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# CERTAIN RESULTS ON THREE-DIMENSIONAL $f$-KENMOTSU MANIFOLDS WITH CONFORMAL RICCI SOLITONS

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Abstract. In the present paper, we have studied conformal Ricci solitons on $f$-Kenmotsu manifolds of dimension three. Also we have studied $\phi$-Ricci symmetry, $\eta$-parallel Ricci tensor, cyclic parallel Ricci tensor and second order parallel tensor in $f$-Kenmotsu manifolds of dimension three admitting conformal Ricci solitons. Finally, we give an example.

## 1. Introduction

In 1972, the notion of Kenmotsu manifolds was introduced by K. Kenmotsu in the paper [16]. $f$-Kenmotsu manifold is the generalization of Kenmotsu manifold. $f$-Kenmotsu manifolds has been studied by several authors such as Hui,Yadav and Patra [15], Venkatesha and Divyashree [23], Yildiz, De and Turan [24]. De and Sarkar introduced the notion of $\phi$-Ricci symmetric manifolds in the paper [6]. $\eta$-parallel Ricci tensor and cyclic parallel Ricci tensor on three-dimensional quasi-Sasakian manifolds have been studied by De and Sarkar [5]. In 1926, Levy introduced the notion of second order parallel tensors. Later many author such as R. Sharma [21], [22], Chandra, Hui and Shaikh [4], Mondal and De [17] have studied second order parallel tensors on several manifolds.

The notion of Ricci solitons was introduced by R. S. Hamilton [14] which is the generalization of the Einstein metrics and is defined by

$$(\mathcal{L}_X g)(Y,Z) + 2S(Y,Z) + 2\lambda g(Y,Z) = 0,$$

where $\mathcal{L}_X$ denotes the Lie-derivative along the vector field $X$, $\lambda$ is a constant, $S$ the Ricci tensor of type $(0,2)$ and $Y$, $Z$ are arbitrary vector fields on the manifold. Here $X$ is called the potential vector field. A Ricci soliton is called shrinking, steady or expanding according as $\lambda$ is negative, zero or positive. A Ricci soliton is the limit of the solutions of Ricci flow equation given by

$$\frac{\partial g}{\partial t} = -2S.$$

Many authors have studied Ricci solitons on different kind of manifolds. For details see ( [3], [7], [8], [18]).

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Conformal Ricci flow equation was introduced by A. E. Fisher [13] in the year 2005 which is a variation of the classical Ricci flow equation and the equation is given by

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg$$

and $r = -1$, where $p$ is a time dependent non-dynamical scalar field, $r$ is the scalar curvature of the manifold and $n$ is the dimension of the manifold.

In 2015, the notion of conformal Ricci soliton was introduced by N. Basu and A. Bhattacharyya [2] which is the generalization of the Ricci soliton and the equation is given by

$$\mathcal{L}_X g + 2S = [2\lambda - (p + \frac{2}{n})]g.$$

Conformal Ricci solitons have been studied in the paper [9] [10], [11].

In this paper we would like to study some properties of conformal Ricci soliton on three-dimensional $f$-Kenmotsu manifolds.

The paper is organized as follows: After introduction, we give some preliminaries in the Section 2. In Section 3, we have studied the conformal Ricci soliton on $f$-Kenmotsu manifolds. In Section 4, we heve derived some conditions of $\phi$-Ricci symmetric $f$-Kenmotsu manifold admitting conformal Ricci soliton. Section 5 is devoted to study $\eta$-parallel Ricci tensor of an $f$-Kenmotsu manifold admitting conformal Ricci soliton. In Section 6, we have deduced some results of cyclic parallel Ricci tensor of an $f$-Kenmotsu manifold admitting conformal Ricci soliton. In Section 7, we heve studied second order parallel tensor and conformal Ricci solitons. In the last Section, we give an example.

## 2. Preliminaries

Let $M$ be a $(2n+1)$-dimensional smooth manifold endowed with an almost contact metric structure $(\phi, \xi, \eta, g)$, where $\phi$ is a tensor field of type $(1,1)$, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is the Riemannian metric on $M$ such that [15]

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1.$$

As a consequence, we get the following:

$$\phi\xi = 0, \quad g(X, \xi) = \eta(X), \quad \eta(\phi X) = 0,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0,$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y),$$

for all vector fields $X$, $Y \in \chi(M)$. A differentiable manifold $M$ of dimension $(2n+1)$ with almost contact metric structure is said to be an almost contact metric manifold. $M$ is said to be an $f$-Kenmotsu manifold if the covariant differentiation of $\phi$ satisfies

$$(\nabla_X \phi)(Y) = f\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{1}$$

where $f \in C^\infty(M)$ is such that $df \wedge \eta = 0$. If $f = \beta$ is non-zero constant, then the manifold is called a $\beta$-Kenmotsu manifold. An $f$-Kenmotsu manifold is called

