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CHARACTER ANALOGUES OF INFINITE SERIES IDENTITIES RELATED TO GENERALIZED NON-HOLOMORPHIC EISENSTEIN SERIES

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Abstract. In this paper, we derive analogues of a couple of classes of infinite series identities with the confluent hypergeometric functions involving Dirichlet characters.

1. Introduction

In [3], the author found character analogues of infinite series identities which originally come from modular transformation formula for generalized Eisenstein series. One of them shows the following symmetric identity([3], Corollary 3.4);
Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$ and let $\left(\frac{\cdot}{p}\right)$ be the Legendre symbol modulo p, where p is a prime with $p \equiv 1 \pmod 4$. Then, for any integer $M > 0$,

$$\begin{aligned}&\alpha^{2M}\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{4M-1}(\left(\frac{\cdot}{p}\right), n)\cos(2\pi n/p)e^{-2\alpha n/p}\\&=\beta^{2M}\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{4M-1}(\left(\frac{\cdot}{p}\right), n)\cos(2\pi n/p)e^{-2\beta n/p},\end{aligned}$$

where

$$\sigma_s(\left(\frac{\cdot}{p}\right), n) = \sum_{d|n}\left(\frac{d}{p}\right)d^s.$$

The study to find this type of character analogues was motivated by the works of B. C. Berndt, A. Dixit and J. Sohn in [2]. For example, a character analogue of Guinand's formula shows the following elegant symmetric identity (Corollary 3.2 in [2]);

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{-1}(n)e^{-2n\alpha/p} = \sqrt{\beta}\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{-1}(n)e^{-2n\beta/p},$$

where $\sigma_s(n)$ is the sum of the s-th powers of the positive divisors of n.
In this paper, we establish character analogues of certain classes of infinite series identities which stem from a modular transformation formula for a class of functions related to generalized non-holomorphic Eisenstein series. We start with introducing

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necessary notations and then shall state the principal theorem which shows a modular transformation formula for a large class of functions coming from generalized non-holomorphic Eisenstein series. In fact, the theorem that we shall use in this paper is a twisted version of the theorem in [4] and so some notations are twisted versions of those in [4].

Let $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of integers, real numbers and complex numbers, respectively. Throughout this paper, let the branch of the argument of $z \in \mathbb{C}$ be defined by $-\pi \le \arg z < \pi$. For any non-negative integer n, the rising factorial $(x)_n$ is defined by

$$(x)_n = x(x+1)\cdots(x+n-1),\ n>0 \text{ and } (x)_0 = 1.$$

Let $\Gamma(s)$ denote the gamma function. It is easy to see that

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}. \tag{1.1}$$

The confluent hypergeometric function of the first kind ${}_1F_1(\alpha;\beta;z)$ is defined by

$${}_1F_1(\alpha;\beta;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n n!} z^n$$

and the confluent hypergeometric function of the second kind $U(\alpha,\beta,z)$ is defined by

$$U(\alpha,\beta,z) = \frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}\ {}_1F_1(\alpha;\beta;z) + \frac{\Gamma(\beta-1)}{\Gamma(\alpha)} z^{1-\beta}\ {}_1F_1(1+\alpha-\beta;2-\beta;z).$$

The function $U(\alpha,\beta,z)$ can be analytically continued to all values of $\alpha,\ \beta,\ z \in \mathbb{C}$ [6]. Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\ \tau > 0\}$ be the upper half-plane. For $r_k,\ h_k \in \mathbb{R}$ $(k=1,2)$, let $\mathbf{r} = (r_1, r_2)$ and $\mathbf{h} = (h_1, h_2)$. Let $e(x) = e^{2\pi i x}$ and let N be a positive integer. For $\tau \in \mathbb{H}$ and $s_1,\ s_2 \in \mathbb{C}$ with $s = s_1 + s_2$, define

$$\mathcal{A}_N(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) = \sum_{Nm+r_1>0} \sum_{n-h_2>0} \frac{e\left(Nmh_1 + ((Nm+r_1)\tau + r_2)(n-h_2)\right)}{(n-h_2)^{1-s}} \times U(s_2; s; 4\pi(Nm+r_1)(n-h_2)\mathrm{Im}(\tau))$$

and

$$\bar{\mathcal{A}}_N(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) = \sum_{Nm+r_1>0} \sum_{n+h_2>0} \frac{e\left(Nmh_1 - ((Nm+r_1)\bar{\tau} + r_2)(n+h_2)\right)}{(n+h_2)^{1-s}} \times\ U(s_1; s; 4\pi(Nm+r_1)(n+h_2)\mathrm{Im}(\tau)).$$

Let

$$\mathcal{H}_N(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) = \mathcal{A}_N(\tau, s_1, s_2; r, h) + e^{\pi i s} \mathcal{A}_N(\tau, s_1, s_2; -\mathbf{r}, -\mathbf{h})$$

and

$$\bar{\mathcal{H}}_N(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) = \bar{\mathcal{A}}_N(\tau, s_1, s_2; r, h) + e^{\pi i s} \bar{\mathcal{A}}_N(\tau, s_1, s_2; -\mathbf{r}, -\mathbf{h}).$$

Let

$$\mathbf{H}_N(\tau, \bar{\tau}, s_1, s_2; \mathbf{r}, \mathbf{h}) = \frac{1}{\Gamma(s_1)} \mathcal{H}_N(\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) + \frac{1}{\Gamma(s_2)} \bar{\mathcal{H}}_N(\tau, s_1, s_2; r, h).$$

The functions $\mathcal{A}_N, \bar{\mathcal{A}}_N, \mathcal{H}_N, \bar{\mathcal{H}}_N$ and $\mathbf{H}_N$ are twisted versions of those in [4]. In fact, the function $\mathbf{H}_N$ comes from generalized non-holomorphic Eisenstein series and the relation between $\mathbf{H}_N$ and the generalized non-holomorphic Eisenstein series is given in [4]. For $x,\ \alpha \in \mathbb{R}$ and $t \in \mathbb{C}$ with $\operatorname{Re} t > 1$, let

$$\psi(x, \alpha, t) = \sum_{n+\alpha>0} \frac{e(nx)}{(n+\alpha)^t}$$

and let

$$\begin{aligned} \Psi(x, \alpha, t) &= \psi(x, \alpha, t) + e^{\pi i t}\psi(-x, -\alpha, t), \\ \Psi_{-1}(x, \alpha, t) &= \psi(x, \alpha, t-1) + e^{\pi i t}\psi(-x, -\alpha, t-1). \end{aligned}$$

Let λ_N denote the characteristic function of the integers modulo N. For $x \in \mathbb{R}$, $[x]$ denotes the greatest integer less than or equal to x and $\{x\} = x - [x]$. Let

$$V\tau = \frac{a\tau + b}{c\tau + d}$$

denote a modular transformation with $c > 0$ and $c \equiv 0 \pmod{N}$ for $\tau \in \mathbb{C}$. Let

$$\mathfrak{R} = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2)$$

and

$$\mathfrak{H} = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2).$$

Put

$$\varrho_N = c\{R_2\} - Nd\left\{\frac{R_1}{N}\right\}.$$

We now state a twisted version of Theorem 3.4 in [4] which we shall use to obtain our results.

