known that the gradient-projection algorithm (GPA) can be used to solve (1.3). If g is continuously differentiable, then GPA generates a sequence $\{x_n\}$ via the recursive formula:

$$x_{n+1} = P_C(x_n - \lambda_n \nabla g(x_n)), \quad n \geq 0, \tag{1.4}$$

where the initial guess $x_0 \in C$ is chosen arbitrarily, and the parameters $\{\lambda_n\}$ are positive real numbers satisfying certain conditions. The convergence of GPA (1.4) depends on the behavior of the gradient ∇g. As a matter of fact, it is known that if ∇g is α-strongly monotone and L-Lipschitzian with constants $\alpha > 0$ and $L \geq 0$, then the operator

$$W_\lambda := P_C(I - \lambda\nabla g) \tag{1.5}$$

is a contraction for $0 < \lambda < 2\alpha/L^2$. Consequently, the sequence $\{x_n\}$ generated by PGA (1.4) converges in norm to the unique minimizer of (1.3) provided (λ_n) is contained in a compact subset of $(0, 2\alpha/L^2)$, in particular, $\lambda_n \equiv \lambda \in (0, 2\alpha/L^2)$.

However, if the gradient ∇g is not strongly monotone, the operator W_λ is no longer contractive (in general). As a result, PGA (1.4) fails to converge strongly in an infinite-dimensional Hilbert space (see a counterexample in [23]). Nevertheless, if ∇g is Lipschitzian, PGA (1.4) can still converge in the weak topology.

In 2012, Tian and Liu [21] studied a composite iterative scheme by the viscosity approximation method [15, 26] for finding a common solution of an equilibrium problem and a constrained convex minimization problem:

$$\begin{cases} \phi(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_n u_n, & n \geq 1, \end{cases} \tag{1.6}$$

where $\phi : C \times C \to \mathbb{R}$ is a bifunction, ∇g is L-Lipschitzian with $L \geq 0$ such that $U \cap EP(\phi) \neq \emptyset$, f is a contraction, $x_1 \in C$, $\{\alpha_n\} \subset (0,1)$, $\{r_n\} \subset (0, \infty)$, $u_n = Q_{r_n} x_n$, T_n is determined by the relation: $P_C(I - \lambda_n \nabla g) = s_n I + (1 - s_n)T_n$, $s_n = \frac{2-\lambda_n L}{4}$ and $\{\lambda_n\} \subset (0, \frac{2}{L})$. They proved that the sequence $\{x_n\}$, generated by (1.6), converges strongly to a point in $U \cap EP(\phi)$ under certain conditions.

In this paper, motivated by the above results, we will propose an iterative fixed point algorithm to find a point which is a common solution to an equilibrium problem associated with a bifunction and a constrained convex minimization problem. We will prove strong convergence of the algorithm under certain conditions and identify the limit of the sequence generated by the algorithm as the unique solution of some variational inequality.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Weak and strong convergence are denoted by the symbols $\rightharpoonup$ and $\to$, respectively. In a real Hilbert space H, we have the identity

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$. Let C be a nonempty closed convex subset of H. Then the (nearest point or metric) projection on C is defined by

$$P_C x := \arg\min_{y\in C} \|x-y\|^2, \quad x \in H.$$

Note that P_C is nonexpansive and characterized by the inequality (for $z \in C$)

$$z = P_C x \iff \langle x - z, z - y\rangle \geq 0, \quad y \in C.$$

The following inequality is convenient in applications. Recall that for all x, y in a Hilbert space H we have

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle.$$

Next, given a Hilbert space H and a function $\varphi : H \to \mathbb{R}$, we recal that φ is weakly lower semicontinuous l.s.c. considering H with its weak topology, that is, $\varphi(\hat{u}) \leq \liminf \varphi(u_n)$ whenever u_n converges weakly to $\hat{u}$.

Lemma 2.1. ([10]) *Let H be a real Hilbert space, C a closed convex subset of H and $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $(I-T)x_n \to y$, then $(I-T)x = y$. [This is known as the demiclosedness principle of nonexpansive mappings.]*

Let $\phi : C \times C \to \mathbb{R}$ be a bifunction. Throughout the paper we always assume that ϕ satisfies the following (standard) conditions [2]:

- (A_1) $\phi(x, x) = 0$ for all $x \in C$;
- (A_2) ϕ is monotone, i.e., $\phi(x, y) + \phi(y, x) \leq 0$, for all $x, y \in C$;
- (A_3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} \phi(tz + (1-t)x, y) \leq \phi(x, y)$;
- (A_4) for each $x \in C$, the mapping $y \mapsto \phi(x, y)$ is convex, weakly lower semicontinuous (l.s.c.).

Under these conditions, it is easy to see that the solution set $EP(\phi)$ of the equilibrium problem (1.1) is closed and convex.

Following Combettes and Hirstoaga [6], we can define, for each fixed $r > 0$, a mapping $Q_r : H \to C$ by the equation

$$Q_r x = z \tag{2.1}$$

for $x \in H$, where $z \in C$ is the unique solution of the inequality:

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad y \in C. \tag{2.2}$$

Lemma 2.2. ([6]) *The mapping Q_r possesses the properties:*

(i) *Q_r is firmly nonexpansive, namely,*

$$\|Q_r x - Q_r y\|^2 \leq \langle Q_r x - Q_r y, x - y\rangle, \quad x, y \in H;$$

(ii) *$Fix(Q_r) = EP(\phi)$.*

Property (ii) shows an equivalence of EP (1.1) to the fixed point problem of $Q_r x = x$ and thus EP (1.1) can be solved by fixed point methods.

