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SEMI-CONFORMAL L-HARMONIC MAPS AND LIOUVILLE TYPE THEOREM

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Abstract. In this paper, we prove that every semi-conformal harmonic map between Riemannian manifolds is L-harmonic map. We also prove a Liouville type theorem for L-harmonic maps.

1. Introduction

Consider a smooth map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ between Riemannian manifolds and a smooth positive function $L : M^m \times N^n \times \mathbb{R}_+ \longrightarrow (0, \infty)$, $(x, y, r) \longmapsto L(x, y, r)$. Then for any compact domain D of M^m, the L-energy functional of φ is defined by

$$E_L(\varphi; D) = \int_D L\big(x, \varphi(x), e(\varphi)(x)\big)\, v^g, \tag{1}$$

where $e(\varphi)$ is the energy density of φ defined by

$$e(\varphi) = \frac{1}{2}\sum_{i=1}^{m} h\big(d\varphi(e_i), d\varphi(e_i)\big) \tag{2}$$

and v^g is the volume element. Here $\{e_1, ..., e_m\}$ is an orthonormal frame on (M^m, g). A map is called L-harmonic if it is a critical point of the L-energy functional over any compact subset D of M^m. We denote by $\partial_r = \partial/\partial r$, $L' = \partial_r(L)$ and

$$L'_\varphi(x) = L'\big(x, \varphi(x), e(\varphi)(x)\big), \quad \forall x \in M^m. \tag{3}$$

Theorem 1.1 (The first variation of E_L, [9]). *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map and let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D. Then*

$$\frac{d}{dt}E_L(\varphi_t; D)\Big|_{t=0} = -\int_D h(\tau_L(\varphi), v)\, v^g, \tag{4}$$

where $v = \dfrac{\partial\varphi_t}{\partial t}\Big|_{t=0}$ denotes the variation vector field of φ,

$$\tau_L(\varphi) = L'_\varphi\, \tau(\varphi) + d\varphi\big(\operatorname{grad}^{M^m} L'_\varphi\big) - (\operatorname{grad}^{N^n} L) \circ \varphi, \tag{5}$$

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and $\tau(\varphi)$ is the tension field of φ given by

$$\tau(\varphi) = \operatorname{trace}_g \nabla d\varphi. \tag{6}$$

From the first variation formula (4), a map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is L-harmonic if and only if $\tau_L(\varphi) = 0$.
Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds. Let $x \in M^m$, the tangent space at x splits $T_xM^m = H_x \oplus V_x$, where $V_x = \operatorname{Ker} d_x\varphi$ and $H_x = V_x^{\perp}$ is the orthogonal complement of the vertical space V_x. The map φ is called semi-conformal if for each $x \in M^m$ with $d_x\varphi \neq 0$, the restriction $d_x\varphi : H_x \longrightarrow T_{\varphi(x)}N^n$ is conformal and surjective. Setting $\lambda(x) = 0$ at points x with $d_x\varphi = 0$, we obtain a continuous function $\lambda : M^m \longrightarrow \mathbb{R}_+$ such that for any $X, Y \in H_x$

$$h(d_x\varphi(X), d_x\varphi(Y)) = \lambda^2(x)g(X, Y).$$

The function λ is called the dilation of φ. Note that the generalized conformal maps are discussed in [10]
The purpose of this paper is to provide a proof of the Liouville type theorem for L-harmonic maps from complete noncompact Riemannian manifold (M^m, g) with positive Ricci curvature into a Riemannian manifold (N^n, h) with non-positive sectional curvature, where $L \in C^\infty(M^m \times N^n \times \mathbb{R}_+)$ is a smooth positive function which satisfies some suitable conditions.

2. Semi-conformal L-harmonic maps

Let (M^m, g) be a Riemannian manifold and let N^n the Euclidian space $\mathbb{R}^n$ equipped with the Riemannian metric $h = dy_1^2 + ... + dy_n^2$. We have the following results.