THEOREM 1.1. [4]. *Let $Q = \{\tau \in \mathbb{H} \mid \operatorname{Re}\tau > -d/c\}$. Let $s_1, s_2 \in \mathbb{C}$ with $s = s_1 + s_2$ and assume that s is not an integer less than or equal to 1. Then, for $\tau \in Q$,*

$$\begin{aligned} &z^{-s_1}\bar{z}^{-s_2}\mathbf{H}_N(V\tau, V\bar{\tau}, s_1, s_2; \mathbf{r}, \mathbf{h}) \\ &= \mathbf{H}_N(\tau, \bar{\tau}, s_1, s_2; \mathfrak{R}, \mathfrak{H}) + \lambda_N(R_1)e(-R_1H_1)(2\pi i)^{-s}e^{-\pi i s_2}\Psi(-H_2, -R_2, s) \\ &\quad -\lambda_N(r_1)e(-r_1h_1)(2\pi i)^{-s}e^{\pi i s_1}z^{-s_1}\bar{z}^{-s_2}\Psi(h_2, r_2, s) \\ &\quad +\lambda_N(H_2)(4\pi \operatorname{Im}(\tau))^{1-s}\frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)}\Psi_{-1}(H_1, R_1, s) \\ &\quad -\lambda_N(h_2)(4\pi \operatorname{Im}(\tau))^{1-s}\frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)}z^{s_2-1}\bar{z}^{s_1-1}\Psi_{-1}(h_1, r_1, s) \\ &\quad +\frac{(2\pi i)^{-s}e^{-\pi i s_2}}{\Gamma(s_1)\Gamma(s_2)}\mathbf{L}_N(\tau, \bar{\tau}, s_1, s_2; \mathfrak{R}, \mathfrak{H}), \end{aligned}$$

where $z = c\tau + d$ and

$$\begin{aligned} &\mathbf{L}_N(\tau, \bar{\tau}, s_1, s_2; \mathfrak{R}, \mathfrak{H}) \\ &= \sum_{j=1}^{c} e(-H_1(Nj + N[R_1/N] - c) - H_2([R_2] + 1 + [(Njd + \varrho_N)/c] - d)) \\ &\quad \times \int_0^1 v^{s_1-1}(1-v)^{s_2-1}\int_C u^{s-1}\frac{e^{-(zv+\bar{z}(1-v))(Nj-N\{R_1/N\})u/c}}{e^{-(zv+\bar{z}(1-v))u} - e(cH_1 + dH_2)}\frac{e^{\{(Njd+\varrho_N)/c\}u}}{e^u - e(-H_2)}\,du dv, \end{aligned}$$

where C is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in a counterclockwise direction so that $u = 0$ is the only zero of

$$(e^{-(zv+\bar{z}(1-v))u} - e(cH_1 + dH_2))(e^u - e(-H_2))$$

lying inside the loop, and then returning to $+\infty$ in the lower half plane. Here, we choose the branch of u^s with $0 < \arg u < 2\pi$.

Let $B_n(x)$ denote the n-th Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad (|t| < 2\pi).$$

The n-th Bernoulli number B_n, $n \geq 0$, is defined by $B_n = B_n(0)$. Put $\bar{B}_n(x) = B_n(\{x\})$, $n \geq 0$. Let ${}_2F_1(\alpha, \beta; \gamma; z)$ be a hypergeometric function defined by

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n.$$

The function $\frac{1}{\Gamma(\gamma)}\ {}_2F_1(\alpha, \beta; \gamma; z)$ can be analytically continued to all $\alpha,\ \beta,\ \gamma \in \mathbb{C}$ and all $z \in \mathbb{C}$ with $|z| < 1$ ([1]).

Remark 1.2. Let $s = s_1 + s_2$ be an integer and let $h_1 = h_2 = 0$. By the residue theorem, we find that

$$\begin{aligned}
&\int_C u^{s-1} \frac{e^{-(zv+\bar{z}(1-v))(Nj-N\{R_1/N\})u/c}}{e^{-(zv+\bar{z}(1-v))u}-1} \frac{e^{\{(Njd+\varrho_N)/c\}u}}{e^u-1}\, du \\
&= 2\pi i \sum_{k=0}^{-s+2} \frac{B_k((Nj-N\{R_1/N\})/c)\bar{B}_{-s+2-k}((Njd+\varrho_N)/c)}{k!(-s+2-k)!} (-zv-\bar{z}(1-v))^{k-1}.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_0^1 v^{s_1-1}(1-v)^{s_2-1} \int_C u^{-s-1} \frac{e^{-(zv+\bar{z}(1-v))(Nj-N\{R_1/N\})u/c}}{e^{-(zv+\bar{z}(1-v))u}-1} \frac{e^{\{(Njd+\varrho_N)/c\}u}}{e^u-1}\, dudv \\
&= 2\pi i \sum_{k=0}^{-s+2} \frac{B_k((Nj-N\{R_1/N\})/c)\bar{B}_{-s+2-k}((Njd+\varrho_N)/c)}{k!(-s+2-k)!} (-z)^{k-1} \\
&\qquad \times \int_0^1 (1-v)^{s_1-1} v^{s_2-1} \left(1 - \frac{z-\bar{z}}{z} v\right)^{k-1} dv \\
&= 2\pi i \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s)} \sum_{k=0}^{-s+2} \frac{B_k((Nj-N\{R_1/N\})/c)\bar{B}_{-s+2-k}((Njd+\varrho_N)/c)}{k!(-s+2-k)!} \\
&\qquad \times (-z)^{k-1}\ {}_2F_1\left(s_2, 1-k; s; \frac{z-\bar{z}}{z}\right).
\end{aligned}$$

The last equality holds due to the integral representation of the hypergeometric function. Hence we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(s_1)\Gamma(s_2)} \mathbf{L}_N(\tau, \bar{\tau}, s_1, s_2; \mathfrak{R}, \mathfrak{H}) \\
&= \frac{2\pi i}{\Gamma(s)} \sum_{k=0}^{-s+2} \frac{B_k((Nj-N\{R_1/N\})/c)\bar{B}_{-s+2-k}((Njd+\varrho_N)/c)}{k!(-s+2-k)!} \\
&\qquad \times (-z)^{k-1}\ {}_2F_1\left(s_2, 1-k; s; \frac{z-\bar{z}}{z}\right).
\end{aligned}$$

We now see that $\frac{1}{\Gamma(s_1)\Gamma(s_2)}\mathbf{L}_N(\tau,\bar{\tau},s_1,s_2;\mathfrak{R},\mathfrak{H})$ vanishes for $s>2$. Let $s=2$ and $s_2=-B$ be a non-positive integer. Then, applying

$$(s_2)_n=\begin{cases}(-1)^n\frac{B!}{(B-n)!}, & n\le B,\\ 0, & n>B\end{cases}$$

and using the binomial expansion

$$(1+x)^B=\sum_{n=0}^{B}\binom{B}{n}x^n,$$

we find

$$\begin{aligned}{}_2F_1\left(s_2,1;2;\frac{z-\bar{z}}{z}\right)&=\sum_{n=0}^{B}\binom{B}{n}\frac{1}{n+1}\left(\frac{\bar{z}-z}{z}\right)^n\\ &=\frac{1}{B+1}\frac{z}{\bar{z}-z}\left(\left(\frac{\bar{z}}{z}\right)^{B+1}-1\right).\end{aligned}$$

Thus, for $s=2$ and s_2 a non-positive integer, after the evaluation of $\mathbf{L}_N(\tau,\bar{\tau},s_1,s_2;\mathfrak{R},\mathfrak{H})$, the relevant formula will be valid for all $z\in\mathbb{H}$ by analytic continuation.