Definition 2.3. ([21]) A mapping $T : H \to H$ is said to be an averaged mapping (av, for short) if $T = (1-\alpha)I + \alpha S$, where $\alpha \in (0,1)$, I is the identity map and $S : H \to H$ is nonexpansive. In this case, we say that T is α-averaged (α-av, for short).

Note that firmly nonexpansive mappings (e.g., projections) are $\frac{1}{2}$-av.

Proposition 2.4. ([4, 23]) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \cdots T_N$. In particular, if T_1 is α_1-av, and T_2 is α_2-av, where $\alpha_1, \alpha_2 \in (0,1)$, then the composite T_1T_2 is α-av, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

Definition 2.5. A nonlinear operator G with domain $D(G)$ and range $R(G)$ both in H is said to be:

(i) β-strongly monotone if there exists $\beta > 0$ such that

$$\langle x-y, Gx-Gy\rangle \geq \beta\|x-y\|^2, \quad x, y \in D(G),$$

(ii) ν-inverse strongly monotone (for short, ν-ism) if there exists $\nu > 0$ such that

$$\langle x-y, Gx-Gy\rangle \geq \nu\|Gx-Gy\|^2, \quad x, y \in D(G).$$

It can be easily seen that any projection P_C is a 1-ism.

Inverse strongly monotone (also referred to as co-coercive) operators have extensively been used in practical problems from various areas, for instance, traffic assignment problems; see, for example, [3, 11] and references therein.

Proposition 2.6. ([4, 23]) *Let T be an operator of H into itself.*

(i) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.*

(ii) *If T is ν-ism, then for $\mu > 0$, μT is $\frac{\nu}{\mu}$-ism.*

(iii) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0,1)$, T is α-av if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Lemma 2.7. ([1, 25]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\nu_n + \mu_n,$$

where $\{\gamma_n\}$ is a sequence in $[0,1]$, $\{\mu_n\}$ is a sequence of nonnegative real numbers and $\{\nu_n\}$ is a sequence in $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty}\gamma_n = \infty, \quad \limsup_{n\to\infty}\nu_n \leq 0, \quad \sum_{n=1}^{\infty}\mu_n < \infty.$$

Then $\lim_{n\to\infty} a_n = 0$.

3. An Iterative Scheme

In this section we introduce an iterative fixed point algorithm for finding a point in $U \cap EP(\phi)$, that is, a common solution to the equilibrium and optimization problems (1.1) and (1.3).

In the rest of this paper, we always assume that C is a nonempty closed convex subset of a Hilbert space H and $g : C \to \mathbb{R}$ is a real-valued continuously differentiable, convex function such that the gradient ∇g is L-Lipschitz for some $L \geq 0$. Recall that U denotes the set of solutions of the minimization problem (1.3). Then we have that a point $x^* \in U$ if and only if $x^* \in Fix(W_\lambda)$, where W_λ is defined in (1.5); that is,

$$x^* = P_C(I - \lambda \nabla g)x^*$$

for each $\lambda > 0$. Further helpful properties of the mapping $P_C(I - \lambda\nabla g)$ are outlined in [23] and summarized below.

Lemma 3.1. *Suppose a continuously differentiable, convex function g has L-Lipschitz continuous gradient ∇g and let $\lambda \in (0, 2/L)$ be given.*

(*i*) *The mapping $P_C(I-\lambda\nabla g)$ is α-av, where $\alpha = \frac{2+\lambda L}{4} \in (0,1)$; namely, one can write $P_C(I - \lambda\nabla g) = sI + (1 - s)T$, where $s = \frac{2-\lambda L}{4}$ and $T = \frac{1}{2+\lambda L}[4P_C(I - \lambda\nabla g) - (2 - \lambda L)I]$ is nonexpansive.*

(*ii*) *The function $h(\lambda)x := P_C(I - \lambda\nabla g)x$ is uniformly Lipschitz continuous in $\lambda \in [0, \infty)$ over bounded $x \in C$.*

Proof. (i) has been proved in [23]. To prove (ii) take λ, $\lambda' \in [0, \infty)$ and let $r > 0$ be given. Setting $M_r := \sup\{\|\nabla g(x)\| : \|x\| \leq r, x \in C\}$ and noting that P_C is nonexpansive, we get

$$\begin{aligned}\|h(\lambda) - h(\lambda')\| &= \|P_C(I - \lambda\nabla g)x - P_C(I - \lambda'\nabla g)x\| \\ &\leq |\lambda - \lambda'|\|\nabla g(x)\| \leq M_r|\lambda - \lambda'|.\end{aligned}$$

□

Lemma 3.2. ([27, Lemma 3.1]) *Suppose $F : C \to H$ is α-Lipschitz and η-strongly monotone and let $0 < \mu < \frac{2\eta}{\alpha^2}$. Then, for $\nu \in (0, 1)$, the mapping U^ν defined by*

$$U^\nu x := x - \mu\nu Fx, \quad x \in C \tag{3.1}$$

is a $(1 - \xi\nu)$-contraction from C into H, with $\xi := 1 - \sqrt{1 - \mu(2\eta - \mu\alpha^2)} \in (0, 1]$.