Theorem 2.1. *Any semi-conformal harmonic map $\varphi : M^m \longrightarrow \mathbb{R}^n$ is L-harmonic with $L(x, y, r) = F\big(2\,y + (n-2)\,\varphi(x)\big)r$, for all $(x, y, r) \in M^m \times \mathbb{R}^n \times \mathbb{R}_+$, where $F \in C^\infty(\mathbb{R}^n)$ is a smooth positive function.*

Proof. A semi-conformal harmonic map φ is L-harmonic if and only if

$$d\varphi\big(\operatorname{grad}^{M^m} L'_\varphi\big) - (\operatorname{grad}^{\mathbb{R}^n} L) \circ \varphi = 0,$$

where $L'_\varphi : M^m \longrightarrow (0, +\infty)$ is a smooth positive function given by

$$L'_\varphi(x) = F\big(n\,\varphi(x)\big).$$

Let us choose $\{e_1, ..., e_m\}$ to be an orthonormal frame on a domain of M^m such that the vectors $\{e_1, ..., e_n\}$ are horizontal and the vectors $\{e_{n+1}, ..., e_m\}$ are vertical, so that $d\varphi(e_i) = \lambda\,(\widetilde{e}_i \circ \varphi)$ for $i = 1, ..., n$, where $\{\widetilde{e}_1, ..., \widetilde{e}_n\}$ is an orthonormal frame on a domain of $\mathbb{R}^n$. Then, we get

$$\begin{aligned} d\varphi(\operatorname{grad}^{M^m} L'_\varphi) &= \sum_{i=1}^{m} e_i(L'_\varphi)\, d\varphi(e_i) \\ &= n \sum_{i=1}^{n} d\varphi(e_i)(F)\, d\varphi(e_i) \\ &= n\,\lambda^2 \sum_{i=1}^{n} (\widetilde{e}_i \circ \varphi)(F)\,(\widetilde{e}_i \circ \varphi) \\ &= n\,\lambda^2 (\operatorname{grad}^{\mathbb{R}^n} F) \circ \varphi, \end{aligned} \tag{7}$$

and the term $(\operatorname{grad}^{\mathbb{R}^n} L) \circ \varphi$ is given by

$$\begin{aligned} (\operatorname{grad}^{\mathbb{R}^n} L) \circ \varphi &= \sum_{i=1}^{n} \left[\frac{\partial L}{\partial y_i} \frac{\partial}{\partial y_i}\right] \circ \varphi \\ &= 2e(\varphi) \sum_{i=1}^{n} \left[\frac{\partial F}{\partial y_i} \frac{\partial}{\partial y_i}\right] \circ \varphi \\ &= 2e(\varphi)\,(\operatorname{grad}^{\mathbb{R}^n} F) \circ \varphi, \end{aligned} \tag{8}$$

since $e(\varphi) = \frac{n}{2}\lambda^2$, we get $(\operatorname{grad}^{\mathbb{R}^n} L) \circ \varphi = n\,\lambda^2 (\operatorname{grad}^{\mathbb{R}^n} F) \circ \varphi$. □

Using Theorem 2.1, we can construct many examples for semi conformal L-harmonic maps.

Example 2.2. The Hopf construction map $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$, defined by

$$\varphi(x) = (x_1^2 + x_2^2 - x_3^2 - x_4^2, 2\,x_1x_3 + 2\,x_2x_4, 2\,x_2x_3 - 2\,x_1x_4),$$

is a semi conformal harmonic map with dilation

$$\lambda(x) = 2|x|,\ \forall x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4,$$

see [3]. According to Theorem 2.1 the map φ is L-harmonic, where L is the form

$$F\left(x_1^2 + x_2^2 - x_3^2 - x_4^2 + 2\,y_1, 2\,x_1x_3 + 2\,x_2x_4 + 2\,y_2, 2\,x_2x_3 - 2\,x_1x_4 + 2\,y_3\right) r,$$

for all $(x, y, r) \in \mathbb{R}^4 \times \mathbb{R}^3 \times \mathbb{R}_+$, where $F \in C^\infty(\mathbb{R}^3)$ is a smooth positive function.

