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Truncated Multi-index Sequences Have an Interpolating Measure

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Abstract. In this note we observe that any truncated multi-index sequence has an interpolating measure supported in Euclidean space. It is well known that the consistency of a truncated moment sequence is equivalent to the existence of an interpolating measure for the sequence. When the moment matrix of a moment sequence is nonsingular, the sequence is naturally consistent; a proper perturbation to a given moment matrix enables us to confirm the existence of an interpolating measure for the moment sequence. We also illustrate how to find an explicit form of an interpolating measure for some cases.

1. Introduction

We first discuss finite sequences of real numbers and then introduce the result of infinite sequences. Let $\beta \equiv \beta^{(m)} = \left\{\beta_{\mathbf{i}} \in \mathbb{R} : \mathbf{i} \in \mathbb{Z}_+^d,\ |\mathbf{i}| \le m\right\}$, with $\beta_{\mathbf{0}} \neq 0$, be a d-dimensional multisequence of degree m. It is called a *truncated moment sequence.* For a closed set $K \subseteq \mathbb{R}^d$, the *truncated K-moment problem* (TKMP) entails finding necessary and sufficient conditions for the existence of a positive Borel measure μ on $\mathbb{R}^d$ with supp $\mu \subseteq K$ such that

$$\beta_{\mathbf{i}} = \int \mathbf{x}^{\mathbf{i}}\ d\mu(\mathbf{x})\ \ (\mathbf{i} \in \mathbb{Z}_+^d,\ |\mathbf{i}| \le m),$$

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where $\mathbf{x} \equiv (x_1, \ldots, x_d)$, $\mathbf{i} \equiv (i_1, \ldots, i_d) \in \mathbb{Z}_+^d$, and $\mathbf{x}^{\mathbf{i}} := x_1^{i_1} \cdots x_d^{i_d}$. The measure μ is said to be a *K-representing measure* for β. For the typical case $K = \mathbb{R}^d$, the problem is referred to as the *truncated real moment problem* (TRMP) and μ is called simply a *representing measure*.

In a similar way, we consider the *full* moment problem for an infinite sequence $\beta \equiv \beta^{(\infty)} = \left\{\beta_{\mathbf{i}} : \mathbf{i} \in \mathbb{Z}_+^d\right\}$. As well known by H. L. Hamburger for $d = 1$, the sequence has a representing measure supported on $\mathbb{R}$ if and only if the Hankel matrice, $[\beta_{i+j}]_{0\le i\le k,\ 0\le j\le k}$ is positive semidefinite (or simply, positive). Furthermore, T. J. Stieltjes showed that the single-index sequence has a representing measure supported in $[0, \infty)$ if and only if both Hankel matrices, $[\beta_{i+j}]_{0\le i\le k,\ 0\le j\le k}$ and $[\beta_{i+j+1}]_{0\le i\le k,\ 0\le j\le k}$ for $k \ge 0$, are positive.

When $m = 2n$, we define a *moment matrix* $M_d(n)$ of $\beta \equiv \beta^{(2n)}$ as

$$M_d(n) \equiv M_d(n)(\beta) := (\beta_{\ \mathbf{i}+\mathbf{j}})_{\ \mathbf{i},\mathbf{j}\in\mathbb{Z}_+^d:\ |\mathbf{i}|,\,|\mathbf{j}|\le n}.$$

Some properties of $M_d(n)$ have been important factors for the existence of a representing measure for β; for example, $M_d(n)$ is necessarily positive (obviously the positivity of $M_d(n)$ is sufficient for $d = 1$ but not sufficient for $d \ge 2$ as well known). R. Curto and L. Fialkow have established many elegant results for various moment problems based on a positive extension of $M_d(n)$. They also have used the functional calculus in the column space of $M_d(n)$; to introduce the functional calculus, we label the columns and rows of $M_d(n)$ with monomials $X^{\mathbf{i}} := X_1^{i_1} \cdots X_d^{i_d}$ in the degree-lexicographic order. Note that each block with the moments of the same order in $M_d(n)$ is Hankel and that $M_d(n)$ is symmetric. In addition, one can define a sesquilinear form: for $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_+^d$,

$$\langle X^{\mathbf{i}}, X^{\mathbf{j}}\rangle_{M_d(n)} := \langle M_d(n)\widehat{X^{\mathbf{i}}}, \widehat{X^{\mathbf{j}}}\rangle = \beta_{\mathbf{i}+\mathbf{j}},$$

where $\widehat{X^{\mathbf{i}}}$ is the column vector associated to the monomial $X^{\mathbf{i}}$.

