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SOME RESULTS ON INVARINAT SUBMANIFOLDS OF LORENTZIAN PARA-KENMOTSU MANIFOLDS

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Abstract. The purpose of this paper is to study invariant submanifolds of a Lorentzian para Kenmotsu manifold. We obtain the necessary and sufficient conditions for an invariant submanifold of a Lorentzian para Kenmotsu manifold to be totally geodesic. Finally, a non-trivial example is built in order to verify our main results.

1. Introduction

The geometry of almost paracontact manifolds is a naturel extension of the almost para Hermitian manifolds. The study of almost paracontact metric manifolds started in [6]. After then, these manifolds were classified by many geometers(see references).

Let $\widetilde{M}$ be an n-dimensional Lorentzian metric manifold. This means that it is endowed with a structure (φ, ξ, η, g), where φ is a (1,1)-type tensor field, ξ is a vector field, η is a 1-form on $\widetilde{M}$ and g is a Lorentzian metric tensor satisfying;

$$\varphi^2 X = X + \eta(X)\xi, \quad g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{1}$$

$$\eta(\xi) = -1, \quad \eta(X) = g(X, \xi), \tag{2}$$

for all vector fields X, Y on $\widetilde{M}$. Then $\widetilde{M}^n(\varphi, \xi, \eta, g)$ is said to be Lorentzian almost paracontact manifold [13].

A Lorentzian almost paracontact manifold $\widetilde{M}^n(\varphi, \xi, \eta, g)$ is called Lorentzian para Kenmotsu manifold if

$$(\widetilde{\nabla}_X \varphi)Y = -g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{3}$$

for all $X, Y \in \Gamma(T\widetilde{M})$, where $\widetilde{\nabla}$ and $\Gamma(T\widetilde{M})$ denote the Levi-Civita connection and differentiable vector fields set on $\widetilde{M}$, respectively.

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In a Lorentzian para Kenmotsu manifold $\widetilde{M}^n(\varphi,\xi,\eta,g)$, we have

$$\widetilde{\nabla}_X\xi = -\varphi^2 X = -X-\eta(X)\xi \tag{4}$$

$$(\widetilde{\nabla}_X\eta)Y = -g(X,Y)-\eta(X)\eta(Y), \tag{5}$$

for all $X,Y\in\Gamma(T\widetilde{M})$.

By $\widetilde{R}$ and S, we denote the Riemannian curvature tensor and Ricci tensor of Lorentzian para Kenmotsu manifold $\widetilde{M}^n(\varphi,\xi,\eta,g)$, then we have

$$R(X,Y)\xi = \eta(Y)X-\eta(X)Y, \tag{6}$$

$$\widetilde{R}(\xi,X)Y = g(X,Y)\xi-\eta(Y)X, \tag{7}$$

$$S(\xi,X) = (n-1)\eta(X), \tag{8}$$

The concircular curvature tensor of $\widetilde{M}^n(\varphi,\xi,\eta,g)$ is given by

$$\widetilde{C}(X,Y)Z = \widetilde{R}(X,Y)Z-\frac{\tau}{n(n-1)}\{g(Y,Z)X-g(X,Z)Y\}, \tag{9}$$

for all $X,Y,Z\in\Gamma(T\widetilde{M})$, where τ is the scalar curvature of $\widetilde{M}$.

Now, let M be an immersed submanifold of a Lorentzian para Kenmotsu manifold $\widetilde{M}^n$. By $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, we denote the tangent and normal subspaces of M in $\widetilde{M}$. Then the Gauss and Weingarten formulae are, respectively, given by

$$\widetilde{\nabla}_XY = \nabla_XY+\sigma(X,Y), \tag{10}$$

and

$$\widetilde{\nabla}_XV = -A_VX+\nabla^{\perp}_XV, \tag{11}$$

for all $X,Y\in\Gamma(TM)$ and $V\in\Gamma(T^{\perp}M)$, where ∇ and $\nabla^{\perp}$ are the connections on M and $\Gamma(T^{\perp}M)$ and σ and A are called the second fundamental form and shape operator of M, respectively. They are related by

$$g(A_VX,Y) = g(\sigma(X,Y),V). \tag{12}$$

The covariant derivative of σ is defined by

$$(\widetilde{\nabla}_X\sigma)(Y,Z) = \nabla^{\perp}_X\sigma(Y,Z)-\sigma(\nabla_XY,Z)-\sigma(Y,\nabla_XZ), \tag{13}$$

for all $X,Y,Z\in\Gamma(TM)$. If $\widetilde{\nabla}\sigma=0$, then submanifold M is said to be its second fundamental form is parallel [6].