If $n = 1$, we have the following corollary.

Corollary 2.3. *A smooth function $\varphi \in C^\infty(M^m)$ is harmonic if and only if it is L-harmonic for $L(x, y, r) = F\big(2\,y - \varphi(x)\big)r$, for all $(x, y, r) \in M^m \times \mathbb{R} \times \mathbb{R}_+$ where $F \in C^\infty(\mathbb{R})$ is a smooth positive function.*

Proof. First note that, the function φ is L-harmonic if and only if

$$\tau_L(\varphi) = L'_\varphi \tau(\varphi) + d\varphi(\operatorname{grad}^{M^m} L'_\varphi) - (\operatorname{grad}^{\mathbb{R}} L) \circ \varphi = 0, \tag{9}$$

where $L'_\varphi(x) = F(\varphi(x))$ for all $x \in M^m$. We compute

$$\begin{aligned} d\varphi(\operatorname{grad}^M L'_\varphi) &= \sum_{i=1}^m e_i(L'_\varphi) d\varphi(e_i) \\ &= \sum_{i=1}^m e_i(F \circ \varphi) e_i(\varphi) \\ &= \sum_{i=1}^m e_i(\varphi)(F' \circ \varphi) e_i(\varphi) \\ &= (F' \circ \varphi)|\operatorname{grad}^{M^m} \varphi|^2. \end{aligned} \tag{10}$$

Here $\{e_1, ..., e_m\}$ is an orthonormal frame in M^m and $e_i(\varphi) = d\varphi(e_i)$. The term $-(\operatorname{grad}^{\mathbb{R}} L) \circ \varphi$ of (9) is given by

$$\begin{aligned} -(\operatorname{grad}^{\mathbb{R}} L) \circ \varphi &= -\frac{1}{2} \sum_{i=1}^m e_i(\varphi)^2 [2(F' \circ \varphi)] \\ &= -|\operatorname{grad}^{M^m} \varphi|^2 (F' \circ \varphi). \end{aligned} \tag{11}$$

Then the proof follows from (9), (10) and (11). □

Example 2.4. The harmonic function

$$\varphi : \mathbb{R}^2 \backslash \{0\} \longrightarrow \mathbb{R}, \quad (x_1, x_2) \longmapsto \frac{x_1}{x_1^2 + x_2^2},$$

is L-harmonic with

$$L(x_1, x_2, y, r) = F\left(2\,y - \frac{x_1}{x_1^2 + x_2^2}\right) r, \quad \forall (x_1, x_2, y, r) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+,$$

where $F \in C^\infty(\mathbb{R})$ is a smooth positive function.

For $n = 2$, we have the following result.

Theorem 2.5. *A semi-conformal map $\varphi : M^m \longrightarrow N^2$ from a Riemannian manifold to a Riemannian 2-manifold is L-harmonic with*

$$L(x, y, r) = r e^{\alpha(x)\beta(y)}, \quad \forall (x, y, r) \in M^m \times N^2 \times \mathbb{R}_+,$$

where $\alpha \in C^\infty(M^m)$ and $\beta \in C^\infty(N^2)$ if and only if

$$\tau(\varphi) + (\beta \circ \varphi)\, d\varphi(\operatorname{grad}^M \alpha) = 0.$$

Proof. First, note that the function L'_φ is given by

$$L'_\varphi(x) = e^{\alpha(x)\,\beta(\varphi(x))}, \quad \forall x \in M^m.$$

Let us choose $\{e_1, ..., e_m\}$ to be an orthonormal frame on a domain of M^m such that the vectors $\{e_1, e_2\}$ are horizontal and the vectors $\{e_3, ..., e_m\}$ are vertical, so that $d\varphi(e_i) = \lambda\,(\widetilde{e}_i \circ \varphi)$ for $i = 1, 2$, where $\{\widetilde{e}_1, \widetilde{e}_2\}$ is an orthonormal frame on a domain of N^2. Then we get