Kenmotsu manifold if $f = 1$. If $f = 0$, then the manifold is cosymplectic. An $f$-Kenmotsu manifold is said to be regular if $f^2 + f' \neq 0$, where $f' = \xi f$. For an $f$-Kenmotsu manifold, we get from (1)

$$\nabla_X \xi = f\{X - \eta(X)\xi\}. \tag{2}$$

The condition $df \wedge \eta = 0$ holds if $dim(M) \geqslant 5$ and does not hold if $dim(M) = 3$. In a three-dimensional $f$-Kenmotsu manifold, we have

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2f^2 + 2f')\{g(Y,Z)X - g(X,Z)Y\} \\ &- (\frac{r}{2} + 3f^2 + 3f')\{(g(Y,Z)\xi - \eta(Z)Y)\eta(X) \\ &- (g(X,Z)\xi - \eta(Z)X)\eta(Y)\}, \end{aligned} \tag{3}$$

$$S(X,Y) = (\frac{r}{2} + f^2 + f')g(X,Y) - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\eta(Y), \tag{4}$$

$$QX = (\frac{r}{2} + f^2 + f')X - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\xi, \tag{5}$$

where $r$ is the scalar curvature of $M$. Also we get the following equations by using (3), (4) and (5)

$$R(X,Y)\xi = -(f^2 + f')(\eta(Y)X - \eta(X)Y), \tag{6}$$

$$S(X,\xi) = -2(f^2 + f')\eta(X), \tag{7}$$

$$Q\xi = -2(f^2 + f')\xi. \tag{8}$$

As a consequence of (2), we have

$$(\nabla_X \eta)(Y) = f(g(X,Y) - \eta(X)\eta(Y)), \tag{9}$$

for all vector fields $X$ and $Y$.

Definition 2.1. [11] A Riemannian manifold $M$ of dimension $n$ is called an $\eta$-Einstein manifold if

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

for all vector fields $X$, $Y$ and $a$, $b$ are functions on $M$.

Definition 2.2. [17] Let $M$ be a Riemannian manifold of dimension $n$ with metric $g$. A tensor field $\gamma$ of type $(0,2)$ is called parallel tensor if $\nabla\gamma = 0$, where $\nabla$ is the operator of covariant differentiation with respect to the metric tensor $g$.

## 3. Conformal Ricci solitons on three-dimensional $f$-Kenmotsu manifold

Let $M$ be an $f$-Kenmotsu manifold of dimension three. Then the conformal Ricci soliton is given by

$$\mathcal{L}_V g + 2S = [2\lambda - (p + \frac{2}{3})]g. \tag{10}$$

Let the potential vector field $V$ be the Reeb vector field $\xi$, then we get

$$(\mathcal{L}_\xi g)(X, Y) = 2f(g(X, Y) - \eta(X)\eta(Y)). \tag{11}$$

Therefore, from (10) and (11), we get

$$S(X, Y) = Ag(X, Y) + f\eta(X)\eta(Y), \tag{12}$$

where $A = \frac{1}{2}[2\lambda - (p + \frac{2}{3}) - 2f]$.

Thus we can state the following

THEOREM 3.1. *An $f$-Kenmotsu manifold of dimension three admitting conformal Ricci soliton is an $\eta$-Einstein manifold.*

From (12), we get

$$S(\xi, X) = \frac{1}{2}[2\lambda - (p + \frac{2}{3})]\eta(X), \tag{13}$$

$$QX = \frac{1}{2}[2\lambda - (p + \frac{2}{3}) - 2f]X + f\eta(X)\xi, \tag{14}$$

$$S(\xi, \xi) = \frac{1}{2}[2\lambda - (p + \frac{2}{3})], \tag{15}$$

$$Q\xi = \frac{1}{2}[2\lambda - (p + \frac{2}{3})]\xi. \tag{16}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space of the manifold $M$. Taking $X = Y = e_i$ in (12) and summing over $i$, we get

$$r = \frac{3}{2}[2\lambda - (p + \frac{2}{3}) - 2f] + f. \tag{17}$$

But for conformal Ricci soliton, $r = -1$. Thus from (17), we get

$$-1 = \frac{3}{2}[2\lambda - (p + \frac{2}{3}) - 2f] + f,$$

which gives

$$\lambda = \frac{p}{2} + \frac{2}{3}f. \tag{18}$$

Thus we can state the following

PROPOSITION 3.2. *If an $f$-Kenmotsu manifold of dimension three admits conformal Ricci soliton, then the value of $\lambda$ is $\frac{p}{2} + \frac{2}{3}f$.*

## 4. $\phi$-Ricci symmetric $f$-Kenmotsu manifold of dimension 3 admitting conformal Ricci soliton

Definition 4.1. [6] An $f$-Kenmotsu manifold of dimension three is said to be $\phi$-Ricci symmetric if

$$\phi^2((\nabla_X Q)(Y)) = 0, \tag{19}$$

for all vector fields $X$, $Y$ on $M$, where $Q$ is the Ricci operator.

