KYUNGPOOK Math. J. 62(2022), 167-177
https://doi.org/10.5666/KMJ.2022.62.1.167
pISSN 1225-6951 eISSN 0454-8124

Classifications of Tubular Surface with L_1-Pointwise 1-Type Gauss Map in Galilean 3-space $\mathbb{G}_3$

İlim Kişi*
Department of Mathematics, Kocaeli University, Kocaeli 41380, Turkey
e-mail : ilim.ayvaz@kocaeli.edu.tr

Günay Öztürk
Department of Mathematics, İzmir Democracy University, İzmir 35140, Turkey
e-mail : gunay.ozturk@idu.edu.tr

Abstract. In this manuscript, we handle a tubular surface whose Gauss map G satisfies the equality $L_1G = f(G + C)$ for the Cheng-Yau operator L_1 in Galilean 3-space $\mathbb{G}_3$. We give an example of a tubular surface having L_1-harmonic Gauss map. Moreover, we obtain a complete classification of tubular surface having L_1-pointwise 1-type Gauss map of the first kind in $\mathbb{G}_3$ and we give some visualizations of this type surface.

1. Introduction

Finite type immersions are first given by Chen [6]. Let M be a submanifold in m-dimensional Euclidean space $\mathbb{E}^m$. An isometric immersion $x : M \to \mathbb{E}^m$ is of *finite type* if it can be written as a finite sum of eigenvectors of the Laplacian Δ of M for a constant map x_0, and non-constant maps $x_1, x_2, ..., x_k$, i.e.,

$$x = x_0 + \sum_{i=1}^{k} x_i.$$

Here, $\Delta x = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$. The submanifold is said to be of $k-$*type* if the numbers λ_is are different [6].

Chen and Piccinni generalised these immersions to the Gauss map G of M

$$\Delta G = a(G + C)$$

for a constant vector C and a real number a in [7]. A submanifold that satisfies the last equality are said to have a 1-*type Gauss map.*

* Corresponding Author.

Received December 26, 2020; accepted March 23, 2021.
2020 Mathematics Subject Classification: 53A35, 53B30.
Key words and phrases: Cheng-Yau operator, Gauss map, tubular surface, Galilean 3-space.

In the last equality, one can take a non-constant differentiable function f instead of a. Namely, one can generalise the last equality to

$$\Delta G = f(G + C). \tag{1.1}$$

A submanifold that satisfies the equation (1.1) is said to have a *pointwise* 1-*type Gauss map*. Also, if the vector C is zero, the pointwise 1-type Gauss map is said to be of *the first kind*. Otherwise, it is of *the second kind*. If $\Delta G = 0$, the Gauss map is harmonic. Surfaces satisfying the equation (1.1) are the subject of many studies such as [3, 4, 13].

In [2, 10], the notion of finite type submanifolds is generalised by replacing the Laplacian operator with operators L_k $(k = 1, 2, ..., n-1)$ that represent the linear operators of the first variation of the $(k+1)$-th mean curvature of a submanifold. Here, $L_0 = -\Delta$ and L_1 is the Cheng-Yau operator. Recently, some papers have been published about surfaces having L_1-pointwise 1-type Gauss map in some spaces, such as [11, 12, 18].

Tubular surfaces are special cases of canal surfaces which are the envelopes of a family of spheres. In canal surfaces, the center of the spheres are on a given space curve (spine curve), and the radius of the spheres are different. In tubular surfaces, the radius functions are constant. These surfaces have been widely studied in recent times [5, 13, 14, 15, 16]. In Galilean 3-space, tubular surfaces are studied in [9].

2. Basic Concepts

Here, some preliminaries about Galilean geometry are given. For more detailed information, the studies [19, 20] can be examined.