The following is the main result of this paper.

Theorem 3.3. *Assume the bifunction $\phi : C \times C \to \mathbb{R}$ satisfies the standard conditions (A_1)-(A_4), and the objective function $g : C \to \mathbb{R}$ is continuously differentiable and convex such that the gradient ∇g is L-Lipschitz with $L \geq 0$. Assume*

$U \cap EP(\phi) \neq \emptyset$. Let Q_r be defined by (2.1) for $r > 0$. Let f be a κ-contraction of C with $\kappa \in [0,1)$ and $F : C \to H$ an α-Lipschitz and η-strongly monotone operator on C with $\alpha > 0$ and $\eta > 0$. Assume $0 < \mu < \frac{2\eta}{\alpha^2}$ and $0 < \gamma < \frac{\xi}{\kappa}$ with $\xi = 1 - \sqrt{1 - \mu(2\eta - \mu\alpha^2)}$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ are real sequences satisfying the following conditions:

(B_1) $\{\alpha_n\} \subset [0,1]$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(B_2) $\{\beta_n\} \subset [0,1]$, $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(B_3) $\{r_n\} \subset (0,\infty)$, $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(B_4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

Let $\{x_n\}$ be a sequence generated by the iteration process:

$$\begin{cases} u_n = Q_{r_n} x_n, \\ y_n = P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T_n u_n), \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n T_n y_n, \qquad n \geq 1, \end{cases} \tag{3.2}$$

where the initial point $x_1 \in C$ is selected arbitrarily, $\{\lambda_n\} \subset (0, \frac{2}{L})$, and T_n is determined by the relation $P_C(I - \lambda_n \nabla g) = s_n I + (1 - s_n) T_n$ with $s_n = \frac{2 - \lambda_n L}{4}$ (cf. Lemma 3.1(i)). Namely,

$$T_n = \frac{1}{1 - s_n} (P_C(I - \lambda_n \nabla g) - s_n I). \tag{3.3}$$

Suppose $\{\lambda_n\}$ satisfies the condition

(B_5) $0 < \underline{\lambda} \leq \lambda_n < \frac{2}{L}$ *for all n, and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ defined by (3.2) converge strongly to a point $q \in U \cap EP(\phi)$, where $q = P_{U \cap EP(\phi)}(I - \mu F + \gamma f)(q)$ is the unique solution to the following variational inequality:

$$\langle (\mu F - \gamma f) q, q - x \rangle \leq 0, \quad x \in U \cap EP(\phi).$$

To prove Theorem 3.3 we first establish some lemmas.

Lemma 3.4. *Let T_n be defined by* (3.3)*. Let $r > 0$ and set*

$$M(r) := \sup\{4\|\nabla g(y)\| + L\|P_C y - y\| : \|y\| \leq r,\, y \in C\} < \infty.$$

Then, for $y \in C$ such that $\|y\| \leq r$, we have

$$\|T_{n+1} y - T_n y\| \leq M(r) |\lambda_{n+1} - \lambda_n|.$$

Proof. By definition of T_n, we have for $y \in C$

$$\begin{aligned}
T_{n+1}y - T_n y &= \frac{1}{1-s_{n+1}} P_C(y - \lambda_{n+1}\nabla g(y)) - \frac{s_{n+1}}{1-s_{n+1}} y \\
&\quad - \frac{1}{1-s_n} P_C(y - \lambda_n \nabla g(y)) + \frac{s_n}{1-s_n} y \\
&= \frac{1}{1-s_{n+1}} [P_C(y - \lambda_{n+1}\nabla g(y)) - P_C(y - \lambda_n \nabla g(y))] \\
&\quad + \left(\frac{1}{1-s_{n+1}} - \frac{1}{1-s_n} \right) P_C(y - \lambda_n \nabla g(y)) + \left(\frac{s_n}{1-s_n} - \frac{s_{n+1}}{1-s_{n+1}} \right) y \\
&= \frac{1}{1-s_{n+1}} [P_C(y - \lambda_{n+1}\nabla g(y)) - P_C(y - \lambda_n \nabla g(y))] \\
&\quad + \frac{s_{n+1} - s_n}{(1-s_n)(1-s_{n+1})} (P_C(y - \lambda_n \nabla g(y)) - y).
\end{aligned}$$

Since P_C is nonexpansive and $s_n = (2 - \lambda_n L)/4$ (thus $1 - s_n = \frac{2+\lambda_n}{4} \geq \frac{1}{2}$), it turns out that (noting $\lambda_n L < 2$)

$$\begin{aligned}
\|T_{n+1}y - T_n y\| &\leq 2\|\nabla g(y)\| |\lambda_{n+1} - \lambda_n| \\
&\quad + L|\lambda_{n+1} - \lambda_n| (\|P_C(y - \lambda_n \nabla g(y)) - P_C y\| + \|P_C y - y\|) \\
&\leq 2\|\nabla g(y)\| |\lambda_{n+1} - \lambda_n| \\
&\quad + L|\lambda_{n+1} - \lambda_n| (\lambda_n \|\nabla g(y)\| + \|P_C y - y\|) \\
&\leq (4\|\nabla g(y)\| + L\|P_C y - y\|) |\lambda_{n+1} - \lambda_n|.
\end{aligned}$$