By R, we denote the Riemannian curvature tensor of the submanifold M, we have the following Gauss equation;

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z+A_{\sigma(X,Z)}Y-A_{\sigma(Y,Z)}X+(\widetilde{\nabla}_X\sigma)(Y,Z)\\ &- (\widetilde{\nabla}_Y\sigma)(X,Z),\end{aligned} \tag{14}$$

for all $X,Y,Z\in\Gamma(T\widetilde{M})$, where if $(\widetilde{\nabla}_X\sigma)(Y,Z)-(\widetilde{\nabla}_Y\sigma)(X,Z)=0$, then it is called curvature-invariant submanifold.

For a $(0,k)$-type tensor field T, $k \geq 1$ and a $(0,2)$-type tensor field A on a Riemannian manifold (M,g), $Q(A,T)$-Tachibana tensor field is defined by

$$\begin{aligned} Q(A,T)(X_1,X_2,...,X_k;X,Y) &= -T((X\Lambda_A Y)X_1,X_2,...,X_k)... \\ &- T(X_1,X_2,...X_{k-1},(X\Lambda_A Y)X_k),, \end{aligned} \tag{15}$$

for all $X_1,X_2,...,X_k,X,Y \in \Gamma(TM)$, where

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{16}$$

Definition 1.1. A submanifold of a Riemannian manifold (M,g) is said to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci-generalized pseudoparallel if

$$\begin{aligned} \widetilde{R}\cdot\sigma \ \ &and \ \ Q(g,\sigma) \\ \widetilde{R}\cdot\widetilde{\nabla}\sigma \ \ &and \ \ Q(g,\widetilde{\nabla}\sigma) \\ \widetilde{R}\cdot\sigma \ \ &and \ \ Q(S,\sigma) \\ \widetilde{R}\cdot\widetilde{\nabla}\sigma \ \ &and \ \ Q(S,\widetilde{\nabla}\sigma) \end{aligned}$$

are linearly dependent, respectively [1].

Equivalently, these can be formulated by the following equations;

$$\widetilde{R}\cdot\sigma = L_1Q(g,\sigma), \tag{17}$$

$$\widetilde{R}\cdot\widetilde{\nabla}\sigma = L_2Q(g,\widetilde{\nabla}\sigma), \tag{18}$$

$$\widetilde{R}\cdot\sigma = L_3Q(S,\sigma), \tag{19}$$

$$\widetilde{R}\cdot\widetilde{\nabla}\sigma = L_4Q(S,\widetilde{\nabla}\sigma), \tag{20}$$

where functions L_1, L_2, L_3 and L_4 are, respectively, defined on
$M_1 = \{x \in M : \sigma(x) \neq g(x)\}$, $M_2 = \{x \in M : \widetilde{\nabla}\sigma(x) \neq g(x)\}$, $M_3 = \{x \in M : S(x) \neq \sigma(x)\}$ and $M_4 = \{x \in M : S(x) \neq \widetilde{\nabla}\sigma(x)\}$.
Particularly, if $L_1 = 0$(resp. $L_2 = 0$), then submanifold is said to be semiparallel(resp. 2-semiparallel) [11].

2. Some Results on Invarinat Submanifolds of a Lorentzian Para Kenmotsu Manifold

Now, we will investigate the above cases for the invariant submanifold M of a para-Kenmotsu manifold $\widetilde{M}^n(\varphi,\xi,\eta,g)$.

Now, let M be an immersed submanifold of a para-Kenmotsu manifold manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. If $\varphi(T_xM) \subseteq T_xM$, for each point at $x \in M$, then M is said to be invariant submanifold. We note that all of the properties of an invariant submanifold inherit the ambient manifold.

In the rest of this paper, we will assume that M is invariant submanifold of a Lorentzian para Kenmotsu manifold $\widetilde{M}^n(\varphi,\xi,\eta,g)$. Thus by using (3) and (10), we have

$$\sigma(X,\xi) = 0, \quad \sigma(\varphi X,Y) = \sigma(X,\varphi Y) = \varphi\sigma(X,Y), \tag{21}$$

and

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{22}$$

for all $X, Y \in \Gamma(TM)$.

LEMMA 2.1. *Let M be an invariant submanifold of a Lorentzian para Kenmotsu manifold $\widetilde{M}^n(\varphi, \xi, \eta, g)$. The second fundamental form σ of M is parallel if and only if M is totally geodesic.*

Proof. Let's assume that σ is parallel. Then (13) yields to

$$\nabla^{\perp}_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) = 0,$$

for all $X, Y, Z \in \Gamma(TM)$. Here, taking $Z = \xi$, by virtue of (21) and (22), we reach at

$$-\sigma(\nabla_X Y, \xi) - \sigma(Y, \nabla_X \xi) = -\sigma(Y, -X - \eta(X)\xi) = \sigma(Y, X) = 0$$

This proves our assertion. The converse is obvious. □

Lemma 2.1 is important for later theorems and propositions.