The scalar product and the cross product of the two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $\mathbb{G}_3$ are defined as

$$\langle a, b\rangle = \begin{cases} a_1 b_1, & if \quad a_1 \neq 0 \quad or \quad b_1 \neq 0 \\ a_2 b_2 + a_3 b_3 & if \quad a_1 = 0 \quad and \quad b_1 = 0, \end{cases}$$

and

$$a \times b = \begin{vmatrix} 0 & \mathbf{e_2} & \mathbf{e_3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

respectively. Here, $\mathbf{e_2}$=$(0, 1, 0)$ and $\mathbf{e_3}$=$(0, 0, 1)$ are the orthonormal unit vectors. The length (norm) of the vector $a = (a_1, a_2, a_3)$ is given as follows:

$$\|a\| = \begin{cases} |a_1|, & if \quad a_1 \neq 0 \\ \sqrt{a_2^2 + a_3^2}, & if \quad a_1 = 0 \end{cases}$$

[17].

An admissible unit speed curve $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3$ is given with the parametrization

$$\alpha(u) = (u, y(u), z(u)).$$

The associated Frenet frame on the curve is given as

$$\begin{aligned} t(u) &= (1, y'(u), z'(u)), \\ n(u) &= \frac{1}{\kappa(u)}(0, y''(u), z''(u)), \\ b(u) &= \frac{1}{\kappa(u)}(0, -z''(u), y''(u)), \end{aligned}$$

where $\kappa(u) = \sqrt{(y''(u))^2 + (z''(u))^2}$ and $\tau(u) = \frac{\det\left(\alpha'(u),\alpha''(u),\alpha'''(u)\right)}{\kappa^2(u)}$ are the curvature and the torsion of the curve, respectively. Thus, the famous Frenet formulas can be written as

$$\begin{aligned} t' &= \kappa n, \\ n' &= \tau b, \\ b' &= -\tau n. \end{aligned}$$

Definition 2.1. ([1]) A regular curve in Galilean space $\mathbb{G}_3$ with constant curvature and non-constant torsion is called a *Salkowski curve*.

For an isometric immersion $X : M \to \widetilde{M}$ from a hypersurface M from an $(n+1)$-dimensional Riemannian manifold $\widetilde{M}$, and for the Levi-Civita connections $\widetilde{\nabla}$ of $\widetilde{M}$ and ∇ of M, the Gauss formula is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle S(X), Y \rangle,$$

where $X, Y \in \chi(M)$ and S is the shape operator of M. It is known that the eigenvalues $\kappa_1, \kappa_2, ..., \kappa_n$ of S are the *principal curvatures* of M. For a smooth function f on M, linear operators L_k are defined

$$L_k(f) = div(P_k(\nabla f)), \tag{2.1}$$

where ∇ is the gradient, *div* is the divergence operator and

$$P_k = \sum_{i=0}^{k} (-1)^i s_{k-i} S^i$$

is the Newton k-th transformation, $s_k = \begin{pmatrix} n \\ k \end{pmatrix} H_k$ is the k-th mean curvature [8]. Thus, for $k = 0$, $P_0 = I_n$ (I_n is the identity matrix), and for $k = 1$, $P_1 = tr(S)I_n - S$.

Now, let M be a surface, e_1, e_2 be the principal directions correspond to the curvatures k_1, k_2 of M. From (2.1), for a smooth function f the Cheng-Yau operator $L_1 f$ can be given as

$$\begin{aligned} L_1 f &= div(P_1(\nabla f)) \\ &= e_1[k_2]e_1 f + e_2[k_1]e_2 f + k_2\left(e_1 e_1 - \nabla_{e_2} e_2\right) f + k_1\left(e_2 e_2 - \nabla_{e_1} e_1\right) f. \end{aligned}$$

Hence, the Cheng-Yau operator L_1 can be given

$$L_1 = e_1[k_2]\widetilde{\nabla} e_1 + e_2[k_1]\widetilde{\nabla} e_2 + k_2\left(\widetilde{\nabla} e_1 \widetilde{\nabla} e_1 - \widetilde{\nabla}_{\nabla_{e_2} e_2}\right) + k_1\left(\widetilde{\nabla} e_2 \widetilde{\nabla} e_2 - \widetilde{\nabla}_{\nabla_{e_1} e_1}\right)$$

[11].