This finishes the proof. □

Lemma 3.5. *For $x, x' \in C$ and $r, r' > 0$, we have*

$$\|Q_r x - Q_{r'} x\| \leq \left| 1 - \frac{r}{r'} \right| \|Q_{r'} x - x\| \tag{3.4}$$

and

$$\|Q_r x - Q_{r'} x'\| \leq \|x - x'\| + \left| 1 - \frac{r}{r'} \right| \|Q_{r'} x - x\|. \tag{3.5}$$

Proof. Set $u = Q_r x$ and $u' = Q_{r'} x$. By definition, we have

$$\phi(u, y) + \frac{1}{r} \langle y - u, u - x \rangle \geq 0, \quad \text{for all} \quad y \in C.$$

In particular,

$$\phi(u, u') + \frac{1}{r} \langle u' - u, u - x \rangle \geq 0.$$

Similarly, we have

$$\phi(u', u) + \frac{1}{r'}\langle u - u', u' - x\rangle \geq 0.$$

Adding up the last two relations and using the monotonicity $\phi(u, u') + \phi(u', u) \leq 0$, we obtain

$$\langle u' - u, \frac{1}{r}(u - x) - \frac{1}{r'}(u' - x)\rangle \geq 0.$$

This can be rewritten as

$$\begin{aligned}\frac{1}{r}\|u' - u\|^2 &\leq \left(\frac{1}{r} - \frac{1}{r'}\right)\langle u' - u, u' - x\rangle \\ &\leq \left|\frac{1}{r} - \frac{1}{r'}\right| \|u' - u\|\|u' - x\|\end{aligned}$$

and (3.4) follows immediately.

Next using the nonexpansivity of $Q_{r'}$, we get

$$\begin{aligned}\|Q_r x - Q_{r'} x'\| &\leq \|Q_{r'} x' - Q_{r'} x\| + \|Q_r x - Q_{r'} x\| \\ &\leq \|x - x'\| + \left|1 - \frac{r}{r'}\right| \|Q_{r'} x - x\|.\end{aligned}$$

The proof of the lemma is complete. □

Proof of Theorem 3.3. First we make a convention: For the sake of convenience, we will use $M > 0$ to stand for an appropriate constant for several estimates from various places throughout the proof which is divided into five steps.

Step 1. The sequences $\{x_n\}$ and $\{u_n\}$ are bounded. To see this, we take a point $p \in U \cap EP(\phi)$ to derive from (3.2) that (noting $Q_r p = p$ and $T_n p = p$ for all $r > 0$ and $n \geq 1$, and P_C is nonexpansive)

$$\begin{aligned}\|x_{n+1} - p\| &\leq (1 - \beta_n\|y_n - p\| + \beta_n\|T_n y_n - p\| \leq \|y_n - p\| \\ &\leq \|\alpha_n \gamma f(x_n) + (I - \alpha_n \mu F)T_n u_n - p\| \\ &= \|\alpha_n \gamma[f(x_n) - f(p)] + \alpha_n[\gamma f(p) - \mu F(p)] \\ &\quad + (I - \alpha_n \mu F)T_n u_n - (I - \alpha_n \mu F)(p)\|.\end{aligned}$$

Now since f is κ-contraction, $(I - \alpha_n \mu F)$ is $(1 - \alpha_n \xi)$-contraction, T_n is nonexpansive, and $\|u_n - p\| = \|Q_{r_n} x_n - p\| \leq \|x_n - p\|$, it follows from the last relation that

$$\begin{aligned}\|x_{n+1} - p\| &\leq (1 - \alpha_n(\xi - \gamma\kappa))\|x_n - p\| + \alpha_n\|\gamma f(p) - \mu F(p)\| \\ &\leq \max\{\|x_n - p\|, \frac{\|\gamma f(p) - \mu F(p)\|}{\xi - \gamma\kappa}\}.\end{aligned}$$

As a result, we get, by induction,

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{\|\gamma f(p) - \mu F(p)\|}{\xi - \gamma\kappa}\}$$

for all $n \geq 1$. Hence, $\{x_n\}$ is bounded, which implies that $\{u_n\}$, $\{y_n\}$, $\{f(x_n)\}$, $\{\mu F(T_n u_n)\}$ and $\{T_n y_n\}$ are all bounded.

Step 2. Asymptotic regularity of $\{x_n\}$: $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. To see this we use the definition of the algorithm (3.2) to derive that, after some manipulations,

$$\begin{aligned} x_{n+2} - x_{n+1} &= (1 - \beta_{n+1})y_{n+1} + \beta_{n+1}T_{n+1}y_{n+1} - (1 - \beta_n)y_n - \beta_n T_n y_n \\ &= (1 - \beta_{n+1})(y_{n+1} - y_n) + \beta_{n+1}(T_{n+1}y_{n+1} - T_{n+1}y_n) \\ &\quad + (\beta_n - \beta_{n+1})(y_n - T_{n+1}y_n) + \beta_n(T_{n+1}y_n - T_n y_n). \end{aligned} \tag{3.6}$$

Since T_{n+1} is nonexpansive and (y_n) is bounded, and by Lemma 3.4, we obtain from (3.6),

$$\|x_{n+2} - x_{n+1}\| \leq \|y_{n+1} - y_n\| + (|\beta_{n+1} - \beta_{n+1}| + \beta_n|\lambda_{n+1} - \lambda_n|)M, \tag{3.7}$$

where $M > 0$ is a big enough constant.