From (19), we get

$$-(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi = 0.$$

Taking inner product, we obtain

$$-g((\nabla_X Q)(Y), Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0.$$

Simlplifying, we obtain

$$\begin{aligned} -g(\nabla_X Q(Y), Z) &+ S(\nabla_X Y, Z) + \eta(\nabla_X Q(Y))\eta(Z) \\ &- \eta(Q(\nabla_X Y))\eta(Z) = 0. \end{aligned} \tag{20}$$

Putting $Y = \xi$ in (20), we get

$$\begin{aligned} -g(\nabla_X Q(\xi), Z) &+ S(\nabla_X \xi, Z) + \eta(\nabla_X Q(\xi))\eta(Z) \\ &- \eta(Q(\nabla_X \xi))\eta(Z) = 0. \end{aligned} \tag{21}$$

Using (14) and (16) in (21), we get

$$-Bg(\nabla_X \xi, Z) + S(\nabla_X \xi, Z) = 0, \tag{22}$$

where $B = \frac{1}{2}[2\lambda - (p + \frac{2}{3})]$.
From (2) and (22), we have

$$-Bf[g(X, Z) - \eta(X)\eta(Z)] + f[S(X, Z) - \eta(X)S(\xi, Z)] = 0. \tag{23}$$

Using (12) and (13) in (23), we get

$$-f^2[g(X, Z) - \eta(X)\eta(Z)] = 0.$$

Since, $g(X, Z) - \eta(X)\eta(Z) \neq 0$ for all vector fields $X$ and $Z$, we get $f = 0$.

Thus we can state the following

Theorem 4.1. *If an $f$-Kenmotsu manifold of dimension three admitting conformal Ricci soliton is $\phi$-Ricci symmetric, then the manifold is cosymplectic.*

## 5. $\eta$-parallel Ricci tensor of an $f$-Kenmotsu manifold admitting conformal Ricci soliton

Definition 5.1. [5] The Ricci tensor $S$ of an $f$-Kenmotsu manifold of dimension three is called $\eta$-parallel if it satisfies

$$(\nabla_W S)(\phi X, \phi Y) = 0,$$

for all vector fields $X$, $Y$ and $W$ on $M$.

From (12), we get

$$S(\phi X, \phi Y) = \frac{1}{2}[2\lambda - (p + \frac{2}{3}) - 2f](g(X, Y) - \eta(X)\eta(Y)). \tag{24}$$

Differentiating (24) covarianty with respect to $W$, we obtain

$$(\nabla_W S)(\phi X, \phi Y) = -\frac{1}{2}[2\lambda - (p + \frac{2}{3}) - 2f]((\nabla_W \eta)(X)\eta(Y) + \eta(X)(\nabla_W \eta)(Y)). \tag{25}$$

Using (9) in (25), we get

$$(\nabla_W S)(\phi X, \phi Y) = -\frac{f}{2}[2\lambda - (p + \frac{2}{3}) - 2f][g(X,W)\eta(Y) + g(Y,W)\eta(X) - 2\eta(X)\eta(Y)\eta(W)]. \tag{26}$$

Let the Ricci tensor of the manifold be $\eta$-parallel, then from (26), we get

$$\frac{f}{2}[2\lambda - (p + \frac{2}{3}) - 2f][g(X,W)\eta(Y) + g(Y,W)\eta(X) - 2\eta(X)\eta(Y)\eta(W)] = 0. \tag{27}$$

Using (18), we get from (27), either $f$=0 or $f = -1$.

Thus we can state the following

Theorem 5.1. *If Ricci tensor of a non-cosymplectic $f$-Kenmotsu manifold of dimension three admitting conformal Ricci soliton is $\eta$-parallel, then the manifold is $\beta$-Kenmotsu.*

## 6. Cyclic parallel Ricci tensor on an $f$-Kenmotsu manifold admitting conformal Ricci tensor

Definition 6.1. [5] Ricci tensor $S$ of an $f$-Kenmotsu manifold $M$ of dimension three is called cyclic-parallel if

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0,$$

for all vector fields $X$, $Y$, $Z$ on $M$ and $\nabla$ denote the Riemannian connection.