THEOREM 2.2. *Let M be an invariant pseudoparallel submanifold of a Lorentzian para-Kenmotsu manifold $\widetilde{M}^n(\varphi, \xi, \eta, g)$. Then M is either totally geodesic or $L_1 = 1$.*

Proof. Let M be pseudoparallel submanifold of a Lorentzian para Kenmotsu manifold $\widetilde{M}(\varphi, \xi, \eta, g)$, then from (17) we have

$$(\widetilde{R}(X, Y) \cdot \sigma)(U, V) = L_1 Q(g, \sigma)(U, V; X, Y),$$

for all $X, Y, U, V \in \Gamma(TM)$. This implies that

$$\begin{aligned} R^{\perp}(X, Y)\sigma(U, V) &- \sigma(R(X, Y)U, V) - \sigma(U, R(X, Y)V) \\ &= -L_1\{\sigma((X \wedge_g Y)U, V) + \sigma(U, (X \wedge_g Y)V)\} \\ &= -L_1\{\sigma(g(Y, U)X - g(X, U)Y, V) \\ &+ \sigma(U, g(Y, V)X - g(X, V)Y)\} \end{aligned} \tag{23}$$

for all $X, Y, U, V \in \Gamma(TM)$. Taking $V = \xi$ in (23) and taking into account of (21), we obtain

$$\begin{aligned} \sigma(R(X, Y)\xi, U) &= L_1\{\eta(Y)\sigma(X, U) - \eta(X)\sigma(U, Y)\} \\ \sigma(\eta(X)Y - \eta(Y)X, U) &= L_1\{\eta(Y)\sigma(X, U) - \eta(X)\sigma(U, Y)\} \end{aligned}$$

This completes the proof. □

From the Theorem 2.2, we have the following proposition.

PROPOSITION 2.3. *Let M be an invariant pseudoparallel submanifold of a Lorentzian para-Kenmotsu manifold $\widetilde{M}^n(\varphi, \xi, \eta, g)$. Then M is semiparallel if and only if M is totally geodesic.*

THEOREM 2.4. *Let M be an invariant 2-pseudoparallel submanifold of a Lorentzian para Kenmotsu manifold $\widetilde{M}^n(\varphi, \xi, \eta, g)$. Then M is either totally geodesic or $L_2 = 1$.*

Proof. Let M be 2-pseudoparallel of a Lorentzian para Kenmotsu manifold $\widetilde{M}^n(\varphi,\xi,\eta,g)$. Then from (18), we have

$$(\widetilde{R}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,Z) \;=\; L_2Q(g,\widetilde{\nabla}\sigma)(U,V,Z;X,Y),$$

for all $X,Y,U,V,Z\in\Gamma(TM)$. This means that

$$\begin{aligned} & R^{\perp}(X,Y)(\widetilde{\nabla}_U\sigma)(V,Z)-(\widetilde{\nabla}_{R(X,Y)U}\sigma)(V,Z)-(\widetilde{\nabla}_U\sigma)(R(X,Y)V,Z)\\ - & (\widetilde{\nabla}_U\sigma)(V,R(X,Y)Z)=-L_2\{(\widetilde{\nabla}_{(X\wedge_g Y)U}\sigma)(V,Z)+(\widetilde{\nabla}_U\sigma)((X\wedge_g Y)V,Z)\\ + & (\widetilde{\nabla}_U\sigma)(V,(X\wedge_g Y)Z)\},\end{aligned}$$

that is,

$$\begin{aligned} & R^{\perp}(X,Y)(\widetilde{\nabla}_U\sigma)(V,Z)-(\widetilde{\nabla}_{R(X,Y)U}\sigma)(V,Z)-(\widetilde{\nabla}_U\sigma)(R(X,Y)V,Z)\\ - & (\widetilde{\nabla}_U\sigma)(V,R(X,Y)Z)=-L_2\{g(Y,U)(\widetilde{\nabla}_X\sigma)(V,Z)-g(X,U)(\widetilde{\nabla}_Y\sigma)(V,Z)\\ + & (\widetilde{\nabla}_U\sigma)(g(Y,V)X-g(X,V)Y,Z)+(\widetilde{\nabla}_U\sigma)(V,g(Y,Z)X-g(X,Z)Y)\}.\end{aligned}$$

In the last equality, taking $X=Z=\xi$ and the after necessary arrangements are made, we obtain

$$\begin{aligned} R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(V,\xi) \;-\; & (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(V,\xi)-(\widetilde{\nabla}_U\sigma)(R(\xi,Y)V,\xi)\\ - \; & (\widetilde{\nabla}_U\sigma)(V,R(\xi,Y)\xi)=-L_2\{g(Y,U)(\widetilde{\nabla}_\xi\sigma)(V,\xi)\\ - \; & \eta(U)(\widetilde{\nabla}_Y\sigma)(V,\xi)+(\widetilde{\nabla}_U\sigma)(g(Y,V)\xi-\eta(V)Y,\xi)\\ + \; & (\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi+Y)\}. \end{aligned} \tag{24}$$