To estimate $\|y_{n+1} - y_n\|$, we again use (3.2) and the nonexpansiveness of P_C to get

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \|\alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}\mu F)T_{n+1}u_{n+1} \\ &\qquad - \alpha_n\gamma f(x_n) - (I - \alpha_n\mu F)T_n u_n\| \\ &= \|(\alpha_{n+1} - \alpha_n)[\gamma f(x_{n+1}) - \mu F(T_{n+1}u_{n+1})] \\ &\qquad + \alpha_n\gamma[f(x_{n+1}) - f(x_n)] \\ &\qquad + (I - \alpha_n\mu F)T_{n+1}u_{n+1} - (I - \alpha_n\mu F)T_n u_n\|. \end{aligned} \tag{3.8}$$

Observe by Lemmas 3.4 and 3.5

$$\begin{aligned} \|T_{n+1}u_{n+1} - T_n u_n\| &\leq \|T_{n+1}u_{n+1} - T_{n+1}u_n\| + \|T_{n+1}u_n - T_n u_n\| \\ &\leq \|u_{n+1} - u_n\| + \|T_{n+1}u_n - T_n u_n\| \\ &= \|Q_{r_{n+1}}x_{n+1} - Q_{r_n}x_n\| + \|T_{n+1}u_n - T_n u_n\| \\ &\leq \|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|\|Q_{r_{n+1}}x_n - x_n\| + M|\lambda_{n+1} - \lambda_n| \\ &\leq \|x_{n+1} - x_n\| + \left(|1 - \frac{r_n}{r_{n+1}}| + |\lambda_{n+1} - \lambda_n|\right)M. \end{aligned} \tag{3.9}$$

Now since f is κ-contraction and $I - \alpha_n\mu F$ is $(1 - \xi\alpha_n)$-contraction, also using (3.9), we get from (3.8)

$$\begin{aligned}\|y_{n+1}-y_n\| &\le M|\alpha_{n+1}-\alpha_n| + \kappa\gamma\alpha_n\|x_{n+1}-x_n\| \\ &\quad + (1-\xi\alpha_n)[\|x_{n+1}-x_n\| + (|1-\frac{r_n}{r_{n+1}}| + |\lambda_{n+1}-\lambda_n|)M] \\ &= [1-(\xi-\kappa\gamma)\alpha_n]\|x_{n+1}-x_n\| \\ &\quad + (|\alpha_{n+1}-\alpha_n| + |1-\frac{r_n}{r_{n+1}}| + |\lambda_{n+1}-\lambda_n|)M. \end{aligned} \tag{3.10}$$

Substituting (3.10)) into (3.7) yields

$$\|x_{n+2}-x_{n+1}\| \le (1-\hat{\alpha}_n)\|x_{n+1}-x_n\| + \nu_n + \delta_n, \tag{3.11}$$

where $\hat{\alpha}_n = (\xi-\kappa\gamma)\alpha_n$, $\nu_n = M|\alpha_{n+1}-\alpha_n|$ and

$$\delta_n = (|\beta_{n+1}-\beta_n| + |1-\frac{r_n}{r_{n+1}}| + (1+\beta_n)|\lambda_{n+1}-\lambda_n|)M.$$

By condition (B_1), we have $\sum_n \hat{\alpha}_n = \infty$. By conditions (B_2)-(B_5), we always have $\sum_n \delta_n < \infty$, and moreover, $\sum_n \nu_n < \infty$ if $\sum_n |\alpha_{n+1}-\alpha_n| < \infty$ or $\lim_n(\nu/\hat{\alpha}_n) = 0$ if $\lim_n(\alpha_n/\alpha_{n+1}) = 1$. So Lemma 2.7 is applicable to (3.11), which yields $\|x_{n+1} - x_n\| \to 0$, as claimed.

Step 3. We claim that

(3a) $\|x_n - y_n\| \to 0$,

(3b) $\|x_n - u_n\| \to 0$,

(3c) $\|P_C(I-\lambda_n\nabla g)x_n - x_n\| \to 0$.

Indeed, since

$$\begin{aligned}\|x_n-y_n\| &\le \|x_n-x_{n+1}\| + \|x_{n+1}-y_n\| \\ &= \|x_n-x_{n+1}\| + \beta_n\|y_n-T_ny_n\| \\ &= \|x_n-x_{n+1}\| + M\beta_n \to 0 \quad (\text{as } \beta_n \to 0)\end{aligned}$$

and (3a) follows. To show (3b), we first observe that the firm nonexpansivity of Q_{r_n} implies that (noting $u_n = Q_{r_n}x_n$ and $Fix(Q_{r_n}) = EP(\phi)$)

$$\|u_n-p\|^2 \le \|x_n-p\|^2 - \|u_n-x_n\|^2, \quad p \in EP(\phi).$$

For the sake of brevity, we shall use the $O(\alpha_n)$ notation in our argument below. For

$p \in U \cap EP(\phi)$, we have (noting again that P_C is nonexpansive)