2. A class of character analogues of infinite series identities

In this section, let χ be a Dirichlet character of modulus N and χ_{o} be the principal Dirichlet character of modulus N. From now on, we set

$$V\tau=\frac{\tau-1}{N\tau-N+1}.$$

The function φ denotes Euler's phi function, i.e., $\varphi(N)$ is the number of positive integers up to N that is relatively prime to N. Let $\zeta_N=e^{2\pi i/N}$ and let

$$\sigma_t(\chi,n)=\sum_{d|n}\chi(d)d^t.$$

Theorem 2.1. *Let χ be even. For any integers $B\ge 0$, $M\ge 1$ and for $z\in\mathbb{H}$,*

$$\begin{aligned}&z^{-B-2M}\bar{z}^B\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(4\pi\mathrm{Im}(z^{-1})/N)^\ell}{(2M+k-1)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\zeta_N^n\sigma_{2M-1}(\bar{\chi},n)e\left(\frac{-nz^{-1}}{N}\right)\\ &=\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\pi\mathrm{Im}(z)/N)^\ell}{(2M+k-1)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\zeta_N^{-n}\sigma_{2M-1}(\bar{\chi},n)e\left(\frac{nz}{N}\right)+\frac{1}{2}\delta_M(B,z)\nu_\chi(N),\end{aligned}$$

where

$$\delta_M(B,z)=\begin{cases}\dfrac{(-1)^{B+1}}{2\pi(B+1)}\dfrac{1}{z-\bar{z}}\left(\left(\dfrac{\bar{z}}{z}\right)^{B+1}-1\right), & M=1,\\ 0, & M>1\end{cases}$$

and

$$\nu_\chi(N)=\begin{cases}\varphi(N), & \chi=\chi_{\mathrm{o}},\\ 0, & \chi\ne\chi_{\mathrm{o}}.\end{cases}$$

Proof. Let $s_1 = A \geq 1$, $s_2 = -B \leq 0$ and $s = 2M \geq 2$ for $A, B, M \in \mathbb{Z}$. Put $\mathbf{r} = (k, 0)$ for any integer k with $1 \leq k < N$ and put $\mathbf{h} = (0,0)$ in Theorem 1.1. Then $\mathfrak{R} = (k, -k)$ and $\mathfrak{H} = (0,0)$. We see that $\lambda_N(r_1) = \lambda_N(R_1) = \lambda_N(k) = 0$. Put $z = N\tau - N + 1$. Using Remark 1.2, we have

$$\frac{(2\pi i)^{1-s}e^{-\pi i s_2}}{\Gamma(s_1)\Gamma(s_2)}\mathbf{L}_N(\tau,\bar{\tau},s_1,s_2;\mathfrak{R},\mathfrak{H}) = \begin{cases} \dfrac{(-1)^{B+1}}{2\pi(B+1)}\dfrac{1}{z-\bar{z}}\left(\left(\dfrac{\bar{z}}{z}\right)^{B+1}-1\right), & M=1,\\ 0, & M>1. \end{cases}$$

Note that $\frac{1}{\Gamma(s_2)} = \frac{1}{\Gamma(-B)} = 0$. Thus, in Theorem 1.1, the terms with $\lambda_N(h_2)$ and $\lambda_N(H_2)$ are equal to 0. Thus we have

$$z^{-A}\bar{z}^{B}\mathbf{H}_N(V\tau, V\bar{\tau}, A, -B; \mathbf{r}, \mathbf{h}) = \mathbf{H}_N(\tau,\bar{\tau},A,-B;\mathfrak{R},\mathfrak{H}) + \delta_M(B,z). \tag{2.1}$$

Multiplying both sides in (2.1) by $\chi(k)$ and summing over k, we find that

$$\begin{aligned} &z^{-A}\bar{z}^{B}\sum_{k=1}^{N-1}\chi(k)\mathbf{H}_N(V\tau,V\bar{\tau},A,-B;\mathbf{r},\mathbf{h}) \\ &= \sum_{k=1}^{N-1}\chi(k)\mathbf{H}_N(\tau,\bar{\tau},A,-B;\mathfrak{R},\mathfrak{H}) + \sum_{k=1}^{N-1}\chi(k)\delta_M(B,z). \end{aligned} \tag{2.2}$$

Since $\frac{1}{\Gamma(s_2)} = \frac{1}{\Gamma(-B)} = 0$,

$$\mathbf{H}_N(V\tau,V\bar{\tau},A,-B;\mathbf{r},\mathbf{h}) = \frac{1}{\Gamma(A)}\mathcal{H}_N(V\tau,A,-B;\mathbf{r},\mathbf{h}).$$

It is easy to see that

$$\begin{aligned} &\mathcal{A}_N(V\tau,A,-B;\mathbf{r},\mathbf{h}) \\ &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{e((Nm+k)nV\tau)}{n^{1-2M}}U(-B;2M;4\pi(Nm+k)n\mathrm{Im}(V\tau)) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\mathcal{A}_N(V\tau,A,-B;-\mathbf{r},-\mathbf{h}) \\ &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{e((Nm+N-k)nV\tau)}{n^{1-2M}}U(-B;2M;4\pi(Nm+N-k)n\mathrm{Im}(V\tau)). \end{aligned} \tag{2.4}$$

Using (2.3) and (2.4), we obtain that

$$\begin{aligned} &\sum_{k=1}^{N-1}\chi(k)\mathbf{H}_N(V\tau,V\bar{\tau},A,-B;\mathbf{r},\mathbf{h}) \\ &= \frac{1}{\Gamma(A)}\sum_{k=1}^{N-1}\chi(k)\left(\mathcal{A}_N(V\tau,A,-B;\mathbf{r},\mathbf{h}) + \mathcal{A}_N(V\tau,A,-B;-\mathbf{r},-\mathbf{h})\right) \\ &= \frac{2}{\Gamma(A)}\sum_{k=1}^{N-1}\chi(k)\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{e((Nm+k)nV\tau)}{n^{1-2M}}U(-B;2M;4\pi(Nm+k)n\mathrm{Im}(V\tau)) \\ &= \frac{2}{\Gamma(A)}\sum_{n=1}^{\infty}\frac{1}{n^{1-2M}}\sum_{m=1}^{\infty}\chi(m)e(mnV\tau)U(-B;2M;4\pi mn\mathrm{Im}(V\tau)) \\ &= \frac{2}{\Gamma(A)}\sum_{n=1}^{\infty}\chi(n)\sigma_{2M-1}(\bar{\chi},n)e(nV\tau)U(-B;2M;4\pi n\mathrm{Im}(V\tau)). \end{aligned}$$

Recall $\frac{1}{\Gamma(s_2)} = \frac{1}{\Gamma(-B)} = 0$ and apply (1.1) to obtain

$$U(-B; 2M; 4\pi n\mathrm{Im}(V\tau)) = (-1)^B(A-1)! \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\pi n\mathrm{Im}(V\tau))^\ell}{(2M+\ell-1)!}.$$