Let the surface M parametrized with

$$X(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2))$$

in $\mathbb{G}_3$. To represent the partial derivatives, we use

$$x_{,i} = \frac{\partial x}{\partial u_i} \quad and \quad x_{,ij} = \frac{\partial^2 x}{\partial u_i \partial u_j}, \qquad 1 \leq i, j \leq 2.$$

If $x_{,i} \neq 0$ for some $i = 1, 2$, then the surface is admissible (i.e. having not any Euclidean tangent planes). The first fundamental form I of the surface M is defined as

$$I = (g_1 d_{u_1} + g_2 d_{u_2})^2 + \varepsilon(h_{11} d_{u_1}^2 + 2h_{12} d_{u_1} d_{u_2} + h_{22} d_{u_2}^2),$$

where $g_i = x_{,i}$, $h_{ij} = y_{,i}\, y_{,j} + z_{,i}\, z_{,j}$; $i, j = 1, 2$ and

$$\varepsilon = \begin{cases} 0, & if \quad d_{u_1} : d_{u_2} \quad is \quad non-isotropic, \\ 1, & if \quad d_{u_1} : d_{u_2} \quad is \quad isotropic. \end{cases}$$

Let a function W is given by

$$W = \sqrt{\left(x_{,1}\, z_{,2} - x_{,2}\, z_{,1}\right)^2 + \left(x_{,2}\, y_{,1} - x_{,1}\, y_{,2}\right)^2}. \tag{2.2}$$

Then, the unit normal vector field is given as

$$G = \frac{1}{W}(0, -x_{,1}\, z_{,2} + x_{,2}\, z_{,1}\, , x_{,1}\, y_{,2} - x_{,2}\, y_{,1}\,). \tag{2.3}$$

Similarly, the second fundamental form II of the surface M is defined as

$$II = L_{11} d_{u_1}^2 + 2L_{12} d_{u_1} d_{u_2} + L_{22} d_{u_2}^2,$$

where

$$L_{ij} = \frac{1}{g_1}\left\langle g_1(0, y_{,ij}\, , z_{,ij}\,) - g_{i,j}(0, y_{,1}\, , z_{,1}\,), N\right\rangle, \qquad g_1 \neq 0$$

or

$$L_{ij} = \frac{1}{g_2} \left\langle g_2(0, y_{,ij}\,, z_{,ij}\,) - g_{i,j}(0, y_{,2}\,, z_{,2}\,), N \right\rangle, \qquad g_2 \neq 0.$$

The Gaussian and the mean curvatures of M are defined as

$$K = \frac{L_{11}L_{22} - L_{12}^2}{W^2} \qquad \textit{and } H = \frac{g_2^2 L_{11} - 2g_1g_2L_{12} + g_1^2 L_{22}}{2W^2}. \tag{2.4}$$

A surface is *flat* (resp. *minimal*) if its Gaussian (resp. mean) curvatures vanish [19].

Lemma 2.2. ([11]) *Let M be an oriented surface in $\mathbb{E}^3$ and K and H be the Gaussian and the mean curvatures of M, respectively. Then the Gauss map G of M satisfies*

$$L_1 G = -\nabla K - 2HKG. \tag{2.5}$$

Definition 2.3. ([11]) Let M be an oriented surface in $\mathbb{E}^3$. Then, M is said to have an L_1-*harmonic Gauss map* if its Gauss map satisfies $L_1 G = 0$.

Definition 2.4. ([11]) Let M be an oriented surface in $\mathbb{E}^3$. Then, M is said to have an L_1-*pointwise 1-type Gauss map* if its Gauss map satisfies

$$L_1 G = f\,(G + C) \tag{2.6}$$

for a smooth function f and a constant vector C. If the vector C is zero, the pointwise L_1-type Gauss map is of the *first kind*, otherwise, it is of the *second kind*.