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\beta_n)(y_n - p) + \beta_n(T_n y_n - p)\|^2 \leq \|y_n - p\|^2 \\ &\leq \|\alpha_n \gamma f(x_n) + (I - \alpha_n \mu F)T_n u_n - p\|^2 \\ &= \|\alpha_n(\gamma f(x_n) - \mu F(p)) + (I - \alpha_n \mu F)T_n u_n - (I - \alpha_n \mu F)p\|^2 \\ &= \alpha_n^2 \|\gamma f(x_n) - \mu F(p)\|^2 + \|(I - \alpha_n \mu F)T_n u_n - (I - \alpha_n \mu F)p\|^2 \\ &\quad + 2\alpha_n \langle \gamma f(x_n) - \mu F(p), (I - \alpha_n \mu F)T_n u_n - (I - \alpha_n \mu F)p\rangle \\ &\leq \alpha_n^2 M + (1 - \xi\alpha_n)^2 \|u_n - p\|^2 + 2\alpha_n M(1 - \xi\alpha_n)\|u_n - p\| \\ &\leq \|u_n - p\|^2 + O(\alpha_n) \\ &\leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + O(\alpha_n).\end{aligned}$$

It turns out that

$$\begin{aligned}\|x_n - u_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + O(\alpha_n) \\ &= O(\|x_{n+1} - x_n\|) + O(\alpha_n) \to 0.\end{aligned}$$

This proves (3b). To verify (3c), it suffices to verify that $\|P_C(I - \lambda_n \nabla g)u_n - u_n\| \to 0$. Since $P_C(I - \lambda_n \nabla g) = s_n I + (1 - s_n)T_n$, it follows that

$$\begin{aligned}\|P_C(I - \lambda_n \nabla g)u_n - u_n\| &= (1 - s_n)\|T_n u_n - u_n\| \\ &\leq \|T_n u_n - y_n\| + \|y_n - u_n\| \\ &\leq \alpha_n \|\gamma f(x_n) - \mu F(T_n x_n)\| + \|y_n - u_n\| \\ &= O(\alpha_n) + \|y_n - u_n\| \to 0.\end{aligned}$$

Step 4. We have the following asymptotic variational inequality:

$$\limsup_{n\to\infty} \langle \gamma f(q) - \mu F(q), y_n - q\rangle \leq 0, \tag{3.12}$$

where q is the unique fixed point of the contraction $P_{U\cap EP(\phi)}(I - \mu F + \gamma f)$; namely, $q =_{U\cap EP(\phi)} (I - \mu F + \gamma f)q$. Alternatively, q is the unique solution of the variational inequality

$$\langle \gamma f(q) - \mu F(q), y - q\rangle \leq 0, \quad y \in U \cap EP(\phi). \tag{3.13}$$

To prove (3.12), take a subsequence (y_{n_i}) of (y_n) weakly convergent to a point $\hat{y} \in C$ and such that

$$\begin{aligned}\limsup_{n\to\infty} \langle \gamma f(q) - \mu F(q), y_n - q\rangle &= \lim_{i\to\infty} \langle \gamma f(q) - \mu F(q), y_{n_i} - q\rangle \\ &= \langle \gamma f(q) - \mu F(q), \hat{y} - q\rangle.\end{aligned}$$

By virtue of VI (3.13), it suffices to show $\hat{y} \in U \cap EP(\phi)$. To see $\hat{y} \in U$, we use (3c) to get (noticing $\|x_n - y_n\| \to 0$)

$$\|P_C(I - \lambda_{n_i} \nabla g)y_{n_i} - y_{n_i}\| \to 0. \tag{3.14}$$

We may assume $\lambda_{n_i} \to \hat{\lambda}$; note that $\hat{\lambda} \in (0, 2/L]$ due to condition (B_5). It is then not hard (see Lemma 3.1(ii)) to find from (3.14) that

$$\|P_C(I - \hat{\lambda}\nabla g)y_{n_i} - y_{n_i}\| \to 0. \tag{3.15}$$

The demiclosedness principle of nonexpansive mappings then ensures that $\hat{y} \in Fix(P_C(I - \hat{\lambda}\nabla g) = U$. It remains to show $\hat{y} \in EP(\phi)$. Since $u_n = Q_{r_n}x_n$, it follows from the definition of Q_r and the monotonicity of ϕ that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq -\phi(u_n, y) \geq \phi(y, u_n), \quad y \in C.$$

This results in $\limsup_{n\to\infty} \phi(y, u_n) \leq 0$ for each $y \in C$, due to the facts $\|u_n - x_n\| \to 0$ and $\inf_n r_n > 0$. As $\phi(y, \cdot)$ is l.s.c. and $u_{n_i} \rightharpoonup \hat{y}$, it turns out that $\phi(y, \hat{y}) \leq 0$ for each $y \in C$. Now set $y_t = ty + (1-t)\hat{y} \in C$ with $t \in (0, 1)$. We then have by the standard conditions (A_1)-(A_4)

$$0 = \phi(y_t, ty + (1-t)\hat{y}) \leq t\phi(y_t, y) + (1-t)\phi(y_t, \hat{y}) \leq t\phi(y_t, y).$$

Hence, $\phi(y_t, y) \geq 0$ and $\phi(y, y_t) \leq 0$. Letting $t \to 0$ yields $\phi(y, \hat{y}) \leq 0$ for each $y \in C$. This asserts $\hat{y} \in EP(\phi)$ and the proof of Step 4 is complete.