$$\begin{aligned} d\varphi(\operatorname{grad}^{M^m} L'_\varphi) &= \sum_{i=1}^m e_i(L'_\varphi)\, d\varphi(e_i) \\ &= \sum_{i=1}^m e^{\alpha(\beta\circ\varphi)}\, e_i\big(\alpha(\beta\circ\varphi)\big)\, d\varphi(e_i) \\ &= e^{\alpha(\beta\circ\varphi)}\big\{(\beta\circ\varphi)\, d\varphi(\operatorname{grad}^{M^m}\alpha) + \alpha\, d\varphi(\operatorname{grad}^{M^m}(\beta\circ\varphi))\big\}. \end{aligned}$$

The term $d\varphi(\operatorname{grad}^{M^m}(\beta\circ\varphi))$ is given by

$$\begin{aligned} d\varphi(\operatorname{grad}^{M^m}(\beta\circ\varphi)) &= \sum_{i=1}^m e_i(\beta\circ\varphi)\, d\varphi(e_i) \\ &= \sum_{i=1}^2 d\varphi(e_i)(\beta)\, d\varphi(e_i) \\ &= \sum_{i=1}^2 \lambda^2\, (\widetilde{e}_i\circ\varphi)(\beta)\, (\widetilde{e}_i\circ\varphi) \\ &= \lambda^2\, (\operatorname{grad}^{N^2}\beta)\circ\varphi. \end{aligned}$$

Hence we conclude that

$$d\varphi(\operatorname{grad}^{M^m} L'_\varphi) = e^{\alpha(\beta\circ\varphi)}\big\{(\beta\circ\varphi)\, d\varphi(\operatorname{grad}^{M^m}\alpha) + \alpha\,\lambda^2\,(\operatorname{grad}^{N^2}\beta)\circ\varphi\big\}.$$

Since $e(\varphi)=\lambda^2$, we get

$$\begin{aligned} (\operatorname{grad}^{N^2} L)\circ\varphi &= \sum_{i=1}^2 (\widetilde{e}_i\circ\varphi)(L)(\widetilde{e}_i\circ\varphi) \\ &= \lambda^2 \sum_{i=1}^2 (\widetilde{e}_i\circ\varphi)(\alpha\,\beta)\, e^{\alpha\,(\beta\circ\varphi)}\,(\widetilde{e}_i\circ\varphi) \\ &= \alpha\lambda^2\, e^{\alpha\,(\beta\circ\varphi)}\,(\operatorname{grad}^{N^2}\beta)\circ\varphi. \end{aligned}$$

Hence the L-tension field of φ is given by

$$\tau_L(\varphi) \;=\; e^{\alpha\,(\beta\circ\varphi)}\big[\tau(\varphi) + (\beta\circ\varphi)\, d\varphi(\operatorname{grad}^{M^m}\alpha)\big].$$

This completes the proof of Theorem 2.5. □

Example 2.6 (The foliation by the circles of Villarceau, [2]). Let M^3 be the manifold $\mathbb{R}\times\mathbb{R}^2\backslash\{0\}$ and let $\varphi : M^3 \longrightarrow \mathbb{R}^2$ be defined by

$$\varphi(x_1,x_2,x_3) = \left(\frac{\left(1-\frac{|x|^2}{2}\right)x_2 + \sqrt{2}x_1x_3}{x_2^2+x_3^2}, \frac{\left(1-\frac{|x|^2}{2}\right)x_3 - \sqrt{2}x_1x_2}{x_2^2+x_3^2}\right).$$

Then φ is semi-conformal and its dilation is given by the function

$$\lambda(x) = \frac{\left(1-\frac{|x|^2}{2}\right)^2}{(x_2^2+x_3^2)^2}, \quad \forall x=(x_1,x_2,x_3)\in M^3.$$

The tension field of φ is

$$\tau(\varphi)(x) = \left(-\frac{x_2}{x_2^2 + x_3^2}, -\frac{x_3}{x_2^2 + x_3^2} \right).$$