Now, let us calculate each of these expressions. Making use of (13), (21) and (22), we obtain

$$\begin{aligned} R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(V,\xi) &= R^{\perp}(\xi,Y)\{\nabla^{\perp}_U\sigma(V,\xi)-\sigma(\nabla_UV,\xi)\\ &- \sigma(V,\nabla_U\xi)\}\\ &= R^{\perp}(\xi,Y)\{-\sigma(V,\nabla_U\xi)\}\\ &= -R^{\perp}(\xi,Y)\sigma(V,-U-\eta(U)\xi)\\ &= R^{\perp}(\xi,Y)\sigma(V,U). \end{aligned} \tag{25}$$

Moreover, taking into account of (7) and (22), we have

$$\begin{aligned} (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(V,\xi) &= \nabla^{\perp}_{R(\xi,Y)U}\sigma(V,\xi)-\sigma(\nabla_{R(\xi,Y)U}V,\xi)\\ &- \sigma(\nabla_{R(\xi,Y)U}\xi,V)\\ &= -\sigma(-R(\xi,Y)U-\eta(R(\xi,Y)U)\xi,V)\\ &= \sigma(R(\xi,Y)U,V)=\sigma(g(Y,U)\xi-\eta(U)Y,V)\\ &= -\eta(U)\sigma(Y,V). \end{aligned} \tag{26}$$

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(R(\xi,Y)V,\xi) &= \nabla^{\perp}_U\sigma(R(\xi,Y)V,\xi)-\sigma(\nabla_UR(\xi,Y)V,\xi)\\ &- \sigma(R(\xi,Y)V,\nabla_U\xi)\\ &= -\sigma(R(\xi,Y)V,-U-\eta(U)\xi)\\ &= \sigma(g(Y,V)\xi-\eta(V)Y,U)=-\eta(V)\sigma(Y,U). \end{aligned} \tag{27}$$

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(V,R(\xi,Y)\xi) &= (\widetilde{\nabla}_U\sigma)(V,Y+\eta(Y)\xi)\\
&= (\widetilde{\nabla}_U\sigma)(V,Y)+(\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi)\\
&= (\widetilde{\nabla}_U\sigma)(V,Y)+\nabla^{\perp}_U\sigma(V,\eta(Y)\xi)\\
&- \sigma(\nabla_U V,\eta(Y)\xi)-\sigma(V,\nabla_U\eta(Y)\xi)\\
&= (\widetilde{\nabla}_U\sigma)(V,Y)-\sigma(V,U\eta(Y)\xi+\eta(Y)\nabla_U\xi)\\
&= (\widetilde{\nabla}_U\sigma)(V,Y)+\eta(Y)\sigma(V,U), \qquad (28)
\end{aligned}
$$

and

$$
\begin{aligned}
(\widetilde{\nabla}_{(\xi\wedge_g Y)U}\sigma)(V,\xi) &= \nabla^{\perp}_{(\xi\wedge_g Y)U}\sigma(V,\xi)-\sigma(\nabla_{(\xi\wedge_g Y)U}V,\xi)\\
&- \sigma(V,\nabla_{(\xi\wedge_g Y)U}\xi)=-\sigma(V,\nabla_{g(Y,U)\xi-\eta(U)Y}\xi)\\
&= -\sigma(V,g(Y,U)\nabla_\xi\xi-\eta(U)\nabla_Y\xi)=\eta(U)\sigma(V,\nabla_Y\xi)\\
&= \eta(U)\sigma(V,-Y-\eta(Y)\xi)=-\eta(U)\sigma(V,Y), \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)((\xi\wedge_g Y)V,\xi) &= \nabla^{\perp}_U\sigma((\xi\wedge_g Y)V,\xi)-\sigma(\nabla_U(\xi\wedge_g Y)V,\xi)\\
&- \sigma((\xi\wedge_g Y)V,\nabla_U\xi)\\
&= -\sigma(g(Y,V)\xi-\eta(V)Y,-U-\eta(U)\xi)\\
&= -\eta(V)\sigma(Y,U). \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(V,(\xi\wedge_g Y)\xi) &= (\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi+Y)\\
&= (\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi)+(\widetilde{\nabla}_U\sigma)(V,Y)\\
&= \nabla^{\perp}_U\sigma(V,\eta(Y)\xi)-\sigma(\nabla_U V,\eta(Y)\xi)\\
&- \sigma(V,\nabla_U\eta(Y)\xi)+(\widetilde{\nabla}_U\sigma)(V,Y)\\
&= -\sigma(V,U\eta(Y)\xi+\eta(Y)\nabla_U\xi)+(\widetilde{\nabla}_U\sigma)(V,Y)\\
&= -\eta(Y)\sigma(V,-U-\eta(U)\xi)+(\widetilde{\nabla}_U\sigma)(V,Y)\\
&= \eta(Y)\sigma(V,U)+(\widetilde{\nabla}_U\sigma)(V,Y). \qquad (31)
\end{aligned}
$$