Step 5. Strong convergence of $\{x_n\}$: $x_n \to q$ in norm, where q satisfies (3.12). Setting $z_n := \alpha_n\gamma f(x_n) + (I - \alpha_n\mu F)T_n u_n$ (thus, $y_n = P_C z_n$), we derive that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(1-\beta_n)(y_n - q) + \beta_n(T_n y_n - p)\|^2 \leq \|y_n - q\|^2 \\
&\leq \|\alpha_n\gamma f(x_n) + (I - \alpha_n\mu F)T_n u_n - q\|^2 \\
&= \|[(I - \alpha_n\mu F)T_n u_n - (I - \alpha_n\mu F)q] + \alpha_n(\gamma f(x_n) - \mu F(q))\|^2 \\
&\leq \|(I - \alpha_n\mu F)T_n u_n - (I - \alpha_n\mu F)q\|^2 \\
&\quad + 2\alpha_n\langle\gamma f(x_n) - \mu F(q), z_n - q\rangle \\
&\leq (1 - \xi\alpha_n)^2\|T_n u_n - q\|^2 + 2\gamma\alpha_n\langle f(x_n) - f(q), x_n - q\rangle \\
&\quad + 2\gamma\alpha_n\langle f(x_n) - f(q), z_n - x_n\rangle + 2\alpha_n\langle\gamma f(q) - \mu F(q), z_n - q\rangle \\
&\leq (1 - \xi\alpha_n)^2\|u_n - q\|^2 + 2\gamma\kappa\alpha_n\|x_n - q\|^2 \\
&\quad + 2\gamma\kappa\alpha_n\|x_n - q\|\|z_n - x_n\| + 2\alpha_n\langle\gamma f(q) - \mu F(q), z_n - q\rangle \\
&\leq (1 - 2(\xi - \gamma\kappa)\alpha_n)\|x_n - q\|^2 + \xi^2\alpha_n^2 M^2 \\
&\quad + 2\gamma\kappa\alpha_n M\|z_n - x_n\| + 2\alpha_n\langle\gamma f(q) - \mu F(q), z_n - q\rangle.
\end{aligned}$$

Here $M \geq \|x_n - q\|$ for all n. We can rewrite the last relation as

$$\|x_{n+1} - q\|^2 \leq (1 - \tilde{\alpha}_n)\|x_n - q\|^2 + \tilde{\alpha}_n\tilde{\delta}_n, \tag{3.16}$$

where $\tilde{\alpha}_n = 2(\xi - \gamma\kappa)\alpha_n$ and

$$\tilde{\delta}_n = \frac{\xi^2 M^2\alpha_n + 2\gamma\kappa M\|z_n - x_n\| + 2\langle\gamma f(q) - \mu F(q), z_n - q\rangle}{2(\xi - \gamma\kappa)}.$$

Since $y_n = P_C z_n$ and $T_n u_n \in C$, we obtain

$$\begin{aligned}\|y_n - z_n\| &= \|P_C z_n - z_n\| \le \|P_C z_n - T_n u_n\| + \|T_n u_n - z_n\| \\ &\le 2\|z_n - T_n u_n\| = 2\alpha_n\|\gamma f(x_n) - \mu F(T_n u_n)\| \to 0.\end{aligned}$$

It turns out from (3a) of Step 3 and (3.12) that $\|z_n - x_n\| \to 0$ and

$$\limsup_{n\to\infty}\langle \gamma f(q) - \mu F(q), z_n - q\rangle \le 0, \tag{3.17}$$

It is now immediately clear that $\sum_{n=1}^{\infty} \tilde{\alpha}_n = \infty$ and $\limsup_{n\to\infty} \tilde{\delta}_n \le 0$. This enables us to apply Lemma 2.7 to the relation (3.16) to arrive at $\|x_n - q\| \to 0$, that is, $x_n \to q$ in norm. □

Remark 3.6. The choices of the parameters (α_n), (β_n), (r_n) are easy. For instance, any decreasing null sequence (β_n) satisfies (B_2). More precisely, the choices:

$$\alpha_n = \frac{1}{n^a},\ \beta_n = \frac{1}{n^b},\ r_n = r + \frac{1}{n^c}, \quad n \ge 1,$$

where $0 < a \le 1$ and $b, c, r > 0$ satisfy $(B_1) - (B_4)$. Also, the choice $\lambda_n = \underline{\lambda} + \frac{1}{n^d}$, where $\underline{\lambda} > 0$ and $d > 0$ satisfies (B_5) for n big enough.

Remark 3.7. Theorem 3.3 is a generalization of [21, Theorem 3.2]. In addition, we used the condition $\liminf_{n\to\infty} \lambda_n > 0$ for the stepsizes $\{\lambda_n\}$. However, [21, Theorem 3.2] required that $\lambda_n \to \frac{2}{L}$, which needs the exact value L of the Lipschitz constant of the gradient ∇g of the objective function g. In practical problems, the exact value of L would be unavailable in many circumstances; consequently, verification of the condition $\lambda_n \to \frac{2}{L}$ turns out to be hard.

4. Numerical Test

In this section, we give a numerical example to illustrate the scheme (3.2) given in Theorem 3.3.