3. Tubular Surface with L_1 Pointwise 1-Type Gauss Map in $\mathbb{G}_3$

A tubular surface M in $\mathbb{G}_3$ at a distance r from the points of spine curve $\alpha(u) = (u, y(u), z(u))$ is given with

$$M : X(u, v) = \alpha(u) + r(\cos vn + \sin vb). \tag{3.1}$$

Writing the Frenet vectors of $\alpha(u)$ in (3.1), the parametrization can be given as (3.2)

$$M : X(u, v) = (u, y(u), z(u)) + \frac{r}{\kappa} \left[\cos v(0, y''(u), z''(u)) + \sin v(0, -z''(u), y''(u))\right].$$

From (3.2),

$$g_1 = u_{,1} = 1, \quad g_2 = u_{,2} = 0. \tag{3.3}$$

An orthonormal frame $\{e_1, e_2, G\}$ of M is given by

$$\begin{aligned} e_1 &= \frac{X_u}{\|X_u\|} = t - r\tau \sin vn + r\tau \cos vb, \;\; \|X_u\| = 1 \\ e_2 &= \frac{X_v}{\|X_v\|} = -\sin vn + \cos vb, \quad \|X_v\| = r \end{aligned} \tag{3.4}$$

and

$$G = -\cos vn - \sin vb. \tag{3.5}$$

Here $W = r$. The coefficients of the second fundamental form are obtained as

$$L_{11} = -\kappa \cos v + r\tau^2, \qquad L_{12} = r\tau, \qquad L_{22} = r. \tag{3.6}$$

From, (3.3) and (3.6), the curvature functions of M are obtained as

$$K = \frac{-\kappa \cos v}{r}, \qquad H = \frac{1}{2r} \tag{3.7}$$

[9].

Corollary 3.5. ([9]) *Tubular surfaces are constant mean curvature surfaces in Galilean space.*

By (3.7), we write the gradient of the Gaussian curvature

$$\nabla K = \frac{-\kappa' \cos v}{r} e_1 + \frac{\kappa \sin v}{r} e_2. \tag{3.8}$$

Thus, from (3.4), (3.7) and (3.8), we obtain the Cheng-Yau operator of the Gauss map as

$$L_1 G = -\frac{1}{r^2} \left\{ \begin{array}{c} -\kappa' r \cos vt \\ + \left(\kappa'\tau r^2 \cos v \sin v - \kappa r \sin^2 v + \kappa \cos^2 v\right) n \\ + \left(-\kappa'\tau r^2 \cos^2 v + \kappa r \sin v \cos v + \kappa \cos v \sin v\right) b. \end{array} \right. \tag{3.9}$$

Now, we consider the surface M has L_1-harmonic Gauss map, i.e. $L_1 G = 0$. Then, from (3.9), we have

$$\kappa' r \cos v = 0$$

and

$$\begin{array}{rcl} \kappa'\tau r^2 \cos v \sin v - \kappa r \sin^2 v + \kappa \cos^2 v & = & 0, \\ -\kappa'\tau r^2 \cos^2 v + \kappa r \sin v \cos v + \kappa \cos v \sin v & = & 0. \end{array} \tag{3.10}$$

Writing $\kappa' r \cos v = 0$ in (3.10), we get

$$\begin{array}{rcl} -\kappa r \sin^2 v + \kappa \cos^2 v & = & 0, \\ \kappa r \sin v \cos v + \kappa \cos v \sin v & = & 0. \end{array}$$

Multiplying the first equation with *cosv* and the second with *sinv*, we obtain $\kappa \cos v = 0$, which implies $\kappa = 0$ or *cosv* $= 0$. If *cosv* $= 0$, again from (3.10), $\kappa = 0$.

Then, we give the following theorem:

Theorem 3.6. *Let M be a tubular surface given with the parametrization (3.1) in $\mathbb{G}_3$. M has L_1-harmonic Gauss map if and only if the spine curve α is a straight line and M is an open part of a cylinder. Thus, the surface is flat.*

Example 3.7. Let us consider the tubular surface M, which has L_1-harmonic Gauss map with the parametrization (3.1) in $\mathbb{G}_3$. Taking the straight line $\alpha(u) = (u, u+1, u+2)$ and writing the Frenet vectors of it $n(u) = (0,1,0)$, $b(u) = (0,0,1)$ and $r = 4$ in (3.1), we write the parametrization of the surface M as

$$M : X(u,v) = (u, u+1+4\cos v, u+2+4\sin v). \tag{3.11}$$

By using the software Maple, we plot the graph of the surface in (3.11).