Consequently, if we put (25), (26), (27), (28), (29), (30) and (31) in (24), we reach at

$$
\begin{aligned}
&R^{\perp}(\xi,Y)\sigma(V,U)+\eta(U)\sigma(Y,V)+\eta(V)\sigma(Y,U)-(\widetilde{\nabla}_U\sigma)(V,Y)\\
&- \eta(Y)\sigma(U,V)=L_2\{\eta(U)\sigma(V,Y)+\eta(V)\sigma(Y,U)-\eta(Y)\sigma(V,U)\\
&- (\widetilde{\nabla}_U\sigma)(V,Y)\} \qquad (32)
\end{aligned}
$$

If taking $X=\xi$ in (32), considering (21) and (5), we get

$$-\sigma(Y,U)-(\widetilde{\nabla}_U\sigma)(Y,\xi)=-L_2\{\sigma(U,Y)+(\widetilde{\nabla}_U\sigma)(Y,\xi)\}, \qquad (33)$$

where

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(\xi,Y) &= \nabla^{\perp}_U\sigma(Y,\xi)-\sigma(\nabla_U Y,\xi)-\sigma(Y,\nabla_U\xi)\\
&= -\sigma(Y,-U-\eta(U)\xi)=\sigma(Y,U). \qquad (34)
\end{aligned}
$$

From (33) and (34), we conclude that

$$(L_2-1)\sigma(U,Y)=0,$$

which is proves our assertions. □

From Theorem 2.4, we have the following proposition.

PROPOSITION 2.5. *Let M be an invariant pseudoparallel submanifold of a Lorentzian para Kenmotsu manifold $\widetilde{M}^n(\varphi,\xi,\eta,g)$. Then M is 2-semiparallel if and only if M is totally geodesic.*

THEOREM 2.6. *Let M be an invariant submanifold of a Lorentzian para Kenmotsu manifold $\widetilde{M}^n(\varphi,\xi,\eta,g)$. Then $\widetilde{C}\cdot\sigma=0$ if and only if M is either totally geodesic or the scalar curvature τ of $\widetilde{M}^n$ is $\tau=n(n-1)$.*

Proof. $\widetilde{C}\cdot\sigma=0$ implies that

$$\begin{aligned}(\widetilde{C}(X,Y)\cdot\sigma)(U,V) &= R^{\perp}(X,Y)\sigma(U,V)-\sigma(\widetilde{C}(X,Y)U,V)\\ &- \sigma(U,\widetilde{C}(X,Y)V)=0,\end{aligned}\tag{35}$$

for all $X,Y,U,V\in\Gamma(TM)$. On the other hand, from (6) and (9), we can derive

$$\widetilde{C}(X,Y)\xi=(1-\frac{\tau}{n(n-1)})(\eta(Y)X-\eta(X)Y).\tag{36}$$

Thus, for $V=\xi$, from (35) and (36), we conclude that

$$\sigma(\widetilde{C}(X,Y)\xi,U)=(1-\frac{\tau}{n(n-1)})\sigma(\eta(Y)X-\eta(X)Y,U)=0.$$

The proof is completed. □

THEOREM 2.7. *Let M be an invariant Ricci-generalized pseudoparallel submanifold of a Lorentzian para Kenmotsu manifold $\widetilde{M}^n(\varphi,\xi,\eta,g)$. Then M is either totally geodesic or the function $L_3=\frac{1}{n-1}$.*

Proof. If M is Ricci-generalized pseudoparallel of a Lorentzian para Kenmotsu manifold $\widetilde{M}(\varphi,\xi,\eta,g)$, then from (15) and (19), we have

$$\begin{aligned}(\widetilde{R}(X,Y)\cdot\sigma)(U,V) &= L_3Q(S,\sigma)(U,V;X,Y)\\ &= -L_3\left\{\sigma((X\wedge_S Y)U,V)+\sigma(U,(X\wedge_S Y)V)\right\},\end{aligned}$$

for all $X,Y,U,V\in\Gamma(TM)$. This means that

$$\begin{aligned}R^{\perp}(X,Y)\sigma(U,V) &- \sigma(R(X,Y)U,V)-\sigma(U,R(X,Y)V)\\ &= -L_3\{\sigma(S(Y,U)X-S(X,U)Y,V)\\ &+ \sigma(S(V,Y)X-S(X,V)Y,U)\}.\end{aligned}$$