For a motivation of the main result, let us consider the basic Fibonacci sequence. In particular, take the first six moments and write them as a 2-dimensional moment sequence $\beta : \{\beta_{00}, \beta_{10}, \beta_{01}, \beta_{20}, \beta_{11}, \beta_{02}\} = \{1, 1, 2, 3, 5, 8\}$. Since $M_2(1)(\beta)$ is not positive, β does not admit a representing measure. However, one can find a formula to express β, such as $\beta_{ij} = 1 \cdot \left(\frac{1}{2}\right)^i (1)^j + \frac{1}{7} \cdot \left(\frac{9}{2}\right)^i (7)^j - \frac{1}{7} \cdot (1)^i (0)^j$; that is, there is a signed measure $\mu = 1 \cdot \delta_{\left(\frac{1}{2},1\right)} + \frac{1}{7} \cdot \delta_{\left(\frac{9}{2},7\right)} - \frac{1}{7} \cdot \delta_{(1,0)}$ for β to get an integral representation. The coefficients in the formula of the measure are called *densities* and the points are *atoms* of the measure. This example shows that even though a sequence has no representing measure, it may have a signed measure so that some of the densities might be negative. We define such a measure as an *interpolating* measure μ for β (truncated or full) as a Borel measure (not necessarily positive) such that $\beta_{\mathbf{i}} = \int \mathbf{x}^{\mathbf{i}}\, d\mu(\mathbf{x}),\ \mathbf{i} \in \mathbb{Z}_+^d$.)

Due to the Jordan decomposition theorem, every interpolating measure μ has a decomposition, $\mu = \mu^+ - \mu^-$ of two positive measures μ^+ and μ^-, at least one of which is finite. Interpolating measures appear in many scientific fields. For example, they are useful to represent electric charge; the moment problem about a signed measure is related to quantum physics as in [10]. Furthermore, there is a possibility that Gauss-Jacobi quadratures would be generalized through moment sequences with a signed measure (see [13]). Analog images are stored in a computer in the form of digital information using pixels. Each pixel contains information about color or contrast, which is identified by an integer value. Since the position of the pixel can be expressed with a bi-index, the image data can be considered as a bivariate truncated moment sequence. Our main results show that all image data can be represented by an interpolating measure. Therefore, if one can find a simple measure corresponding to the image, it will be useful in many fields of image processing.

For $d = 1$, R. P. Boas showed that any single-index "infinite" sequence of real numbers admits an interpolating measure supported in $[0, \infty)$; that is, one can always find a measure for any sequence of the form $\mu = \mu^+ - \mu^-$ such that both μ^+ and μ^- are positive Borel measures supported in $[0, \infty]$ [2]. Moreover, G. Flessas, K. Burton, and R. R. Whitehead found an algorithm to find such a measure supported in the real line for a "finite" real sequence $\{s_j\}_{j=0}^{2n-1}$ [10]. As a generalization of these results, we will see that any finite sequence has an interpolating measure supported in $\mathbb{R}^d$ for any $d \geq 2$. Notice that since moment problems about finite sequences are known to be more general than problems about infinite sequences due to the work of J. Stochel [15], the main results may contribute to an investigation of infinite sequences.

We conclude this section with another application of the moment problem to the numerical integration. For more details, readers can refer to [11].

Definition 1.1. A **quadrature** (or **cubature**) rule of size p and precision m is a numerical integration formula which uses p nodes, is exact for all polynomials of degree at most m, and fails to recover the integral some polynomial of degree $m+1$.

Example 1.2. (Gaussian Quadrature; size n, precision $2n-1$) We would like to find nodes $t_0, t_1 \ldots, t_{n-1}$ satisfying

$$\int_{-1}^{1} f(t)\, dt = \sum_{j=0}^{n-1} \rho_j f(t_j) \tag{1.1}$$

for every polynomial f with $\deg f \leq 2n-1$. Now, we consider interpolating equations with polynomials and we get

$$\sum_{j=0}^{n-1} \rho_j t_j^k = \int_{-1}^{1} t^k\, dt = \begin{cases} 0 & k = 1, 3, \ldots, 2n-1; \\ \frac{2}{k+1} & k = 0, 2, \ldots, 2n-2. \end{cases} \tag{1.2}$$