Since $z = N\tau - N + 1$,

$$V\tau = \frac{\tau - 1}{N\tau - N + 1} = \frac{1}{N}(1 - z^{-1}),$$

$$e(nV\tau) = e^{2\pi i(n(1-z^{-1})/N)} = \zeta_N^n e\left(\frac{-nz^{-1}}{N}\right)$$

and

$$\mathrm{Im}(V\tau) = -\frac{1}{N}\mathrm{Im}(z^{-1}).$$

Hence we obtain

$$\sum_{k=1}^{N-1} \chi(k)\mathbf{H}_N(V\tau, V\bar{\tau}, A, -B; \mathbf{r}, \mathbf{h})$$
$$= 2(-1)^B \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(4\pi\mathrm{Im}(z^{-1})/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \zeta_N^n \sigma_{2M-1}(\bar{\chi}, n) e\left(\frac{-nz^{-1}}{N}\right).$$

By the same way, we also obtain

$$\sum_{k=1}^{N-1} \chi(k)\mathbf{H}_N(\tau, \bar{\tau}, A, -B; \mathfrak{R}, \mathfrak{H})$$
$$= 2(-1)^B \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\pi\mathrm{Im}(z)/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \zeta_N^{-n} \sigma_{2M-1}(\bar{\chi}, n) e\left(\frac{nz}{N}\right).$$

Put the last two identities into (2.2) and use

$$\sum_{k=1}^{N-1} \chi(k) = \begin{cases} \varphi(N), & \chi = \chi_o, \\ 0, & \chi \neq \chi_o. \end{cases}$$

to complete the proof. □

Theorem 2.2. *Let χ be even and let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. For any integers $B \geq 0$ and $M \geq 1$,*

$$(-1)^B \alpha^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \zeta_N^n \sigma_{2M-1}(\bar{\chi}, n) e^{-2n\alpha/N}$$
$$= (-1)^M \beta^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \zeta_N^{-n} \sigma_{2M-1}(\bar{\chi}, n) e^{-2n\beta/N} - \frac{1}{2}\delta_M(B)\nu_\chi(N),$$

where

$$\delta_M(B) = \begin{cases} \dfrac{(-1)^B + 1}{4(B+1)}, & M = 1, \\ 0, & M > 1. \end{cases}$$

Proof. Put $z = \frac{\pi}{\alpha} i$ in Theorem 2.1. Then

$$z^{-A}\bar{z}^{B} = (-1)^{B}\alpha^{M}(-\beta)^{-M},\ e\left(\frac{-nz^{-1}}{N}\right) = e^{-2n\alpha/N} \text{ and } e\left(\frac{nz}{N}\right) = e^{-2n\beta/N}.$$

A short calculation shows that

$$\delta_1\left(B, \frac{\pi}{\alpha} i\right) = \frac{(-1)^B + 1}{4\beta(B+1)}.$$

Multiplying both sides of the identity in Theorem 2.1 by $(-\beta)^M$, we complete the proof. □

Let $B = 0$ in Theorem 2.2. If $M \geq 2$ or $\chi \neq \chi_{\mathrm{o}}$, then we have

$$\alpha^M \sum_{n=1}^{\infty} \chi(n)\zeta_N^n \sigma_{2M-1}(\bar{\chi}, n) e^{-2n\alpha/N} = (-\beta)^M \sum_{n=1}^{\infty} \chi(n)\zeta_N^{-n} \sigma_{2M-1}(\bar{\chi}, n) e^{-2n\beta/N},$$

which is given as Corollary 3.3 in [3].
Let $\chi = \chi_{\mathrm{o}}$, $M = 1$ and $\alpha = \beta = \pi$ in Theorem 2.2. If B is even, then

$$\sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\pi/N)^\ell}{\ell!} \sum_{n=1}^{\infty} \frac{\chi_{\mathrm{o}}(n)}{n^{-\ell}} \cos\left(\frac{2n\pi}{N}\right) \sigma_1(\bar{\chi}_{\mathrm{o}}, n) e^{-2n\pi/N} = -\frac{\varphi(N)}{8\pi(B+1)}.$$

Put $B = 0$. Then

$$\sum_{n=1}^{\infty} \chi_{\mathrm{o}}(n) \cos\left(\frac{2n\pi}{N}\right) \sigma_1(\chi_{\mathrm{o}}, n) e^{-2n\pi/N} = -\frac{1}{8\pi}\varphi(N).$$

Let $\chi = \bar{\chi}$ and let $M \geq 2$ or $\chi \neq \chi_{\mathrm{o}}$ in Theorem 2.2. Then, equating the real part and the imaginary part, respectively, we have

$$\begin{aligned}
&(-1)^B \alpha^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \cos\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\bar{\chi}, n) e^{-2n\alpha/N} \\
&= (-1)^M \beta^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \cos\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\bar{\chi}, n) e^{-2n\beta/N}
\end{aligned}$$

and

$$\begin{aligned}
&(-1)^B \alpha^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \sin\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\bar{\chi}, n) e^{-2n\alpha/N} \\
&= (-1)^{M+1} \beta^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \sin\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\bar{\chi}, n) e^{-2n\beta/N}.
\end{aligned}$$

Thus we obtain the following two corollaries which include elegant symmetric identities for α and β.

Corollary 2.3. *Let χ be even and $\chi = \bar{\chi}$. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$ and let B, M be integers with $B \geq 0$ and $M \geq 1$. Suppose that $M = 1$ and $\chi = \chi_{\mathrm{o}}$ cannot be considered simultaneously. If B and M have the same parity, then*

$$\alpha^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \cos\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\alpha/N}$$

$$= \beta^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \cos\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\beta/N},$$

$$\alpha^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \sin\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\alpha/N}$$
$$= -\beta^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \sin\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\beta/N}.$$

If B and M have the different parity, then

$$\alpha^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \cos\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\alpha/N}$$
$$= -\beta^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \cos\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\beta/N},$$

$$\alpha^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \sin\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\alpha/N}$$
$$= \beta^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \sin\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\beta/N}.$$

Corollary 2.3 contains generalizations of Corollary 3.4 and 3.5 in [3].

COROLLARY 2.4. *Let χ be even and $\chi = \bar{\chi}$. Let B, M be integers with $B \geq 0$ and $M \geq 1$. Suppose that $M = 1$ and $\chi = \chi_{\rm o}$ cannot be considered simultaneously. If B and M have the same parity, then*

$$\sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\pi/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \sin\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\pi/N} = 0.$$

If B and M have the different parity, then

$$\sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\pi/N)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \cos\left(\frac{2\pi n}{N}\right) \sigma_{2M-1}(\chi, n) e^{-2n\pi/N} = 0.$$

Proof. Let $\alpha = \beta = \pi$ in Corollary 2.3. □

Let $B = 0$ and replace M by $2M$ in the first equation in Corollary 2.4. Then

$$\sum_{n=1}^{\infty} \chi(n) \sin\left(\frac{2\pi n}{N}\right) \sigma_{4M-1}(\chi, n) e^{-2n\pi/N} = 0.$$

Let $B = 0$ and replace M by $2M-1$ in the second equation in Corollary 2.4. Then

$$\sum_{n=1}^{\infty} \chi(n) \cos\left(\frac{2\pi n}{N}\right) \sigma_{4M-3}(\chi, n) e^{-2n\pi/N} = 0.$$

Let p be a prime with $p \equiv 1 \pmod 4$ and let $(\frac{\cdot}{p})$ be the Legendre symbol. Then $(\frac{\cdot}{p})$ is an even character with real values. Thus we can put $\chi = (\frac{\cdot}{p})$ in Theorem 2.1, Theorem 2.2, Corollary 2.3 and Corollary 2.4. For example, if $\chi = (\frac{\cdot}{p})$ in the first identity in Corollary 2.3, we obtain

$$\alpha^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/p)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) n^\ell \cos\left(\frac{2\pi n}{p}\right) \sigma_{2M-1}(\left(\frac{\cdot}{p}\right), n) e^{-2n\alpha/p}$$
$$= \beta^M \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/p)^\ell}{(2M+\ell-1)!} \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) n^\ell \cos\left(\frac{2\pi n}{p}\right) \sigma_{2M-1}(\left(\frac{\cdot}{p}\right), n) e^{-2n\beta/p}$$

which gives a generalization of Corollary 3.4 in [3].