Here taking $X=V=\xi$ and by using (7) and (15), we reach at

$$\begin{aligned}R^{\perp}(\xi,Y)\sigma(U,\xi) &- \sigma(R(\xi,Y)U,\xi)-\sigma(U,R(\xi,Y)\xi)\\ &= -L_3\{\sigma(S(Y,U)\xi-S(\xi,U)Y,\xi)\\ &+ \sigma(S(\xi,Y)\xi-S(\xi,\xi)Y,U)\}.\end{aligned}\tag{37}$$

By using (8) and (21), we can infer

$$\begin{aligned}-\sigma(U,Y+\eta(Y)\xi) &= -L_3\{-S(\xi,\xi)\sigma(Y,U)\}\\ -\sigma(Y,U) &= -(n-1)L_3\sigma(Y,U).\end{aligned}$$

This proves our assertion. □

THEOREM 2.8. *Let M be an invariant 2-Ricci-generalized pseudoparallel submanifold of a Lorentzian para Kenmotsu manifold $\widetilde{M}^n(\varphi,\xi,\eta,g)$. Then M is either totally geodesic or $L_4=\frac{1}{n-1}$.*

Proof. Let us assume that M is 2-Ricci-generalized pseudoparallel submanifold. Then from (20), we have

$$(\widetilde{R}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,Z) = L_4 Q(S,\widetilde{\nabla}\sigma)(U,V,Z;X,Y),$$

for all $X,Y,U,V,Z\in\Gamma(TM)$. This implies that

$$\begin{aligned} R^{\perp}(X,Y)(\widetilde{\nabla}_U\sigma)(V,Z) &- (\widetilde{\nabla}_{R(X,Y)U}\sigma)(V,Z) - (\widetilde{\nabla}_U\sigma)(R(X,Y)V,Z) \\ &- (\widetilde{\nabla}_U\sigma)(V,R(X,Y)Z) = -L_4\{(\widetilde{\nabla}_{(X\wedge_S Y)U}\sigma)(V,Z) \\ &+ (\widetilde{\nabla}_U\sigma)((X\wedge_S Y)V,Z) + (\widetilde{\nabla}_U\sigma)(V,(X\wedge_S Y)Z)\}. \end{aligned}$$

Here taking $X=V=\xi$, we have

$$\begin{aligned} R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(\xi,Z) &- (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi,Z) - (\widetilde{\nabla}_U\sigma)(R(\xi,Y)\xi,Z) \\ &- (\widetilde{\nabla}_U\sigma)(\xi,R(\xi,Y)Z) = -L_4\{(\widetilde{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi,Z) \\ &+ (\widetilde{\nabla}_U\sigma)((\xi\wedge_S Y)\xi,Z) + (\widetilde{\nabla}_U\sigma)(\xi,(\xi\wedge_S Y)Z)\}. \end{aligned} \tag{38}$$

Now, let's calculate each of these expressions. Also taking into account of (21) and (22), we arrive at

$$\begin{aligned} R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(\xi,Z) &= R^{\perp}(\xi,Y)\{\nabla^{\perp}_U\sigma(\xi,Z) - \sigma(\nabla_U Z,\xi) \\ &- \sigma(Z,\nabla_U\xi)\} = R^{\perp}(\xi,Y)\{-\sigma(Z,-U-\eta(U)\xi)\} \\ &= R^{\perp}(\xi,Y)\sigma(Z,U). \end{aligned} \tag{39}$$

On the other hand, by using (21) and (22), we have

$$\begin{aligned} (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi,Z) &= \nabla^{\perp}_{R(\xi,Y)U}\sigma(\xi,Z) - \sigma(\nabla_{R(\xi,Y)U}\xi,Z) \\ &- \sigma(\xi,\nabla_{R(\xi,Y)U}Z) \\ &= -\sigma(-R(\xi,Y)U - \eta(R(\xi,Y)U)\xi,Z) \\ &= \sigma(R(\xi,Y)U,Z) = \sigma(g(Y,U)\xi - \eta(U)Y,Z) \\ &= -\eta(U)\sigma(Y,Z), \end{aligned} \tag{40}$$