If $n = 2$, (1.2) becomes the system of polynomial equations

$$\begin{cases} \rho_0 + \rho_1 & = & 2; \\ \rho_0 t_0 + \rho_1 t_1 & = & 0; \\ \rho_0 t_0^2 + \rho_1 t_1^2 & = & 2/3; \\ \rho_0 t_0^3 + \rho_1 t_1^3 & = & 0. \end{cases}$$

The solution is $\rho_0 = \rho_1 = 1$, $t_0 = -1/\sqrt{3}$, and $t_1 = 1/\sqrt{3}$. Thus we easily see

$$\begin{aligned}\int_{-1}^{1} &\left(a_0 + a_1 t + a_2 t^2 + a_3 t^3\right) dt \\ &= a_0(\rho_0 + \rho_1) + a_1(\rho_0 t_0 + \rho_1 t_1) + a_2\left(\rho_0 t_0^2 + \rho_1 t_1^2\right) + a_3\left(\rho_0 t_0^3 + \rho_1 t_1^3\right) \\ &= \int_{-1}^{1} \left(a_0 + a_1 t + a_2 t^2 + a_3 t^3\right) d\mu,\end{aligned}$$

where $\mu := \rho_0\delta_{t_0} + \rho_1\delta_{t_1}$. This solution in numerical analysis textbooks is usually based on Legendre polynomials. With an approach via the truncated moment problem, we can find an alternative solution as follows: Let $\beta_0 := 2, \beta_1 := 0, \beta_2 := 2/3, \beta_3 := 0$ and form a Hankel matrix H with a parameter α,

$$H := \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 \\ \beta_1 & \beta_2 & \beta_3 \\ \beta_2 & \beta_3 & \alpha \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & \alpha \end{pmatrix}$$

For the sake of a minimal number of nodes, we want rank $H = 2$; thus, $\alpha = 2/9$. After labeling the columns in H as $\mathit{1}, T, T^2$, the column relation in H can be written as $T^2 = (1/3)\mathit{1}$. In [3], it is known the roots of the equation $t^2 = 1/3$ (that is, $t_0 = -1/\sqrt{3}$ and $t_1 = 1/\sqrt{3}$) are the nodes. We may compute the densities by solving the Vandermonde equation:

$$\begin{pmatrix} 1 & 1 \\ t_0 & t_1 \\ t_0^2 & t_1^2 \\ t_0^3 & t_1^3 \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix},$$

whose solution is obviously $\rho_0 = \rho_1 = 1$.

This method seems to provide an economical way to solve a qudrature problem and we will see the main result of this article gives a technique for more general cases, that is, when a signed measure arises in (1.1).

2. The Consistency and Rank-one Decompositions of Moment Matrices

This Section is designed to introduce some background knowledge for dealing with truncated moment sequences.

2.1 The consistency

We are about to define an algebraic set associated to $M_d(n)$. Let $\mathcal{P} := \mathbb{R}[x_1, \ldots, x_d]$ and let $\mathcal{P}_k := \{p \in \mathcal{P} : \deg p \leq k\}$. Since we labeled columns in $M_d(n)$ with monomials, a column relation in $M_d(n)$ can be written as $p(\mathbf{X}) = \mathbf{0}$ for some

$p \in \mathcal{P}_n$. Let $\mathcal{Z}(p)$ denote the zero set of a polynomial p and we define the *algebraic variety* $\mathcal{V}_\beta$ of β or $M_d(n)$ by

$$\mathcal{V}_\beta \equiv \mathcal{V}_{M_d(n)} := \bigcap_{p(\mathbf{X})=\mathbf{0}} \mathcal{Z}(p). \tag{2.1}$$

Given $\beta \equiv \beta^{(m)}$, define the *Riesz functional* $\Lambda \equiv \Lambda_\beta : \mathcal{P}_m \to \mathbb{R}$ by $\Lambda\left(\sum a_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}\right) := \sum a_{\mathbf{i}}\beta_{\mathbf{i}}$. We also define a notion which is the key to the main result of this note; $\beta \equiv \beta^{(2n)}$ or $M_d(n) \equiv M_d(n)(\beta^{(2n)})$ is said to be V*-consistent* for a set $V \in \mathbb{R}^d$ if the following holds:

$$p \in \mathcal{P}_{2n},\ p|_V \equiv 0 \Longrightarrow \Lambda(p) = 0. \tag{2.2}$$