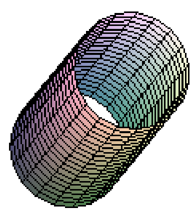

Figure 1: Tubular surfaces M which has L_1-harmonic Gauss map with the spine curve $\alpha(u) = (u, u+1, u+2)$ and the radius $r = 4$.

Now, we assume that the tubular surface M has L_1-pointwise 1-type Gauss map of the first kind, i.e., $L_1 G = fG$ for a smooth function f. Then, from (3.5) and (3.9),

$$\begin{aligned} (3.12) \qquad & - \frac{1}{r^2}\left\{ \begin{array}{c} -\kappa' r \cos v t \\ + \left(\kappa'\tau r^2 \cos v \sin v - \kappa r \sin^2 v + \kappa \cos^2 v\right) n \\ + \left(-\kappa'\tau r^2 \cos^2 v + \kappa r \sin v \cos v + \kappa \cos v \sin v\right) b. \end{array} \right\} \\ & = -f\cos v n - f \sin v b \end{aligned}$$

From (3.12), we have

$$\kappa' \cos v = 0$$

and

$$\begin{aligned} \kappa'\tau r^2 \cos v \sin v - \kappa r \sin^2 v + \kappa \cos^2 v &= f r^2 \cos v, \\ -\kappa'\tau r^2 \cos^2 v + \kappa r \sin v \cos v + \kappa \cos v \sin v &= f r^2 \sin v. \end{aligned} \tag{3.13}$$

Similar to above, writing $\kappa' \cos v = 0$ in (3.13), we get

$$\begin{aligned} -\kappa r \sin^2 v + \kappa \cos^2 v &= f r^2 \cos v, \\ \kappa r \sin v \cos v + \kappa \cos v \sin v &= f r^2 \sin v. \end{aligned} \tag{3.14}$$

Multiplying the first equation with $\cos v$, the second with $\sin v$, and combining them, we obtain $f = \frac{\kappa \cos v}{r^2}$. Moreover, since $\kappa' \cos v = 0$, we have two cases: $\kappa = 0$ or κ is a constant. If $\kappa = 0$, the tubular surface M has L_1-harmonic Gauss map. Thus, κ is a nonzero constant.

Theorem 3.8. *Let M be a tubular surface given with the parametrization (3.1) in $\mathbb{G}_3$. M has L_1-pointwise 1-type Gauss map of the first kind if and only if the curvature κ of the curve is constant and $f = -\frac{K}{r}$.*

Corollary 3.9. *The spine curve of the surface which has L_1-pointwise 1-type Gauss map of the first kind is a Salkowski curve in $\mathbb{G}_3$.*

Example 3.10. Let us consider the tubular surface M, which has L_1-pointwise 1-type Gauss map of the first kind with the parametrization (3.1) in $\mathbb{G}_3$. For the curves $\alpha_1(u) = (u, \cos u, \sin u)$, $\alpha_2(u) = (u, \frac{u^2}{2}, 0)$, and the radius $r = 2$, we write the parametrizations of the surfaces M_1 and M_2 as

$$
\begin{aligned}
(3.15) \qquad M_1 \quad &: \quad X(u,v) = (u, \cos u - 2\cos(u+v), \sin u - 2\sin(u+v)), \\
M_2 \quad &: \quad X(u,v) = \left(u, \frac{u^2}{2} + 2\cos v, 2\sin v\right).
\end{aligned}
$$

We again use Maple to plot the graphs of the surfaces in (3.15).

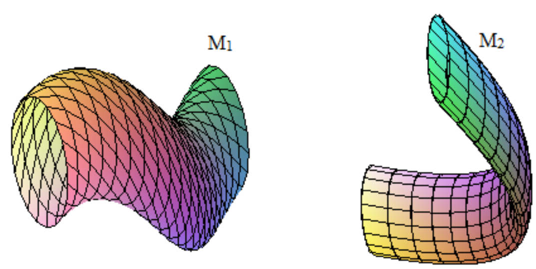

Figure 2: Tubular surfaces M_1 and M_2 which have L_1-harmonic Gauss map with the spine curves $\alpha_1(u) = (u, \cos u, \sin u)$, $\alpha_2(u) = (u, \frac{u^2}{2}, 0)$ and the radius $r = 2$.