From (12), we get

$$(\nabla_X S)(Y,Z) = -f((\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(z).$$

Therefore, we have from above

$$\begin{aligned}&(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y)\\&= -f\{(\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z)\\&+(\nabla_Y \eta)(Z)\eta(X) + \eta(Z)(\nabla_Y \eta)(X)\\&+\nabla_Z \eta)(X)\eta(Y) + \eta(X)(\nabla_Z \eta)(Y)\}.\end{aligned} \tag{28}$$

Let the Ricci tensor is cyclic-parallel, then from (28), we get

$$f\{(\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z) + (\nabla_Y \eta)(Z)\eta(X) + \eta(Z)(\nabla_Y \eta)(X) + (\nabla_Z \eta)(X)\eta(Y) + \eta(X)(\nabla_Z \eta)(Y)\} = 0. \tag{29}$$

From (9) and (29), we get

$$f^2[g(X,Y)\eta(Z) + g(Y,Z)\eta(X) + g(Z,X)\eta(Y) - 3\eta(X)\eta(Y)\eta(Z)] = 0. \tag{30}$$

Putting $X = Y = e_i$ in (30), where $\{e_i\}$ $(i = 1, 2, 3)$ is the orthonormal basis of the tangent space of the manifold and summing over $i$, we get

$$2f^2\eta(Z) = 0,$$

which gives $f = 0$.

Thus we can state the following

Theorem 6.1. *If the Ricci tensor of an $f$-Kenmotsu manifold of dimension three admitting conformal Ricci soliton is cyclic-parallel, then the manifold is cosymplectic.*

## 7. Second order parallel tensor and conformal Ricci soliton

Let $\gamma$ be a second order symmetric tensor field on a regular $f$-Kenmotsu manifold $M$ of dimension three, that is, $\gamma(X, Y) = \gamma(Y, X)$ such that $\nabla\gamma = 0$. Then from the Ricci identity, we have

$$\nabla^2\gamma(X, Y; Z, W) = \nabla^2\gamma(X, Y; W, Z).$$

From above, we obtain

$$\gamma(R(X, Y)Z, W) + \gamma(R(X, Y)W, Z) = 0, \tag{31}$$

for all vector fields $X$, $Y$, $Z$ and $W$ on $M$.
Substituting $X = Z = W = \xi$ in (31), we get

$$\gamma(R(\xi, Y)\xi, \xi) = 0. \tag{32}$$

From (6), we get

$$R(\xi, Y)\xi = -(f^2 + f^{'})(\eta(Y)\xi - Y). \tag{33}$$

Therefore, from (32) and (33), we obtain

$$-(f^2 + f^{'})(\eta(Y)\gamma(\xi, \xi) - \gamma(Y, \xi)) = 0.$$

Since, $f^2 + f^{'} \neq 0$, we get from above

$$\gamma(Y, \xi) = \eta(Y)\gamma(\xi, \xi). \tag{34}$$

Taking differentiation (34) covariantly with respect to $X$, we get

$$\begin{aligned} \gamma(\nabla_X Y, \xi) + \gamma(Y, \nabla_X \xi) &= g(\nabla_X Y, \xi)\gamma(\xi, \xi) \\ &+(Y, \nabla_X \xi)\gamma(\xi, \xi) + 2\eta(Y)\gamma(\nabla_X \xi, \xi). \end{aligned} \tag{35}$$

From (34), we obtain

$$\gamma(\nabla_X Y, \xi) = g(\nabla_X Y, \xi)\gamma(\xi, \xi). \tag{36}$$

From (35) and (36), we get

$$\gamma(Y, \nabla_X \xi) = g(Y, \nabla_X \xi)\gamma(\xi, \xi) + 2\eta(Y)\gamma(\nabla_X \xi, \xi). \tag{37}$$

From (2) and (37), we get

$$\begin{aligned} &f[g(X, Y)\gamma(\xi, \xi) - 3\eta(X)\eta(Y)\gamma(\xi, \xi) \\ &+2\eta(Y)\gamma(X, \xi) - \gamma(X, Y) + \eta(X)\gamma(\xi, Y)] = 0. \end{aligned} \tag{38}$$