This is a property of the moment sequence that guarantees the existence of an interpolating measure. Here is a formal result:

Lemma 2.1. ([5, Lemma 2.3]) *Let $L : \mathcal{P}_{2n} \to \mathbb{R}$ be a linear functional and let $V \subseteq \mathbb{R}^d$. Then the following statements are equivalent:*

(i) *There exist $\alpha_1, \ldots, \alpha_\ell \in \mathbb{R}$ and there exist $\mathbf{w}_1, \ldots, \mathbf{w}_\ell \in V$ such that for all $p \in \mathcal{P}_{2n}$*

$$L(p) = \sum_{k=1}^{\ell} \alpha_k p(\mathbf{w}_k). \tag{2.3}$$

(ii) *If $p \in \mathcal{P}_{2n}$ and $p|_V \equiv 0$, then $L(p) = 0$.*

If L is the Riesz functional of the moment sequence β, then Lemma 2.1(ii) is just as the $\mathcal{V}_\beta$-consistency condition of β and $\sum_{k=1}^{\ell} \alpha_k \delta_{\mathbf{w}_k}$ is an interpolating measure for β. While it seems like Lemma 2.1 gives a concrete solution for β to have an interpolating measure, we should indicate that checking the consistency is a highly nontrivial process. To show that β is V-consistent, it is essential (but, difficult) to find a representation of all the polynomials vanishing on V.

For $M_d(n)$ to have a (positive) representing measure, β must be $\mathcal{V}_\beta$-consistent; in the *extremal* cases (that is, rank $M_d(n) =$ card $\mathcal{V}_\beta$), it is known that $M_d(n)(\beta)$ is consistent if and only if β admits a unique rank $M_d(n)$-atomic representing measure whose support is exactly $\mathcal{V}_\beta$[5].

In particular, when a positive $M_d(n)$ is invertible, we know $\mathcal{V}_\beta = \mathbb{R}^d$ and the only polynomial vanishing on $\mathbb{R}^d$ is the zero polynomial. Thus, $M_d(n)$ is naturally consistent and has an interpolating measure.

2.2 Rank-one decompositions

After rearranging the terms in (2.3) by the sign of densities, we write a measure μ for a consistent $M_d(n)$ as

$$\mu = \sum_{k=1}^{s} \alpha_k \delta_{\mathbf{w}_k} - \sum_{k=s+1}^{\ell} \alpha_k \delta_{\mathbf{w}_k}, \tag{2.4}$$

where $\alpha_k > 0$ for all $k = 1, \dots, \ell$; we denote the first summand in (2.4) as μ^+ and the second as μ^-. Due to this fact, a bound of the cardinality of the support of an interpolating measure is established:

Proposition 2.2. *A minimal interpolating measure for a consistent $M_d(n)$ is at most $(2n+1)(2n+2)$-atomic.*

Proof. If $M_d(n)$ is consistent with a measure $\mu = \mu^+ - \mu^-$ of two positive finitely atomic measures μ^+ and μ^-, we may write $M_d(n) = M[\mu^+] - M[\mu^-]$, where each term is a moment matrix generated by the corresponding measure of the same size as $M_d(n)$. A result [1, Theorem 2] by C. Bayer and J. Teichmann showed that the cardinality of the support of a positive measure is at most $\dim \mathcal{P}_{2n}$ in the presence of a representing measure for a moment matrix associated to a moment sequence of degree $2n$.

Since $M[\mu^+]$ and $M[\mu^-]$ have a positive measure, it follows that a minimal measure for each moment matrix is at most $\dim \mathcal{P}_{2n}$-atomic. Therefore, we conclude that the cardinality of a minimal interpolating measure is at most $2(\dim \mathcal{P}_{2n}) = (2n+1)(2n+2)$. ☐

Many solutions of TRMP for a positive measure depend on finding a positive moment matrix extension of $M_d(n)$. However, this approach needs to allow new parameters and constructing an extension is not handy for most cases when $n \geq 3$. Alternatively, R. Curto and the second-author recently have used a decomposition of $M_d(n)$ for the study of TRMP. To introduce the decomposition, we now define some notations: Let $\mathbf{w} = (w_1, \dots, w_d) \in \mathbb{R}^d$ and let

(i) $\mathbf{v}(\mathbf{w}) := \begin{pmatrix} 1 & w_1 & \cdots & w_d & w_1^2 & w_1 w_2 & w_1 w_3 & \cdots & w_{d-1} w_d & w_d^2 & \cdots & w_1^n & \cdots & w_d^n \end{pmatrix}$, which is a row vector corresponding to the monomials $\mathbf{w}^{\mathbf{i}}$ in the degree-lexicographic order.