According to Theorem 2.5, with $\alpha(x) = c_1 \ln(2 + |x|^2) + c_2$ and $\beta(y) = -\frac{1}{c_1}$, where $c_1 \in \mathbb{R}^*$, $c_2 \in \mathbb{R}$, the map φ is L-harmonic with

$$L(x, y, r) = \frac{e^{-\frac{c_2}{c_1}r}}{2 + |x|^2}, \quad \forall (x, y, r) \in M^3 \times \mathbb{R}^2 \times \mathbb{R}_+.$$

3. A Liouville type theorem for L-harmonic maps

Theorem 3.1. *Let (M^m, g) be a complete noncompact Riemannian manifold with positive Ricci curvature $\operatorname{Ricci}^{M^m} \geq 0$, (N^n, h) a Riemannian manifold with non-positive sectional curvature $\operatorname{Sect}^{N^n} \leq 0$. Consider an L-harmonic map φ from (M^m, g) to (N^n, h), where $L \in C^\infty(M^m \times N^n \times \mathbb{R}_+)$ is a smooth positive function. Suppose that*

$$L'_\varphi > 0, \quad \operatorname{Hess}^{M^m} L'_\varphi \leq 0, \quad \operatorname{Hess}^{N^n} L \geq 0,$$

$$d\varphi(\operatorname{grad}^{M^m} L'_\varphi)(L) \leq 0, \quad \int_{M^m} L'_\varphi \, v^g = \infty, \quad \int_{M^m} L'_\varphi |d\varphi|^2 \, v^g < \infty.$$

Then φ is constant.

We need the following Lemmas to prove Theorem 3.1.

Lemma 3.2 ([6, 11]). *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth mapping between Riemannian manifolds and let $f \in C^\infty(M^m)$. Then*

$$\big\langle d\varphi, \nabla^\varphi d\varphi(\operatorname{grad}^{M^m} f)\big\rangle = \frac{1}{2}\big(\operatorname{grad}^{M^m} f\big)\big(|d\varphi|^2\big) + \big\langle d\varphi, d\varphi(\nabla^{M^m} \operatorname{grad}^{M^m} f)\big\rangle.$$

Here $\langle , \rangle$ denotes the inner product on $T^*M^m \otimes \varphi^{-1}TN^n$.

Lemma 3.3. *Let (M^m, g) and (N^n, h) be two Riemannian manifolds, and $L \in C^\infty(M^m \times N^n \times \mathbb{R}_+)$ a smooth positive function. Then an L-harmonic map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ satisfies*

$$\begin{aligned}
\frac{1}{2}\Delta^{M^m}|d\varphi|^2 \;=\;& |\nabla d\varphi|^2 + \frac{1}{{L'_\varphi}^2}|d\varphi(\operatorname{grad}^{M^m} L'_\varphi)|^2 - \frac{1}{2L'_\varphi}\big(\operatorname{grad}^{M^m} L'_\varphi\big)\big(|d\varphi|^2\big) \\
& -\frac{1}{L'_\varphi}\big\langle d\varphi, d\varphi(\nabla^{M^m} \operatorname{grad}^{M^m} L'_\varphi)\big\rangle - \frac{1}{{L'_\varphi}^2} d\varphi(\operatorname{grad}^{M^m} L'_\varphi)(L) \\
& +\frac{1}{L'_\varphi}\big\langle d\varphi, \nabla^\varphi(\operatorname{grad}^{N^n} L) \circ \varphi\big\rangle + \sum_{i=1}^m h\big(d\varphi\big(\operatorname{Ricci}^{M^m} e_i\big), d\varphi(e_i)\big) \\
& -\sum_{i,j=1}^m h\big(R^{N^n}\big(d\varphi(e_i), d\varphi(e_j)\big)d\varphi(e_j), d\varphi(e_i)\big),
\end{aligned}$$

where $\{e_1, ..., e_m\}$ be an orthonormal frame on (M^m, g).