For an odd character χ, applying the similar method, we obtain the following theorems and corollaries.

THEOREM 2.5. *Let χ be odd. For any integers $B \geq 0$, $M \geq 1$ and for $z \in \mathbb{H}$,*

$$z^{-B-2M-1}\bar{z}^B \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(4\pi \mathrm{Im}(z^{-1})/N)^\ell}{(2M+k)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \zeta_N^n \sigma_{2M}(\bar{\chi}, n) e\left(\frac{-nz^{-1}}{N}\right)$$
$$= \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\pi \mathrm{Im}(z)/N)^\ell}{(2M+k)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \zeta_N^{-n} \sigma_{2M}(\bar{\chi}, n) e\left(\frac{nz}{N}\right).$$

Proof. Let $s_1 = A \geq 1,\ s_2 = -B \leq 0$ and $s = 2M+1 \geq 3$ for $A, B, M \in \mathbb{Z}$. Since $s \geq 3$, we have, by using Remark 1.2,

$$\frac{1}{\Gamma(s_1)\Gamma(s_2)} \mathbf{L}_N(\tau, \bar{\tau}, s_1, s_2; \mathfrak{R}, \mathfrak{H}) = 0.$$

For the other parts of the proof, apply the similar method in the proof of Theorem 2.1. □

THEOREM 2.6. *Let χ be odd and let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. For any integers $B \geq 0$ and $M \geq 1$,*

$$(-1)^B \alpha^{M+1/2} \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/N)^\ell}{(2M+\ell)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \zeta_N^n \sigma_{2M}(\bar{\chi}, n) e^{-2n\alpha/N}$$
$$= (-\beta)^{M+1/2} \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/N)^\ell}{(2M+\ell)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \zeta_N^{-n} \sigma_{2M}(\bar{\chi}, n) e^{-2n\beta/N}.$$

COROLLARY 2.7. *Let χ be odd and $\chi = \bar{\chi}$. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Let $B,\ M$ be integers with $B \geq 0$, $M \geq 1$. If B and M have the same parity, then*

$$\alpha^{M+1/2} \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/N)^\ell}{(2M+\ell)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \cos\left(\frac{2\pi n}{N}\right) \sigma_{2M}(\chi, n) e^{-2n\alpha/N}$$
$$= \beta^{M+1/2} \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\beta/N)^\ell}{(2M+\ell)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \sin\left(\frac{2\pi n}{N}\right) \sigma_{2M}(\chi, n) e^{-2n\beta/N}.$$

If B and M have the different parity, then

$$\alpha^{M+1/2} \sum_{\ell=0}^{B} \binom{B}{\ell} \frac{(-4\alpha/N)^\ell}{(2M+\ell)!} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-\ell}} \cos\left(\frac{2\pi n}{N}\right) \sigma_{2M}(\chi, n) e^{-2n\alpha/N}$$

$$= -\beta^{M+1/2}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\beta/N)^{\ell}}{(2M+\ell)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\sin\left(\frac{2\pi n}{N}\right)\sigma_{2M}(\chi,n)e^{-2n\beta/N}.$$

Corollary 2.8. *Let χ be odd and $\chi=\bar{\chi}$. Let B, M be integers with $B\geq 0$, $M\geq 1$. If B and M have the same parity, then*

$$\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\pi/N)^{\ell}}{(2M+\ell)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\sin\left(\frac{2\pi n}{N}-\frac{\pi}{4}\right)\sigma_{2M}(\chi,n)e^{-2n\pi/N}=0.$$

If B and M have the different parity, then

$$\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(-4\pi/N)^{\ell}}{(2M+\ell)!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-\ell}}\sin\left(\frac{2\pi n}{N}+\frac{\pi}{4}\right)\sigma_{2M}(\chi,n)e^{-2n\pi/N}=0.$$

Proof. Let $\alpha=\beta=\pi$ in Corollary 2.7. □

Put $B=0$ in Corollary 2.8. Then

$$\sum_{n=1}^{\infty}\chi(n)\sin\left(\frac{2\pi n}{N}-\frac{\pi}{4}\right)\sigma_{4M}(\chi,n)e^{-2n\pi/N}=0$$

and

$$\sum_{n=1}^{\infty}\chi(n)\sin\left(\frac{2\pi n}{N}+\frac{\pi}{4}\right)\sigma_{4M-2}(\chi,n)e^{-2n\pi/N}=0.$$

Let p be a prime with $p\equiv 3 \pmod 4$. Then $\left(\frac{\cdot}{p}\right)$ is an odd character with real values. Thus we also put $\chi=\left(\frac{\cdot}{p}\right)$ in Theorem 2.5, Theorem 2.6, Corollary 2.7 and Corollary 2.8.

3. Another class of character analogue of infinite series identities

In this section, we obtain another type of character analogue of infinite series identities. We shall let $s_1\geq 1$, $s_2\geq 1$ and let $s\geq 3$ or $s\geq 4$ for $s=s_1+s_2$. Thus, by Remark 1.2, we see that

$$\frac{1}{\Gamma(s_1)\Gamma(s_2)}\mathbf{L}_N(\tau,\bar{\tau},s_1,s_2;\mathfrak{R},\mathfrak{H})=0.$$

Theorem 3.1. *Let χ be even. For $A,B,M\in\mathbb{Z}$, let $A\geq 0$, $B\geq 0$, $M\geq 1$ with $A+B=2M$. Then, for $z\in\mathbb{H}$,*

$$z^{-A-1}\bar{z}^{-B-1}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(-4\pi\mathrm{Im}(z^{-1})/N)^{2M-\ell+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{n}\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{-nz^{-1}}{N}\right)$$
$$+z^{-A-1}\bar{z}^{-B-1}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(-4\pi\mathrm{Im}(z^{-1})/N)^{2M-\ell+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{-n}\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{n\bar{z}^{-1}}{N}\right)$$
$$=\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(4\pi\mathrm{Im}(z)/N)^{2M-\ell+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{-n}\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{nz}{N}\right)$$

$$+\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(4\pi\mathrm{Im}(z)/N)^{2M-\ell+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^n\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{-n\bar{z}}{N}\right)$$
$$+\mu_\chi(A,B,M,z),$$

where

$$\mu_\chi(A,B,M,z)=\begin{cases}\frac{(2M)!}{(4\pi\mathrm{Im}(z/N))^{2M+1}}(1-z^B\bar{z}^A)\zeta(2M+1)\varphi(N), & \chi=\chi_{\mathrm{o}},\\ 0, & \chi\neq\chi_{\mathrm{o}}.\end{cases}$$