(ii) $P(\mathbf{w}) := \mathbf{v}(\mathbf{w})^T \mathbf{v}(\mathbf{w})$, which is indeed the rank-one moment matrix generated by the measure $\delta_{\mathbf{w}}$.

For example, if $d = n = 2$ and $\mathbf{w} = (a, b)$, then

$$P(\mathbf{w}) = \begin{pmatrix} 1 & a & b & a^2 & ab & b^2 \\ a & a^2 & ab & a^3 & a^2 b & ab^2 \\ b & ab & b^2 & a^2 b & ab^2 & b^3 \\ a^2 & a^3 & a^2 b & a^4 & a^3 b & a^2 b^2 \\ ab & a^2 b & ab^2 & a^3 b & a^2 b^2 & ab^3 \\ b^2 & ab^2 & b^3 & a^2 b^2 & ab^3 & b^4 \end{pmatrix}. \tag{2.5}$$

Thus, if $M_d(n)$ has an interpolating measure μ supported in a set $\{\mathbf{w}_1, \dots, \mathbf{w}_\ell\}$, then one should be able to write $M_d(n) = \sum_{k=1}^{\ell} d_k P(\mathbf{w}_k)$ for some $d_1, \dots, d_\ell \in \mathbb{R} \setminus \{0\}$.

3. Main Result

We will verify that any truncated moment matrix turns out to be $\mathbb{R}^d$-consistent after applying proper perturbations, and so it admits an interpolating measure. To prove the main result, we begin with auxiliary results:

Lemma 3.1. ([12]) *Assume A and B are matrices of the same size. Then* $\text{rank}\,(A+B) = \text{rank}\,A + \text{rank}\,B$ *if and only if* $\text{range}\,A \cap \text{range}\,B = \{\mathbf{0}\}$ *and* $\text{range}\,A^T \cap \text{range}\,B^T = \{\mathbf{0}\}$.

As a special case of Lemma 3.1, one can easily prove:

Lemma 3.2. *Assume A and B are Hermitian matrices of the same size and* $\text{rank}\,B = 1$. *Then* $\text{rank}\,(A+B) = 1+\text{rank}\,A$ *if and only if* $\text{range}\,A \cap \text{range}\,B = \{\mathbf{0}\}$.

We are ready to introduce a crucial lemma:

Lemma 3.3. *A point $\mathbf{w}$ is in $\mathcal{V}_{M_d(n)}$ if and only if the vector $\mathbf{v}(\mathbf{w})$ is in* $\text{range}\,M_d(n)$.

Proof. Assume that $\left\{p_k(\mathbf{X}) \equiv \sum a_{\mathbf{i}}^{(k)} \mathbf{X}^{\mathbf{i}}\right\}_{k=1}^{\ell}$ is the set of polynomials obtained from column relations in $M_d(n)$. Note that $\text{span}\,\{\widehat{p_k}\}_{k=1}^{\ell} = \ker M_d(n)$. Now observe:

$$\begin{aligned}
\mathbf{w} \in \mathcal{V}_{M_d(n)} \quad &\Longleftrightarrow \quad p_k(\mathbf{w}) = \mathbf{0} && \text{for } k = 1, \ldots, \ell \\
&\Longleftrightarrow \quad \sum a_{\mathbf{i}}^{(k)} \mathbf{w}^{\mathbf{i}} = \mathbf{0} && \text{for } k = 1, \ldots, \ell \\
&\Longleftrightarrow \quad \langle \widehat{p_k}, \mathbf{v}(\mathbf{w}) \rangle = 0 && \text{for } k = 1, \ldots, \ell \\
&\Longleftrightarrow \quad \widehat{p_k} \perp \mathbf{v}(\mathbf{w}) && \text{for } k = 1, \ldots, \ell \\
&\Longleftrightarrow \quad \mathbf{v}(\mathbf{w}) \in (\ker M_d(n))^{\perp} = \text{range}\,M_d(n).
\end{aligned}$$

□

Theorem 3.4. *Any truncated moment sequence $\beta \equiv \beta^{(2n)}$ of degree $2n$ has an interpolating measure in $\mathbb{R}^d$ for any positive $d \in \mathbb{Z}_+$.*