Proof. We start recalling the standard Bochner formula for the smooth map φ. Let $\{e_1, ..., e_m\}$ be an orthonormal frame on (M^m, g). Then

$$\begin{aligned}\frac{1}{2}\Delta^{M^m}|d\varphi|^2 = |\nabla d\varphi|^2 + \big\langle d\varphi, \nabla^\varphi\tau(\varphi)\big\rangle + \sum_{i=1}^m h\big(d\varphi\big(\operatorname{Ricci}^{M^m} e_i\big), d\varphi(e_i)\big)\\ -\sum_{i,j=1}^m h\big(R^{N^n}\big(d\varphi(e_i), d\varphi(e_j)\big)d\varphi(e_j), d\varphi(e_i)\big),\end{aligned} \tag{12}$$

where $|\nabla d\varphi|$ and $\big\langle d\varphi, \nabla^\varphi\tau(\varphi)\big\rangle$ are defined by

$$|\nabla d\varphi|^2 = \sum_{i,j=1}^m h\big(\nabla d\varphi(e_i, e_j), \nabla d\varphi(e_i, e_j)\big)$$

and

$$\big\langle d\varphi, \nabla^\varphi\tau(\varphi)\big\rangle = \sum_{i=1}^m h\big(d\varphi(e_i), \nabla^\varphi_{e_i}\tau(\varphi)\big),$$

respectively. Since φ is L-harmonic, i.e.,

$$\tau_L(\varphi) = L'_\varphi\,\tau(\varphi) + d\varphi(\operatorname{grad}^{M^m} L'_\varphi) - (\operatorname{grad}^{N^n} L)\circ\varphi = 0$$

and $L'_\varphi > 0$ on M^m, we obtain

$$\tau(\varphi) = -\frac{1}{L'_\varphi}d\varphi(\operatorname{grad}^{M^m} L'_\varphi) + \frac{1}{L'_\varphi}(\operatorname{grad}^{N^n} L)\circ\varphi. \tag{13}$$

Hence we get the following

$$\begin{aligned}\big\langle d\varphi, \nabla^\varphi\tau(\varphi)\big\rangle &= \frac{1}{{L'_\varphi}^2}|d\varphi(\operatorname{grad}^{M^m} L'_\varphi)|^2 - \frac{1}{L'_\varphi}\big\langle d\varphi, \nabla^\varphi d\varphi(\operatorname{grad}^{M^m} L'_\varphi)\big\rangle\\ &\quad -\frac{1}{{L'_\varphi}^2}h(d\varphi(\operatorname{grad}^{M^m} L'_\varphi), (\operatorname{grad}^{N^n} L)\circ\varphi)\\ &\quad +\frac{1}{L'_\varphi}\big\langle d\varphi, \nabla^\varphi(\operatorname{grad}^{N^n} L)\circ\varphi\big\rangle,\end{aligned} \tag{14}$$

by the Lemma 3.2. Since the second term on the left-hand side of (14) is

$$\begin{aligned}-\frac{1}{L'_\varphi}\big\langle d\varphi, \nabla^\varphi d\varphi(\operatorname{grad}^{M^m} L'_\varphi)\big\rangle &= -\frac{1}{2L'_\varphi}\big(\operatorname{grad}^{M^m} L'_\varphi\big)\big(|d\varphi|^2\big)\\ &\quad -\frac{1}{L'_\varphi}\big\langle d\varphi, d\varphi(\nabla^{M^m}\operatorname{grad}^{M^m} L'_\varphi)\big\rangle,\end{aligned} \tag{15}$$