Lastly, we consider that the tubular surface M has L_1-pointwise 1-type Gauss map of the second kind, i.e., $L_1 G = f(G + C)$ for a smooth function f and a nonzero constant vector C. From the equations (2.6) and (3.9), we can write the vector C as

$$C = -\frac{1}{fr^2}\left(-\kappa' r \cos v t + A(u,v)n + B(u,v)b\right),$$

where

$$\begin{aligned} A(u,v) &= \kappa'\tau r^2\cos v\sin v-\kappa r\sin^2 v+\kappa\cos^2 v-fr^2\cos v,\\ B(u,v) &= -\kappa'\tau r^2\cos^2 v+\kappa r\sin v\cos v+\kappa\cos v\sin v-fr^2\sin v. \end{aligned}$$

Since C is a nonzero constant vector, $\widetilde{\nabla}_{e_1}C=0$ and $\widetilde{\nabla}_{e_2}C=0$. Thus, we have

$$\begin{aligned} 0=\widetilde{\nabla}_{e_1}C &= e_1\left[\frac{\kappa'\cos v}{fr}\right]t\\ &\quad+\left(e_1\left[\frac{-A(u,v)}{fr^2}\right]+\frac{\tau B(u,v)}{fr^2}+\frac{\kappa\kappa'\cos v}{fr}\right)n\\ &\quad+\left(e_1\left[\frac{-B(u,v)}{fr^2}\right]-\frac{\tau A(u,v)}{fr^2}\right)b, \end{aligned}\tag{3.16}$$

and

$$0=\widetilde{\nabla}_{e_2}C=e_2\left[\frac{\kappa'\cos v}{fr}\right]t+e_2\left[\frac{-A(u,v)}{fr^2}\right]n+e_2\left[\frac{-B(u,v)}{fr^2}\right]b.\tag{3.17}$$

Since

$$e_1\left[\frac{\kappa'\cos v}{fr}\right]=e_2\left[\frac{\kappa'\cos v}{fr}\right]=0,$$

we get

$$\begin{aligned} \kappa''f &= \kappa'f_u,\\ -f\sin v &= f_v\cos v. \end{aligned}$$

From the last differential equation system,

$$f=a\kappa'\cos v,\tag{3.18}$$

where a is a real constant. By the equation (3.17),

$$e_2\left[\frac{-A(u,v)}{fr^2}\right]=e_2\left[\frac{-B(u,v)}{fr^2}\right]=0$$

which means

$$A(u,v)=fh_1(u),\quad B(u,v)=fh_2(u).\tag{3.19}$$

Here $h_1(u)$ and $h_2(u)$ are any functions of u. Moreover, from (3.16),

$$e_1\left[\frac{-B(u,v)}{fr^2}\right]-\frac{\tau A(u,v)}{fr^2}=0.$$

Writing (3.19) in the last equation, we obtain

$$e_1\left[-h_2(u)\right]-\tau h_1(u)=0,$$

which has a solution as

$$h_2(u) = -\int \tau h_1(u) du. \tag{3.20}$$

From (3.16), we have

$$e_1 \left[\frac{-A(u,v)}{fr^2} \right] + \frac{\tau B(u,v)}{fr^2} + \frac{\kappa\kappa' \cos v}{fr} = 0.$$

Again writing (3.19) in the last equation, we obtain

$$e_1 \left[-h_1(u) \right] + \tau h_2(u) + \frac{\kappa r}{a} = 0,$$

which has a solution as

$$h_1(u) = \int \left(\tau h_2(u) + \frac{\kappa r}{a} \right) du. \tag{3.21}$$

Then, we give the following theorem:

Theorem 3.11. *Let M be a tubular surface given with the parametrization (3.1) in $\mathbb{G}_3$. M has L_1-pointwise 1-type Gauss map of the second kind if and only if $f = a\kappa' \cos v$ for a real constant a and the equations (3.19)-(3.21) are hold.*

References

[1] A. T. Ali, *Position vectors of curves in the Galilen Space* $\mathbb{G}_3$, Matematički Vesnik, **64**(2012), 200–210.