Proof. Pick a point $\mathbf{w}_1 \in \mathbb{R}^d \setminus \mathcal{V}_{\beta}$. Then we know from Lemma 3.3 that $\mathbf{v}(\mathbf{w}_1) \notin \text{range}\,M_d(n)(\beta)$. Since $\text{range}\,P(\mathbf{w}_1) = \{\alpha \mathbf{w}_1 : \alpha \in \mathbb{R}\}$, it holds that $\text{range}\,M_d(n)(\beta) \cap \text{range}\,P(\mathbf{w}_1) = \{\mathbf{0}\}$. Therefore, it follows from Lemma 3.2 that $\text{rank}\,(M_d(n)(\beta) + P(\mathbf{w}_1)) = 1 + \text{rank}\,M_d(n)(\beta)$. Next, choose a point $\mathbf{w}_2$ which not in the algebraic variety of $M_d(n)(\beta) + P(\mathbf{w}_1)$ and we know from the same argument that $\text{rank}\,(M_d(n)(\beta) + P(\mathbf{w}_1) + P(\mathbf{w}_2)) = 2 + \text{rank}\,M_d(n)(\beta)$. Keep this process until we obtain an invertible matrix $\widetilde{M} := M_d(n)(\beta) + \sum_{k=1}^{\ell} P(\mathbf{w}_k)$ for some ℓ. $\widetilde{M}$ is naturally consistent, and so it admits an interpolating measure, say $\tilde{\mu}$. Thus, $M_d(n)(\beta)$ has an interpolating measure of the form $\tilde{\mu} - \sum_{k=1}^{\ell} \delta_{\mathbf{w}_k}$. □

Theorem 3.5. *Any finite sequence has an interpolating measure.*

Proof. It suffices to cover the cases when the given sequence is not the type of $\beta^{(2n)}$. Such a sequence cannot fill up the associated moment matrix, so we use new parameters to complete the moment matrix. If it is possible to make the moment matrix invertible, then the extended moment sequence is consistent. Thus, the given sequence has an interpolating measure. Otherwise, one can follow the same process in the proof of Theorem 3.4 and verify that the sequence admits an interpolating measure. $\square$

Before we conclude this note, let us discuss how investigate the location of atoms of an interpolating measure. In addition, an algorithmic approach to find an explicit formula of a measure will be presented through a concrete example. Recall that in the presence of a (positive) representing measure μ for a positive $M_d(n)(\beta)$, Proposition 3.1 in [4] states that

$$\hat{p} \in \ker M_d(n)(\beta) \iff p(\mathbf{X}) = \mathbf{0} \iff \text{supp } \mu \subseteq \mathcal{Z}(p).$$

This result provides an evidence that where the atoms of μ lie for a singular $M_d(n)$; that is, the algebraic variety of $M_d(n)$ must contain the support of a representing measure. However, the following example shows such an argument is no longer valid for the moment problem about an interpolating measure; consider

$$M_2(1) \equiv M_2(1)\left(\beta^{(2)}\right) = \begin{pmatrix} -1 & -16 & -4 \\ -16 & -94 & -10 \\ -4 & -10 & 2 \end{pmatrix}. \tag{3.1}$$

Note that $M_2(1)$ has a single column relation $X_2 = -(4/3)\mathit{1} + (1/3)X_1$. Indeed, the sequence can be generated by an interpolating measure $\nu = \delta_{(-2,1)} + \delta_{(-2,-2)} - \delta_{(1,1)} - \delta_{(10,1)}$; but, different from the case for a positive measure, supp $\nu \not\subseteq \mathcal{Z}(x_2 + 4/3 - (1/3)x_1) = \mathcal{V}_{\beta^{(2)}}$. In other words, an interpolating measure for the sequence may have atoms outside of the algebraic variety. Nonetheless, one can still find an interpolating measure supported in the algebraic variety of $M_2(1)$ as follows:

Example 3.6. We illustrate how to find an interpolating measure of the sequence in (3.1). To find an interpolating measure supported in the algebraic variety of $M_2(1)$, select a point $\left(a, \frac{a-4}{3}\right) \in \mathcal{Z}(x_2 + 4/3 - (1/3)x_1)$ for some $a \in \mathbb{R}$. Using the rank-one decomposition, we write

$$M_2(1) = \widetilde{M_2(1)} + u \begin{pmatrix} 1 & a & \frac{a-4}{3} \\ a & a^2 & \frac{a(a-4)}{3} \\ \frac{a-4}{3} & \frac{a(a-4)}{3} & \frac{(a-4)^2}{9} \end{pmatrix} \tag{3.2}$$

for some $u \in \mathbb{R}$. Note that rank $M_2(1) = 2$ and we are attracted to guess that a minimal interpolating measure is 2-atomic (cf. Lemma 3.2 and 3.3). In order to find such a measure, we impose a condition that rank $\widetilde{M_2(1)} = 1$; a calculation shows rank $\widetilde{M_2(1)} = 1$ if and only if $u = 162/(a^2 - 32a + 94)$. If we take $u = 162/(a^2 - 32a + 94)$, then

$$M_2(1) = \frac{-(a-16)^2}{a^2-32a+94}\begin{pmatrix} 1 & \frac{2(8a-47)}{a-16} & \frac{2(2a-5)}{a-16} \\ \frac{2(8a-47)}{a-16} & \frac{4(8a-47)^2}{(a-16)^2} & \frac{4(2a-5)(8a-47)}{(a-16)^2} \\ \frac{2(2a-5)}{a-16} & \frac{4(2a-5)(8a-47)}{(a-16)^2} & \frac{4(2a-5)^2}{(a-16)^2} \end{pmatrix}$$
$$+\frac{162}{a^2-32a+94}\begin{pmatrix} 1 & a & \frac{a-4}{3} \\ a & a^2 & \frac{a(a-4)}{3} \\ \frac{a-4}{3} & \frac{a(a-4)}{3} & \frac{(a-4)^2}{9} \end{pmatrix}.$$

Therefore, we get an interpolating measure $\mu = \frac{-(a-16)^2}{a^2-32a+94}\delta_{\left(\frac{2(8a-47)}{a-16}, \frac{2(2a-5)}{a-16}\right)} + \frac{162}{a^2-32a+94}\delta_{\left(a, \frac{a-4}{3}\right)}$ (with $a^2 - 32a + 94 \neq 0$ and $a \neq 16$), which is supported in $\mathcal{V}_{M_2(1)}$.

Removing noise from the original data is a challenging problem in many different fields. Moment sequences need to be modified since data obtained from physical experiments and phenomena are often corrupt or incomplete. By Theorem 3.5, one can find an interpolating measure μ for the given data, which is $\mu = \mu^+ - \mu^-$ of two positive measures μ^+ and μ^-. Assuming that μ^- is generated by the distribution of noise, μ^+ can be a measure for the denosing data in a sense. In Example 3.7, $M_2(2)$ can be considered as an observed data with noise. After removing the noise, one can find the original data as in the following example.

Example 3.7. Consider a truncated moment sequence $\beta^{(4)}$:

$$\begin{aligned} &\beta_{00} = 6, \quad \beta_{10} = 6, \quad \beta_{01} = 20, \quad \beta_{20} = 18, \quad \beta_{11} = 16, \quad \beta_{02} = 68, \\ &\beta_{30} = 30, \quad \beta_{21} = 56, \quad \beta_{12} = 40, \quad \beta_{03} = 236, \\ &\beta_{40} = 66, \quad \beta_{31} = 88, \quad \beta_{22} = 176, \quad \beta_{13} = 88, \quad \beta_{04} = 836. \end{aligned}$$

Construct its moment matrix as follows:

$$\widetilde{M_2(2)} = \begin{pmatrix} 6 & 6 & 20 & 18 & 16 & 68 \\ 6 & 18 & 16 & 30 & 56 & 40 \\ 20 & 16 & 68 & 56 & 40 & 236 \\ 18 & 30 & 56 & 66 & 88 & 176 \\ 16 & 56 & 40 & 88 & 176 & 88 \\ 68 & 40 & 236 & 176 & 88 & 836 \end{pmatrix}$$