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(R(\xi,Y)\xi,Z) &= (\widetilde{\nabla}_U\sigma)(Y+\eta(Y)\xi,Z) = (\widetilde{\nabla}_U\sigma)(Y,Z) \\ &+ (\widetilde{\nabla}_U\sigma)(\eta(Y)\xi,Z) = (\widetilde{\nabla}_U\sigma)(Y,Z) \\ &+ \nabla^{\perp}_U\sigma(\eta(Y)\xi,Z) - \sigma(\nabla_U\eta(Y)\xi,Z) \\ &- \sigma(\eta(Y)\xi,\nabla_U Z) \\ &= (\widetilde{\nabla}_U\sigma)(Y,Z) - \sigma(U\eta(Y)\xi + \eta(Y)\nabla_U\xi,Z) \\ &= (\widetilde{\nabla}_U\sigma)(Y,Z) - \sigma(U\eta(Y)\xi - \eta(Y)(U+\eta(U)\xi),Z) \\ &= (\widetilde{\nabla}_U\sigma)(Y,Z) + \eta(Y)\sigma(U,Z). \end{aligned} \tag{41}$$

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(\xi,R(\xi,Y)Z) &= \nabla^{\perp}_U\sigma(\xi,R(\xi,Y)Z) - \sigma(\nabla_U\xi,R(\xi,Y)Z) \\ &- \sigma(\xi,\nabla_U R(\xi,Y)Z) = -\sigma(-U-\eta(U)\xi,R(\xi,Y)Z) \\ &= -\sigma(-U,g(Y,Z)\xi - \eta(Z)Y) = -\eta(Z)\sigma(U,Y). \end{aligned} \tag{42}$$

Now, let's calculate the left side of (32). Making use of (13), (21) and (22), we have

$$\begin{aligned}(\widetilde{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi,Z) &= \nabla^{\perp}_{(\xi\wedge_S Y)U}\sigma(\xi,Z)-\sigma(\nabla_{(\xi\wedge_S Y)U}\xi,Z)\\ &- \sigma(\xi,\nabla_{(\xi\wedge_S Y)U}Z)\\ &= -\sigma(\nabla_{S(Y,U)\xi-S(\xi,U)Y}\xi,Z)\\ &= -S(Y,U)\sigma(\nabla_{\xi}\xi,Z)+S(\xi,U)\sigma(\nabla_Y\xi,Z)\\ &= (n-1)\eta(U)\sigma(-Y-\eta(Y)\xi,Z)\\ &= -(n-1)\eta(U)\sigma(Y,Z), \qquad (43)\end{aligned}$$

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)((\xi\wedge_S Y)\xi,Z) &= (\widetilde{\nabla}_U\sigma)(S(Y,\xi)\xi-S(\xi,\xi)Y,Z)\\ &= (\widetilde{\nabla}_U\sigma)((n-1)Y+(n-1)\eta(Y)\xi,Z)\\ &= (n-1)\{(\widetilde{\nabla}_U\sigma)(Y,Z)+(\widetilde{\nabla}_U\sigma)(\eta(Y)\xi,Z)\}\\ &= (n-1)\{(\widetilde{\nabla}_U\sigma)(Y,Z)+\nabla^{\perp}_U\sigma(\eta(Y)\xi,Z)\\ &- \sigma(\nabla_U\eta(Y)\xi,Z)-\sigma(\eta(Y)\xi,\nabla_U Z)\}\\ &= (n-1)\{(\widetilde{\nabla}_U\sigma)(Y,Z)-\sigma(U\eta(Y)\xi+\eta(Y)\nabla_U\xi,Z)\}\\ &= (n-1)\{(\widetilde{\nabla}_U\sigma)(Y,Z)-\eta(Y)\sigma(-U-\eta(U)\xi,Z)\}\\ &= (n-1)\{(\widetilde{\nabla}_U\sigma)(Y,Z)+\eta(Y)\sigma(U,Z)\}. \qquad (44)\end{aligned}$$

Finally,

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)(\xi,(\xi\wedge_S Y)Z) &= (\widetilde{\nabla}_U\sigma)(\xi,S(Y,Z)\xi-S(\xi,Z)Y)\\ &= (\widetilde{\nabla}_U\sigma)(\xi,S(Y,Z)\xi)-(n-1)(\widetilde{\nabla}_U\sigma)(\xi,\eta(Z)Y)\\ &= \nabla^{\perp}_U\sigma(\xi,S(Y,Z)\xi)-\sigma(\nabla_U\xi,S(Y,Z)\xi)\\ &- \sigma(\xi,\nabla_U S(Y,Z)\xi)-(n-1)\{\nabla^{\perp}_U\sigma(\xi,\eta(Z)Y)\\ &- \sigma(\nabla_U\xi,\eta(Z)Y)-\sigma(\xi,\nabla_U\eta(Z)Y)\}\\ &= -(n-1)\sigma(-U-\eta(U)\xi,Y)\eta(Z)\\ &= (n-1)\eta(Z)\sigma(U,Y). \qquad (45)\end{aligned}$$