Proof. Let $s_1=A+1\geq 1$, $s_2=B+1\geq 1$ and $s=2M+2\geq 4$ with $A,B,M\in\mathbb{Z}$. Put $\mathbf{r}=(k,0)$ for any integer k with $1\leq k<N$ and put $\mathbf{h}=(0,0)$ in Theorem 1.1. Then $\mathfrak{R}=(k,-k)$ and $\mathfrak{H}=(0,0)$. We see that $\lambda_N(r_1)=\lambda_N(R_1)=\lambda_N(k)=0$ and $\lambda(h_2)=\lambda(H_2)=1$. Put $z=N\tau-N+1$. Since

$$\Psi_{-1}(0,k,2M+2)=2\sum_{n=1}^{\infty}\frac{1}{n^{2M+1}}=2\zeta(2M+1),$$

we have

$$z^{-A-1}\bar{z}^{-B-1}\mathbf{H}_N(V\tau,V\bar{\tau},A+1,B+1;\mathbf{r},\mathbf{h})$$
$$=\mathbf{H}_N(\tau,\bar{\tau},A+1,B+1;\mathfrak{R},\mathfrak{H})+\frac{2(2M)!}{A!B!}\frac{1-z^B\bar{z}^A}{(4\pi\mathrm{Im}(\tau))^{2M+1}}\zeta(2M+1).\tag{3.1}$$

Note that, for any $b\in\mathbb{C}$,

$$\sum_{k=1}^{N-1}\chi(k)b=\begin{cases}b\varphi(N), & \chi=\chi_{\mathrm{o}},\\ 0, & \chi\neq\chi_{\mathrm{o}}.\end{cases}$$

Thus, multiplying both sides in (3.1) by $\chi(k)$ and summing over k, we find that

$$z^{-A-1}\bar{z}^{-B-1}\sum_{k=1}^{N-1}\chi(k)\mathbf{H}_N(V\tau,V\bar{\tau},A+1,B+1;\mathbf{r},\mathbf{h})$$
$$=\sum_{k=1}^{N-1}\chi(k)\mathbf{H}_N(\tau,\bar{\tau},A+1,B+1;\mathfrak{R},\mathfrak{H})+2\mu_\chi(A,B,M,z),\tag{3.2}$$

where

$$\mu_\chi(A,B,M,z)=\begin{cases}\frac{(2M)!}{A!B!}\frac{1-z^B\bar{z}^A}{(4\pi\mathrm{Im}(z/N))^{2M+1}}\zeta(2M+1)\varphi(N), & \chi=\chi_{\mathrm{o}},\\ 0, & \chi\neq\chi_{\mathrm{o}}.\end{cases}$$

To compute $\sum_{k=1}^{N-1}\chi(k)\mathbf{H}_N(V\tau,V\bar{\tau},A+1,B+1;\mathbf{r},\mathbf{h})$, we shall apply the same method in the proof of Theorem 2.1. Then

$$\sum_{k=1}^{N-1}\chi(k)\mathcal{H}_N(V\tau,A+1,B+1;\mathbf{r},\mathbf{h})$$
$$=2\sum_{n=1}^{\infty}\chi(n)\zeta_N^n\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{-nz^{-1}}{N}\right)U(B+1;2M+2;4\pi n\mathrm{Im}(V\tau))$$

and

$$\sum_{k=1}^{N-1}\chi(k)\bar{\mathcal{H}}_N(V\tau,A+1,B+1;-\mathbf{r},-\mathbf{h})$$

$$= 2\sum_{n=1}^{\infty}\chi(n)\zeta_N^{-n}\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{n\bar{z}^{-1}}{N}\right)U(A+1;2M+2;4\pi n\mathrm{Im}(V\tau)).$$

Apply Lemma 2.1 in [5] to obtain

$$U(B+1;2M+2;4\pi n\mathrm{Im}(V\tau)) = \frac{1}{B!}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(4\pi n\mathrm{Im}(V\tau))^{2M-\ell+1}}$$

and

$$U(A+1;2M+2;4\pi n\mathrm{Im}(V\tau)) = \frac{1}{A!}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(4\pi n\mathrm{Im}(V\tau))^{2M-\ell+1}}.$$

Thus we have

$$\begin{aligned}
&\sum_{k=1}^{N-1}\chi(k)\mathbf{H}_N(V\tau,V\bar{\tau},A+1,B+1;\mathbf{r},\mathbf{h})\\
&=\frac{2}{A!B!}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(4\pi\mathrm{Im}(V\tau))^{2M-\ell+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{n}\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{-nz^{-1}}{N}\right)\\
&\quad+\frac{2}{A!B!}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(4\pi\mathrm{Im}(V\tau))^{2M-\ell+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{-n}\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{n\bar{z}^{-1}}{N}\right).
\end{aligned}$$

By the similar way, we also have

$$\begin{aligned}
&\sum_{k=1}^{N-1}\chi(k)\mathbf{H}_N(\tau,\bar{\tau},A+1,B+1;\mathfrak{R},\mathfrak{H})\\
&=\frac{2}{A!B!}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(4\pi\mathrm{Im}(\tau))^{2M-\ell+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{-n}\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{nz}{N}\right)\\
&\quad+\frac{2}{A!B!}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(4\pi\mathrm{Im}(\tau))^{2M-\ell+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{n}\sigma_{2M+1}(\bar{\chi},n)e\left(\frac{-n\bar{z}}{N}\right).
\end{aligned}$$

Use $\mathrm{Im}(V\tau) = \mathrm{Im}\left(\frac{-z^{-1}}{N}\right)$ and $\mathrm{Im}(\tau) = \mathrm{Im}(\frac{z}{N})$ to complete the proof. □

THEOREM 3.2. *Let χ be even and let $\alpha,\beta > 0$ with $\alpha\beta = \pi^2$. For $A, B, M \in \mathbb{Z}$, let $A \geq 0$, $B \geq 0$, $M \geq 1$ with $A+B = 2M$. Then*

$$\begin{aligned}
&(-1)^B\alpha^{-M}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{n}\sigma_{2M+1}(\bar{\chi},n)e^{-2\alpha n/N}\\
&+(-1)^B\alpha^{-M}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{-n}\sigma_{2M+1}(\bar{\chi},n)e^{-2\alpha n/N}\\
&=(-1)^M\beta^{-M}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{-n}\sigma_{2M+1}(\bar{\chi},n)e^{-2\beta n/N}\\
&\quad+(-1)^M\beta^{-M}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\zeta_N^{n}\sigma_{2M+1}(\bar{\chi},n)e^{-2\beta n/N}\\
&\quad+\nu_\chi(B,M,\alpha,\beta),
\end{aligned}$$

where

$$\nu_\chi(B,M,\alpha,\beta)=\begin{cases}(2M)!((-1)^M\beta^{-M}-(-1)^B\alpha^{-M})\zeta(2M+1)\varphi(N), & \chi=\chi_\circ,\\ 0, & \chi\neq\chi_\circ.\end{cases}$$

Proof. Put $z=\frac{\pi}{\alpha}i$ in Theorem 3.1. Apply

$$e\left(\frac{-nz^{-1}}{N}\right)=e\left(\frac{n\bar{z}^{-1}}{N}\right)=e^{-2\alpha n/N},\ e\left(\frac{nz}{N}\right)=e\left(\frac{-n\bar{z}}{N}\right)=e^{-2\beta n/N}$$

and

$$z^{-A-1}\bar{z}^{-B-1}=(-1)^{B+M}\alpha^{M+1}\beta^{-M-1},\ z^B\bar{z}^A=(-1)^{B+M}\alpha^{-M}\beta^M.$$

Multiplying both sides of the identity in Theorem 3.1 by $(-1)^M\left(\frac{4}{M}\right)^{2M+1}\beta^{M+1}$, we obtain the desired result. □