Example 4.1. Let $C = [-20, 20] \subset \mathbb{R}$ and define $\phi(x, y) = -4x^2 + 3xy + y^2$, where $x, y \in \mathbb{R}$. It can easily be verified that ϕ satisfies the conditions $(A_1) - (A_4)$. Let us deduce a formula for $Q_r x$, where $r > 0$ and $x \in \mathbb{R}$. For $y \in [-20, 20]$, we have

$$\begin{aligned}&\phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0 \\ &\Longleftrightarrow ry^2 + ((3r + 1)z - x)y + xz - (4r + 1)z^2 \ge 0.\end{aligned}$$

Set

$$G(y) := ry^2 + ((3r + 1)z - x)y + xz - (4r + 1)z^2.$$

Then $G(y)$ is a quadratic function of y with coefficients

$$a := r, \quad b := (3r+1)z - x, \quad c := xz - (4r+1)z^2.$$

So its discriminant is $\Delta := b^2 - 4ac = [(5r+1)z - x]^2$. Since $G(y) \geq 0$ for all $y \in C$, it turns out that $\Delta \leq 0$. That is, $[(5r+1)z - x]^2 \leq 0$. Therefore, $z = \frac{x}{5r+1}$, which yields $Q_r(x) = \frac{x}{5r+1}$.

Table 1: The values of the sequences $\{x_n\}$ and $\{u_n\}$

Numerical results for $x_1 = 12$ and $x_1 = -18$					
n	x_n	u_n	n	x_n	u_n
1	12	2	1	-18	-3
2	2.944	0.49067	2	-4.416	-0.736
3	0.42394	0.07065	3	-0.6359	-0.10598
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
15	$2.2373e^{-15}$	$3.7289e^{-16}$	15	$-3.356e^{-15}$	$-5.593e^{-16}$
16	$1.088e^{-16}$	$1.8133e^{-17}$	16	$-1.632e^{-16}$	$-2.72e^{-17}$
17	$5.1872e^{-18}$	$8.6453e^{-19}$	17	$-7.7808e^{-18}$	$-1.2968e^{-18}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
28	$6.1677e^{-33}$	$1.028e^{-33}$	28	$-9.251e^{-33}$	$-1.5419e^{-33}$
29	$2.5625e^{-34}$	$4.2709e^{-35}$	29	$-3.8438e^{-34}$	$-6.4064e^{-35}$
30	$1.0574e^{-35}$	0	30	$-1.5862e^{-35}$	0

Thus, by Lemma 2.2, we get $EP(\phi) = \{0\}$. Let $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{10n}$, $\lambda_n = \frac{1}{4}$, $r_n = 1$, for all $n \in \mathbb{N}$, $F(x) = \frac{1}{4}x$ (hence, $\alpha = \frac{1}{4}$ and $\eta = \frac{1}{4}$), $f(x) = \frac{1}{2}x$, $g(x) = x^2$, $\mu = 2$ and $\gamma = \frac{1}{2}$. Hence $U \cap EP(\phi) = \{0\}$ and $s_n = \frac{2-\lambda_n L}{4} = \frac{3}{8}$. Also, $T_n x = \frac{1}{5}x$, for all $x \in [-20, 20]$. Indeed,

$$P_C(I - \lambda_n \nabla g)x = P_{[-20,20]}(x - \frac{x}{2}) = \frac{x}{2} = \frac{3}{8}x + \frac{5}{8}T_n x$$

for $x \in [-20, 20]$. Then, from Lemma 2.7, the sequences $\{x_n\}$ and $\{u_n\}$, generated by the algorithm

$$\begin{cases} u_n = Q_{r_n} x_n = \frac{1}{6}x_n, \\ y_n = \frac{1}{4n}x_n + (1 - \frac{1}{2n})T_n(\frac{1}{6}x_n) = \dfrac{2n+14}{60n}x_n, \\ x_{n+1} = \dfrac{100n^2 + 692n - 56}{3000n^2}x_n, \end{cases} \tag{4.1}$$

converge to $0 \in U \cap EP(\phi)$, and it is also evident that $0 = P_{U \cap EP(\phi)}(\frac{3}{4}I)(0)$.

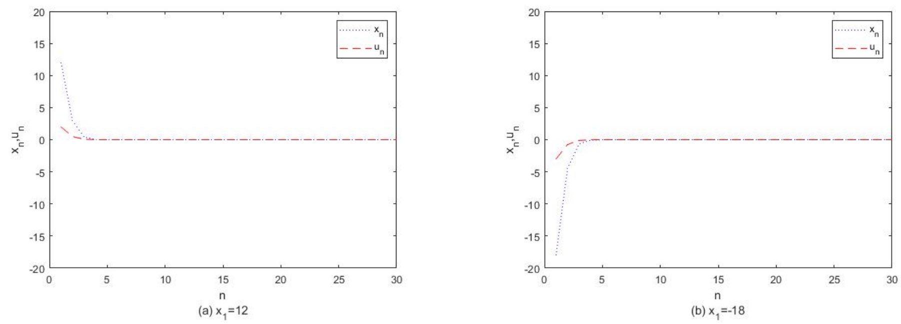

Figure 1: The convergence of $\{x_n\}$ and $\{u_n\}$ with different initial values x_1

Table 1 indicates the values of the sequences $\{x_n\}$ and $\{u_n\}$ generated by the algorithm (4.1) with different initial values of $x_1 = 12$ and $x_1 = -18$, respectively, and $n = 30$.

Figure 1 presents the behavior of the sequences $\{x_n\}$ and $\{u_n\}$ that corresponds to Table 1 and shows that both sequences converge to $0 \in U \cap EP(\phi)$.

Acknowledgements. A part of this research was carried out while the third author was visiting the university of Alberta. The authors are grateful to Professor Hong-Kun Xu for his comments.

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