By substituting (39), (40), (41), (42), (43), (44) and (45) into (38) we reach at

$$\begin{aligned}&R^{\perp}(\xi,Y)\sigma(U,Z)+\eta(U)\sigma(Y,Z)-(\widetilde{\nabla}_U\sigma)(Y,Z)-\eta(Y)\sigma(U,Z)\\ &+ \eta(Z)\sigma(U,Y)=(n-1)L_4\{\eta(U)\sigma(Y,Z)-\eta(Y)\sigma(U,Z)\\ &- (\widetilde{\nabla}_U\sigma)(Y,Z)+\eta(Z)\sigma(U,Y)\}. \qquad (46)\end{aligned}$$

Here if taking $Z=\xi$ in (46), we can easily to see that

$$(n-1)L_4\{(\widetilde{\nabla}_U\sigma)(Y,\xi)+\sigma(U,Y)\}=(\widetilde{\nabla}_U\sigma)(Y,\xi)+\sigma(U,Y).$$

From (34), we conclude that

$$((n-1)L_4-1)\sigma(U,Y)=0,$$

which proves our assertion. □

Example 2.9. Let us consider the 5-dimensional manifold

$$\widetilde{M}^5=\{(x_1,x_2,x_3,x_4,z):z>0\},$$

where (x_1, x_2, x_3, x_4, z) denote the standard coordinates of $\mathbb{R}^5$. Then let e_1, e_2, e_3, e_4, e_5 be vector fields on $\widetilde{M}^5$ by given

$$e_1 = z\frac{\partial}{\partial x_1}, e_2 = z\frac{\partial}{\partial x_2}, e_3 = z\frac{\partial}{\partial x_3}, e_4 = z\frac{\partial}{\partial x_4}, e_5 = z\frac{\partial}{\partial z}$$

which are linearly independent at each point at each point $\widetilde{M}^5$ and we define a Lorentzian metric tensor g on $\widetilde{M}^5$ as

$$\begin{aligned} g(e_i, e_i) &= 1, \quad 1 \leq i \leq 4 \\ g(e_i, e_j) &= 0, \quad 1 \leq i \neq j \leq 5 \\ g(e_5, e_5) &= -1. \end{aligned}$$

Let η be the 1-form defined by $\eta(X) = g(X, e_5)$ for all $X \in \Gamma(T\widetilde{M})$. Now, we define the tensor field $(1,1)$-type φ such that

$$\varphi e_1 = -e_2, \quad \varphi e_3 = -e_4, \quad \varphi e_5 = 0.$$

Then for $X = x_i e_i, Y = y_j e_j \in \Gamma(T\widetilde{M})$, $1 \leq i, j \leq 5$, we can easily see that

$$\varphi^2 X = X + \eta(X)\xi, \quad \xi = e_5, \quad \eta(X) = g(X, \xi)$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y).$$

By direct calculations, only non-vanishing components are

$$[e_i, e_5] = -e_i, \quad 1 \leq i \leq 4.$$

From Kozsul's formula, we can compute

$$\widetilde{\nabla}_{e_i} e_5 = -e_i, \quad 1 \leq i \leq 4.$$

Thus for $X = x_i e_i, Y = y_j e_j \in \Gamma(T\widetilde{M})$, we have

$$\widetilde{\nabla}_X \xi = -X - \eta(X)\xi, \quad \textit{and} \quad (\widetilde{\nabla}_X \varphi)Y = -g(\varphi X, Y)\xi - \eta(Y)\varphi X,$$

that is, $\widetilde{M}^5(\varphi, \xi, \eta, g)$ is a Lorentzian para Kenmotsu manifold.
Now, we consider the 3-dimensional submanifold M^3 of $\widetilde{M}^5(\varphi, \xi, \eta, g)$ given by ψ-immersion

$$\begin{aligned} &\psi : M^3 \longrightarrow \widetilde{M}^5(\varphi, \xi, \eta, g) \\ &\psi(x_1, x_2, z) = (zx_1, zx_2, zx_1, zx_2, \frac{1}{2}z^2). \end{aligned}$$

Then the tangent space of submanifold M is spanned by the vector fields

$$U = z\frac{\partial}{\partial x_1} + z\frac{\partial}{\partial x_3} = e_1 + e_3, \quad V = z\frac{\partial}{\partial x_2} + z\frac{\partial}{\partial x_4} = e_2 + e_4, \quad \xi = z\frac{\partial}{\partial z}.$$

Thus we can see that $\varphi U = \varphi(e_1 + e_3) = -e_2 - e_4 = -V$. This verifies M is a 3-dimensional invariant submanifold of a Lorentzian para Kenmotsu manifold $\widetilde{M}^5(\varphi, \xi, \eta, g)$. On the other hand, we can easily that $\widetilde{\nabla}_U V = \widehat{\nabla}_V U = 0$, $\widetilde{\nabla}_U \xi = -U$, $\widetilde{\nabla}_V \xi = -V$. Also this tells us that M is pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci generalized pseudoparallel.

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