Corollary 3.3. *Let χ be even with $\chi\neq\chi_\circ$. Let $\alpha,\beta>0$ with $\alpha\beta=\pi^2$. For any integer $M\geq 1$,*

$$\alpha^{-M}\sum_{\ell=0}^{M}\binom{M}{\ell}\frac{(2M-\ell)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{2M+1}(\bar{\chi},n)e^{-2\alpha n/N}$$
$$=\beta^{-M}\sum_{\ell=0}^{M}\binom{M}{\ell}\frac{(2M-\ell)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{2M+1}(\bar{\chi},n)e^{-2\beta n/N}.$$

Proof. Let $A=B$ in Theorem 3.2 and use $\zeta_N^n+\zeta_N^{-n}=2\cos(\frac{2\pi n}{N})$. □

Corollary 3.3 shows a fairly good symmetric identity for α and β. If we put $M=1$ in Corollary 3.3, then

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n^3}\cos\left(\frac{2\pi n}{N}\right)\sigma_3(\bar{\chi},n)\left(\frac{1}{\alpha}e^{-2\alpha n/N}-\frac{1}{\beta}e^{-2\beta n/N}\right)$$
$$=-\frac{2}{N}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^2}\cos\left(\frac{2\pi n}{N}\right)\sigma_3(\bar{\chi},n)\left(e^{-2\alpha n/N}-e^{-2\beta n/N}\right).$$

Corollary 3.4. *Let χ be even. For $A,B,M\in\mathbb{Z}$, let $A\geq 0$, $B\geq 0$, $M\geq 1$ with $A+B=2M$. If B and M have the same parity, then*

$$\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(4\pi/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\sin\left(\frac{2\pi n}{N}\right)\sigma_{2M+1}(\bar{\chi},n)e^{-2\pi n/N}$$
$$=\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(4\pi/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\sin\left(\frac{2\pi n}{N}\right)\sigma_{2M+1}(\bar{\chi},n)e^{-2\pi n/N}.$$

Proof. Put $\alpha=\beta=\pi$ in Theorem 3.2. Then $\nu_\chi(B,M,\alpha,\beta)=0$ for any χ. Apply $\zeta_N^n-\zeta_N^{-n}=2i\sin\left(\frac{2\pi n}{N}\right)$, $(-1)^B=(-1)^M$. □

Corollary 3.4 also gives an elegant symmetric identity for A and B.

Corollary 3.5. *Let χ be even. For $A,B,M\in\mathbb{Z}$, let $A\geq 0$, $B\geq 0$, $M\geq 1$ with $A+B=2M$. If B and M have the different parity, then*

$$\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell)!}{(4\pi/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{2M+1}(\bar{\chi},n)e^{-2\pi n/N}$$

$$= -\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell)!}{(4\pi/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell+1}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{2M+1}(\bar{\chi},n)e^{-2\pi n/N}+\nu_\chi(M),$$

where

$$\nu_\chi(M)=\begin{cases}-2(2M)!\zeta(2M+1)\varphi(N), & \chi=\chi_{\mathrm{o}},\\ 0, & \chi\neq\chi_{\mathrm{o}}.\end{cases}$$

Proof. Put $\alpha=\beta=\pi$ in Theorem 3.2. Use $\zeta_N^n+\zeta_N^{-n}=2\cos\left(\frac{2\pi n}{N}\right)$, $(-1)^B=-(-1)^M$. □

COROLLARY 3.6. *Let χ be even. For any integer $M\geq 1$,*

$$\sum_{\ell=1}^{4M}\frac{(4\pi/N)^{\ell}}{\ell!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{4M-\ell+1}}\sin\left(\frac{2\pi n}{N}\right)\sigma_{4M+1}(\bar{\chi},n)e^{-2\pi n/N}=0.$$

Proof. Put $B=0$ in Corollary 3.4 and replace M by $2M$. □

COROLLARY 3.7. *Let χ be even. For any integer $M\geq 1$,*

$$\sum_{\ell=1}^{4M-2}\frac{(4\pi/N)^{\ell}}{\ell!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{4M-\ell-1}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{4M-1}(\bar{\chi},n)e^{-2\pi n/N}$$
$$=-2\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{4M-1}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{4M-1}(\bar{\chi},n)e^{-2\pi n/N}+\nu_\chi(2M-1).$$

Proof. Put $B=0$ in Corollary 3.5 and replace M by $2M-1$. □

For a prime p with $p\equiv 1 \pmod 4$, we can put $\chi=\left(\frac{\cdot}{p}\right)$ in Theorem 3.1, Theorem 3.2, Corollary 3.3 – Corollary 3.7.
Next we find character analogues of infinite series identities for an odd character χ. In this case, we shall let s be any odd integer greater than 2. The process to obtain the results is similar to the case of χ even.

THEOREM 3.8. *Let χ be odd. For $A,B,M\in\mathbb{Z}$, let $A\geq 0$, $B\geq 0$, $M\geq 1$ with $A+B=2M-1$. Then, for $z\in\mathbb{H}$,*

$$z^{-A-1}\bar{z}^{-B-1}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(-4\pi\mathrm{Im}(z^{-1})/N)^{2M-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^n\sigma_{2M}(\bar{\chi},n)e\left(\frac{-nz^{-1}}{N}\right)$$
$$+z^{-A-1}\bar{z}^{-B-1}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(-4\pi\mathrm{Im}(z^{-1})/N)^{2M-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{-n}\sigma_{2M}(\bar{\chi},n)e\left(\frac{n\bar{z}^{-1}}{N}\right)$$
$$=\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(4\pi\mathrm{Im}(z)/N)^{2M-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{-n}\sigma_{2M}(\bar{\chi},n)e\left(\frac{nz}{N}\right)$$
$$+\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(4\pi\mathrm{Im}(z)/N)^{2M-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{n}\sigma_{2M}(\bar{\chi},n)e\left(\frac{-n\bar{z}}{N}\right).$$

Proof. Let $s_1=A+1\geq 1$, $s_2=B+1\geq 1$ and $s=2M+1\geq 3$ with $A,B,M\in\mathbb{Z}$. Put $\mathbf{r}=(k,0)$ for any integer k with $1\leq k<N$ and put $\mathbf{h}=(0,0)$ in Theorem 1.1. The basic process of the proof is similar to the proof of Theorem 3.1. The only

noticeable difference arise from the terms with $\lambda(h_2)$ and $\lambda(H_2)$. Direct calculations show that they are vanished by using

$$\begin{aligned}\Psi_{-1}(0,R_1,s) &= \psi(0,k,2M)+e^{\pi i(2M+1)}\psi(0,-k,2M)\\ &= \sum_{n=1}^{\infty}\frac{e(nk)}{n^{2M}}-\sum_{n=1}^{\infty}\frac{e(-nk)}{n^{2M}}=0.\end{aligned}$$

□

Theorem 3.9. *Let χ be odd and let $\alpha,\beta>0$ with $\alpha\beta=\pi^2$. For $A,B,M\in\mathbb{Z}$, let $A\ge 0$, $B\ge 0$, $M\ge 1$ with $A+B=2M-1$. Then*

$$\begin{aligned}&(-1)^A\alpha^{-M+1/2}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{n}\sigma_{2M}(\bar\chi,n)e^{-2\alpha n/N}\\ &+(-1)^A\alpha^{-M+1/2}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{-n}\sigma_{2M}(\bar\chi,n)e^{-2\alpha n/N}\\ &=(-\beta)^{-M+1/2}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{-n}\sigma_{2M}(\bar\chi,n)e^{-2\beta n/N}\\ &\quad+(-\beta)^{-M+1/2}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{n}\sigma_{2M}(\bar\chi,n)e^{-2\beta n/N}.\end{aligned}$$