It is easy to check that the representing measure is $\mu = 2\delta_{(-1,4)} + 4\delta_{(2,3)}$. Assume that for sufficiently small perturbation we have

$$M_2(2) = \begin{pmatrix} 5.990000 & 5.995000 & 19.998000 & 17.997500 & 15.999000 & 67.999600 \\ 5.995000 & 17.997500 & 15.999000 & 29.998750 & 55.999500 & 39.999800 \\ 19.998000 & 15.999000 & 67.999600 & 55.999500 & 39.999800 & 235.999920 \\ 17.997500 & 29.998750 & 55.999500 & 65.999375 & 87.999750 & 175.999900 \\ 15.999000 & 55.999500 & 39.999800 & 87.999750 & 175.999900 & 87.999960 \\ 67.999600 & 39.999800 & 235.999920 & 175.999900 & 87.999960 & 835.999984 \end{pmatrix},$$

which is not positive semidefinite. So, arbitrarily small perturbations of a given sequence eject one from the cone of positive semidefinite matrices. As a result, this sequence does not have a representing measure. Instead, one can find interpolating measures for the sequence. Concretely, one of them is $\mu = -0.01\delta_{(0.5,0.2)} + 2\delta_{(-1,4)} + 4\delta_{(2,3)}$; here the first term with the negative density can be considered as noise.

Concluding Remark. According to the main results, we can confirm the existence of an interpolating measure for any finite sequence. A proper moment matrix perturbation enables us to obtain an invertible moment matrix which is consistent. However, invertible matrices may not be useful to find a specific representation of the measure. Rather, it is more advantageous to obtain a moment matrix whose complete solution is known; for example, we may try to make the resulting matrix to be flat and positive semidefinite at the same time. Recall that the Flat Extension Theorem in [4] says if $M_d(n)$ is positive semidefinite and rank $M_d(n) =$ rank $M_d(n-1)$, then it has a unique rank $M_d(n)$-atomic (minimal) representing measure. The following example illustrates how this alternative method works.

Example 3.8. Consider a sequence $\{0, 1, 0, -2, 1, 2, 4, -2, 1, -8, -8, 4, -2\}$ and let us try to find an interpolating measure for it on $\mathbb{R}^2$. Note that the sequence even starts with 0 and it is not be a moment sequence of degree 4 due to lack of two terms. So we add two parameters in the tail and construct $\beta \equiv \beta^{(4)}$ with as follows:

$$\begin{array}{llllll} \beta_{00} = 0, & \beta_{10} = 1, & \beta_{01} = 0, & \beta_{20} = -2, & \beta_{11} = 1, & \beta_{02} = 2, \\ \beta_{30} = 4, & \beta_{21} = -2, & \beta_{12} = 1, & \beta_{03} = -8, & & \\ \beta_{40} = -8, & \beta_{31} = 4, & \beta_{22} = -2, & \beta_{13} = g, & \beta_{04} = h. & \end{array}$$

Observe that $M \equiv M_2(2)(\beta)$ has a column relation $X^2 + 2X = \mathbf{0}$. Let $P(a, b)$ be the rank-one moment matrix exactly as in (2.5). Thus, if $(a, b) \notin \mathcal{Z}(x^2 + 2x)$, then $M + P(a, b)$ would be invertible with a proper choice of g and h. However, it is not easy to find an explicit form of an interpolating measure for a moment sequence with an invertible moment matrix. As a different approach, we consider an atom on $x^2 + 2x = 0$ so that the rank of $\widetilde{M(2)} := M + P(a, b)$ will not increase but there is a chance for $\widetilde{M(2)}$ to become positive semidefinite and flat; that is, rank $\widetilde{M(2)} =$ rank $\widetilde{M(1)} = 3$. If we take $(a, b) = (-2, 1)$, $g = 1$, and $h = 62$, then we can easily check that rank $\widetilde{M(2)}$ is as desired. Thus, it follows from the flat extension theorem that $\widetilde{M(2)}$ has a unique 3-atomic representing measure. Using the results in [4], we see the measure is given by

$$\widetilde{\mu} := \frac{1}{2}\delta_{(-2,1)} + \frac{65-7\sqrt{65}}{260}\delta_{\left(0,-\frac{5}{2}-\frac{\sqrt{65}}{2}\right)} + \frac{65+7\sqrt{65}}{260}\delta_{\left(0,-\frac{5}{2}+\frac{\sqrt{65}}{2}\right)}.$$

An interpolating measure μ for M satisfies $\widetilde{\mu} = \mu + \delta_{(-2,1)}$ so that we get

$$\mu = -\frac{1}{2}\delta_{(-2,1)} + \frac{65-7\sqrt{65}}{260}\delta_{\left(0,-\frac{5}{2}-\frac{\sqrt{65}}{2}\right)} + \frac{65+7\sqrt{65}}{260}\delta_{\left(0,-\frac{5}{2}+\frac{\sqrt{65}}{2}\right)}.$$

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