[2] L. J. Alias and N. Gürbüz, *An extension of Takahashi theorem for the linearized operators of the higher order mean curvatures*, Geom. Dedicata, **121**(2006), 113–127.

[3] K. Arslan and B. (Kılıç) Bayram, B. Bulca, Y. H. Kim, C. Murathan and G. Öztürk, *Rotational embeddings in* $\mathbb{E}^4$ *with pointwise 1-type Gauss map*, Turk. J. Math., **35**(2011), 493–499.

[4] K. Arslan, B. (Kılıç) Bayram, B. Bulca, Y. H. Kim, C. Murathan and G. Öztürk, *Vranceanu surface in* $\mathbb{E}^4$ *with pointwise 1-type Gauss map*, Indian J. Pure Appl. Math., **42**(2011), 41–51.

[5] B. Bulca, K. Arslan, B. Bayram and G. Öztürk, *Canal surfaces in 4-dimensional Euclidean Space*, Libertas Mathematica, **32**(2012), 1–13.

[6] B. Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, Series in Pure Mathematics, 1. World Scientific Publishing Co., Singapore(1984).

[7] B. Y. Chen and P. Piccinni, *Submanifolds with finite type Gauss map*, Bull. Austral. Math. Soc., **35**(1987), 161–186.

[8] S. Y. Cheng and S. T. Yau, *Hypersurfaces with constant scalar curvature*, Math. Ann., **225**(1977), 195–204.

[9] M. Dede, *Tubular surfaces in Galilean space*, Math. Commun., **18(1)**(2013), 209–217.

[10] S. M. B. Kashani, *On some L_1-finite type (hyper)surfaces in* $\mathbb{R}^{n+1}$, Bull. Korean Math. Soc., **46**(2009), 35–43.

[11] Y. H. Kim and N. C. Turgay, *Surfaces in* $\mathbb{E}^3$ *with L_1-pointwise 1-type Gauss map*, Bull. Korean Math. Soc., **50(3)**(2013), 935–949.

[12] Y. H. Kim and N. C. Turgay, *On the ruled surfaces with L_1-pointwise 1-type Gauss map*, Kyungpook Math. J., **57(1)**(2017), 133–144.

[13] İ. Kişi and G. Öztürk, *A new approach to canal surface with parallel transport frame*, Int. J. Geom. Methods Mod. Phys., **14(2)**(2017), 1–16.

[14] İ. Kişi and G. Öztürk, *Tubular surface having pointwise 1-type Gauss map in Euclidean 4-space*, Int. Electron. J. Geom., **12**(2019), 202–209.

[15] İ. Kişi, G. Öztürk and K. Arslan, *A new type of canal surface in Euclidean 4-space* $\mathbb{E}^4$, Sakarya University Journal of Science, **23**(2019), 801–809.

[16] G. Öztürk, B. Bulca, B. K. Bayram and K. Arslan, *On canal surfaces in* $\mathbb{E}_3$, Selçuk J. Appl. Math., **11**(2010), 103–108.

[17] B. J. Pavkovic and I. Kamenarovic, *The equiform differential geometry of curves in the Galilean space* $\mathbb{G}_3$, Glas. Mat. Ser. III, **22**(1987), 449–457.

[18] J. Qian and Y. H. Kim, *Classifications of canal surfaces with L_1-pointwise 1-type Gauss map*, Milan J. Math., **83(1)**(2015), 145–155.

[19] O. Röschel, *Die Geometrie Des Galileischen Raumes*, Forschungszentrum Graz Research Centre, Austria(1986).

[20] I. M. Yaglom, *A simple non-Euclidean geometry and its physical basis*, Springer-Verlag Inc., New York(1979).