Proof. Put $z=\frac{\pi}{\alpha}i$ in Theorem 3.8.

$$z^{-A-1}\bar z^{-B-1}=(-1)^{B+1}\alpha^{M+1/2}(-\beta)^{-M-1/2}.$$

Multiplying both sides of the identity in Theorem 3.8 by $\left(\frac{4}{N}\right)^{2M}(-\beta)^{M+1/2}$, we obtain the desired result. □

If we put $s_1=B+1$, $s_2=A+1$ in the proof of Theorem 3.8, then the associated Theorem 3.9 is changed to the following identity;

$$\begin{aligned}&(-1)^B\alpha^{-M+1/2}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{n}\sigma_{2M}(\bar\chi,n)e^{-2\alpha n/N}\\ &+(-1)^B\alpha^{-M+1/2}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{-n}\sigma_{2M}(\bar\chi,n)e^{-2\alpha n/N}\\ &=(-\beta)^{-M+1/2}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{-n}\sigma_{2M}(\bar\chi,n)e^{-2\beta n/N}\\ &\quad+(-\beta)^{-M+1/2}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\zeta_N^{n}\sigma_{2M}(\bar\chi,n)e^{-2\beta n/N}.\end{aligned}\tag{3.3}$$

Note that A and B have the different parity. Adding the identity in Theorem 3.9 and (3.3), we obtain the following theorem.

Theorem 3.10. *Let χ be odd and let $\alpha,\beta>0$ with $\alpha\beta=\pi^2$. For $A,B,M\in\mathbb{Z}$, let $A\ge 0$, $B\ge 0$, $M\ge 1$ with $A+B=2M-1$. Then*

$$(-1)^A\alpha^{-M+1/2}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\sin\left(\frac{2\pi n}{N}\right)\sigma_{2M}(\bar\chi,n)e^{-2\alpha n/N}$$

$$-(-1)^A\alpha^{-M+1/2}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(4\alpha/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\sin\left(\frac{2\pi n}{N}\right)\sigma_{2M}(\bar\chi,n)e^{-2\alpha n/N}$$
$$=(-1)^M\beta^{-M+1/2}\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{2M}(\bar\chi,n)e^{-2\beta n/N}$$
$$+(-1)^M\beta^{-M+1/2}\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(4\beta/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{2M}(\bar\chi,n)e^{-2\beta n/N}.$$

COROLLARY 3.11. *Let χ be odd. For $A,B,M\in\mathbb{Z}$, let $A\geq 0$, $B\geq 0$, $M\geq 1$ with $A+B=2M-1$ and $A-B\equiv 1 \pmod 4$. Then*

$$\sum_{\ell=0}^{A}\binom{A}{\ell}\frac{(2M-\ell-1)!}{(4\pi/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\sin\left(\frac{2\pi n}{N}-\frac{\pi}{4}\right)\sigma_{2M}(\bar\chi,n)e^{-2\pi n/N}$$
$$=\sum_{\ell=0}^{B}\binom{B}{\ell}\frac{(2M-\ell-1)!}{(4\pi/N)^{-\ell}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\sin\left(\frac{2\pi n}{N}+\frac{\pi}{4}\right)\sigma_{2M}(\bar\chi,n)e^{-2\pi n/N}.$$

Proof. Let $\alpha=\beta=\pi$ in Theorem 3.10. Use the fact that $A-B\equiv 1 \pmod 4$ is equivalent to that A and M have the same parity. □

COROLLARY 3.12. *Let χ be odd. For any integer $M\geq 1$,*

$$\sum_{\ell=1}^{4M-3}\frac{(4\pi/N)^{\ell}}{\ell!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{4M-\ell-2}}\sin\left(\frac{2\pi n}{N}-\frac{\pi}{4}\right)\sigma_{4M-2}(\bar\chi,n)e^{-2\pi n/N}$$
$$=\sqrt{2}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{4M-2}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{4M-2}(\bar\chi,n)e^{-2\pi n/N}.$$

Proof. Let $\alpha=\beta=\pi$. Put $A=0$ and $B=2M-1$ in Theorem 3.10. Then

$$\sum_{\ell=0}^{2M-1}\frac{(4\pi/N)^{\ell}}{\ell!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M-\ell}}\left(\sin\left(\frac{2\pi n}{N}\right)+(-1)^M\cos\left(\frac{2\pi n}{N}\right)\right)\sigma_{2M}(\bar\chi,n)e^{-2\pi n/N}$$
$$=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2M}}\left(\sin\left(\frac{2\pi n}{N}\right)-(-1)^M\cos\left(\frac{2\pi n}{N}\right)\right)\sigma_{2M}(\bar\chi,n)e^{-2\pi n/N}.$$

Replace M by $2M-1$ and pull out the term with $\ell=0$ to complete the proof. □

Let $M=1$ in Corollary 3.12. Then we have

$$\frac{4\pi}{N}\sum_{n=1}^{\infty}\frac{\chi(n)}{n}\sin\left(\frac{2\pi n}{N}-\frac{\pi}{4}\right)\sigma_{2}(\bar\chi,n)e^{-2\pi n/N}$$
$$=\sqrt{2}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{2}(\bar\chi,n)e^{-2\pi n/N}.$$

COROLLARY 3.13. *Let χ be odd. For any integer $M\geq 1$,*

$$\sum_{\ell=1}^{4M-1}\frac{(4\pi/N)^{\ell}}{\ell!}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{4M-\ell}}\sin\left(\frac{2\pi n}{N}+\frac{\pi}{4}\right)\sigma_{4M}(\bar\chi,n)e^{-2\pi n/N}$$
$$=-\sqrt{2}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{4M}}\cos\left(\frac{2\pi n}{N}\right)\sigma_{4M}(\bar\chi,n)e^{-2\pi n/N}.$$

Proof. Let $\alpha = \beta = \pi$. Put $A = 0$ and $B = 2M - 1$ in Theorem 3.10. Replace M by $2M$ and pull out the term with $\ell = 0$. □

For a prime p with $p \equiv 3 \pmod 4$, we can put $\chi = \left(\frac{\cdot}{p}\right)$ in Theorem 3.8 – Theorem 3.10, Corollary 3.11 – Corollary 3.13.

References

[1] G. E. Andrews, R. Askey, R. Roy, *Special functions*, Cambridge University Press, 1999.

[2] B. C. Berndt, A. Dixit, J. Sohn *Character analogues of theorems of Ramanujan, Koshliakov and Guinand*, Adv. in Appl. Math. **46** (1) (2011), 54–70.

[3] S. Lim, *Character analogues of infinite series from a certain modular transformation formula*, J. Korean. Math. Soc. **48** (1) (2011), 169–178.

[4] S. Lim, *Analytic continuation of generalized non-holomorphic Eisenstein series*, Korean J. Math. **21** (3) (2013) 285–292.

[5] S. Lim, *Infinite series identities which stem from generalized non-holomorphic Eisenstein series*, J. Chungcheong Math. Soc. **32** (4) (2019), 421–440.

[6] L. J. Slater, *Confluent hypergeometric functions*, Cambridge University Press, 1960.

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