From (34), we get

$$\eta(X)\gamma(Y, \xi) = \eta(X)\eta(Y)\gamma(\xi, \xi). \tag{39}$$

From (38) and (39), we obtain

$$f[g(X,Y)\gamma(\xi,\xi) - \gamma(X,Y)] = 0.$$

Since $f \neq 0$, we get from above

$$\gamma(X,Y) = g(X,Y)\gamma(\xi,\xi).$$

Thus we can state

THEOREM 7.1. *If $\gamma$ is a second order parallel tensor of a regular $f$-Kenmotsu manifold of dimension three, then $\gamma$ is given by*

$$\gamma(X,Y) = g(X,Y)\gamma(\xi,\xi),$$

*for all vector fields $X$, $Y$ on $M$.*

Since $\nabla\{[2\lambda - (p + \frac{2}{3})]g(X,Y)\} = 0$ for all vector fields $X$ and $Y$, we can say that $\{\mathcal{L}_\xi g(X,Y) + 2S(X,Y)\}$ is a second order parallel tensor. So,

$$(\mathcal{L}_\xi g)(X,Y) + 2S(X,Y) = \{(\mathcal{L}_\xi g)(\xi,\xi) + 2S(\xi,\xi)\}g(X,Y). \tag{40}$$

From (7) and (11), we get

$$(\mathcal{L}_\xi g)(\xi,\xi) + 2S(\xi,\xi) = -2(f^2 + f^{'}). \tag{41}$$

From (40) and (41), we obtain

$$(\mathcal{L}_\xi g)(X,Y) + 2S(X,Y) = -2(f^2 + f^{'})g(X,Y). \tag{42}$$

Compairing (10) and (42), we get

$$\lambda = -(f^2 + f^{'}) + \frac{1}{2}(p + \frac{2}{3}). \tag{43}$$

From (18) and (43), we get

$$3(f^2 + f^{'}) + 2f = 1.$$

Thus we can state the following

PROPOSITION 7.2. *If a regular $f$-Kenmotsu manifold of dimension three admits conformal Ricci soliton, then $3(f^2 + f^{'}) + 2f = 1$.*

## 8. Example

Let us consider the manifold $M = \{x, y, z \in \mathbb{R}^3 : z \neq 0\}$ of dimension 3, where $(x, y, z)$ are standard co-ordinates in $\mathbb{R}^3$. We choose the vector fields

$$e_1 = z^{-2}\frac{\partial}{\partial x}, \quad e_2 = z^{-2}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z},$$

which are linearly independent at each point of $M$. We get the following by direct computations

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -\frac{2}{z}e_1, \quad [e_2, e_3] = -\frac{2}{z}e_2.$$

Let the metric tensor $g$ be defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$$

and

$$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

The 1-form $\eta$ is defined by $\eta(X) = g(X, e_3)$, for all $X$ on $M$. Let $\phi$ be the $(1,1)$-tensor field defined by

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$

Then we find that

$$\eta(e_3) = 1, \quad \phi^2 X = -X + \eta(X)e_3,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields $X$, $Y$ on $M$.

Let $\nabla$ be the Levi-Civita connection on $M$, then by Koszul'S formula, we obtain

$$\nabla_{e_1} e_1 = \frac{2}{z} e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\frac{2}{z} e_1,$$
$$\nabla_{e_2} e_2 = \frac{2}{z} e_3, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_3 = -\frac{2}{z} e_2,$$
$$\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0.$$

From the above expressions of $\nabla$, the manifold is an $f$-Kenmotsu manifold of dimension three with $\xi = e_3$ and $f = -\frac{2}{z}$. Also $f^2 + f' \neq 0$, thus the manifold is a regular $f$-Kenmotsu manifold.

Using the formula $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, we get

$$R(e_1, e_2)e_2 = -\frac{4}{z^2} e_1, \quad R(e_2, e_1)e_1 = -\frac{4}{z^2} e_2, \quad R(e_2, e_3)e_3 = -\frac{6}{z^2} e_2,$$
$$R(e_3, e_2)e_2 = -\frac{6}{z^2} e_3, \quad R(e_1, e_3)e_3 = -\frac{6}{z^2} e_1, \quad R(e_3, e_1)e_1 = -\frac{6}{z^2} e_3,$$
$$R(e_1, e_2)e_3 = 0, \qquad R(e_2, e_3)e_1 = 0, \qquad R(e_3, e_1)e_2 = 0.$$

From the above expressions of curvature tensor, we get

$$S(e_1, e_1) = -\frac{10}{z^2}, \quad S(e_2, e_2) = -\frac{10}{z^2}, \quad S(e_3, e_3) = -\frac{12}{z^2}.$$

Let $r$ be the scalar curvature, then from above

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -\frac{32}{z^2}.$$

But for conformal Ricci soliton $r = -1$. Thus the given manifold admits conformal Ricci soliton if $z = 4\sqrt{2}$.

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