Lemma 3.3 follows by (12), (14) and (15). □

Proof. of theorem 3.1. By Lemma 3.3, we get

$$\begin{aligned}\frac{1}{2}L'_\varphi\Delta^{M^m}|d\varphi|^2 &= L'_\varphi|\nabla d\varphi|^2+\frac{1}{L'_\varphi}|d\varphi(\operatorname{grad}^{M^m}L'_\varphi)|^2-\frac{1}{2}\big(\operatorname{grad}^{M^m}L'_\varphi\big)\big(|d\varphi|^2\big)\\ &\quad -\big\langle d\varphi, d\varphi(\nabla^{M^m}\operatorname{grad}^{M^m}L'_\varphi)\big\rangle-\frac{1}{L'_\varphi}d\varphi(\operatorname{grad}^{M^m}L'_\varphi)(L)\\ &\quad +\big\langle d\varphi,\nabla^\varphi(\operatorname{grad}^{N^n}L)\circ\varphi\big\rangle+L'_\varphi\sum_{i=1}^m h\big(d\varphi\big(\operatorname{Ricci}^{M^m}e_i\big),d\varphi(e_i)\big)\\ &\quad -L'_\varphi\sum_{i,j=1}^m h\big(R^{N^n}\big(d\varphi(e_i),d\varphi(e_j)\big)d\varphi(e_j),d\varphi(e_i)\big).\end{aligned}$$

If we denote $\Delta_L^{M^m}\rho\equiv L'_\varphi\Delta^{M^m}\rho+\big(\operatorname{grad}^{M^m}L'_\varphi\big)\big(\rho\big)$ for all $\rho\in C^\infty(M^m)$, then

$$\begin{aligned}\frac{1}{2}\Delta_L^{M^m}|d\varphi|^2 &= L'_\varphi|\nabla d\varphi|^2+\frac{1}{L'_\varphi}|d\varphi(\operatorname{grad}^{M^m}L'_\varphi)|^2-\big\langle d\varphi, d\varphi(\nabla^{M^m}\operatorname{grad}^{M^m}L'_\varphi)\big\rangle\\ &\quad -\frac{1}{L'_\varphi}d\varphi(\operatorname{grad}^{M^m}L'_\varphi)(L)+\big\langle d\varphi,\nabla^\varphi(\operatorname{grad}^{N^n}L)\circ\varphi\big\rangle\\ &\quad +L'_\varphi\sum_{i=1}^m h\big(d\varphi\big(\operatorname{Ricci}^{M^m}e_i\big),d\varphi(e_i)\big)\\ &\quad -L'_\varphi\sum_{i,j=1}^m h\big(R^{N^n}\big(d\varphi(e_i),d\varphi(e_j)\big)d\varphi(e_j),d\varphi(e_i)\big).\end{aligned}$$

Since $\operatorname{Sect}^{N^n}\leq 0$, $\operatorname{Ricci}^{M^m}\geq 0$, $\operatorname{Hess}^{N^n}L\geq 0$, $\operatorname{Hess}^{M^m}L'_\varphi\leq 0$ and $d\varphi(\operatorname{grad}^{M^m}L'_\varphi)(L)\leq 0$ by (16), we obtain the following inequality

$$\frac{1}{2}\Delta_L^{M^m}|d\varphi|^2\geq L'_\varphi\,|\nabla d\varphi|^2. \tag{16}$$

Since $\frac{1}{2}\Delta_L^{M^m}|d\varphi|^2 = |d\varphi|\Delta_L^{M^m}|d\varphi|+L'_\varphi\,|\operatorname{grad}^{M^m}|d\varphi||^2$, by (16) and the Kato's inequality [4], we get the following

$$|d\varphi|\Delta_L^{M^m}|d\varphi|\geq L'_\varphi\big(|\nabla d\varphi|^2-|\operatorname{grad}^{M^m}|d\varphi||^2\big)\geq 0. \tag{17}$$

Let $\rho: M^m\longrightarrow\mathbb{R}$ be a smooth function with compact support. Then

$$\begin{aligned}\rho^2|d\varphi|\Delta_L^{M^m}|d\varphi| &= \rho^2|d\varphi|\operatorname{div}^{M^m}\big(L'_\varphi\operatorname{grad}^{M^m}|d\varphi|\big)\\ &= \operatorname{div}^{M^m}\big(\rho^2|d\varphi|L'_\varphi\operatorname{grad}^{M^m}|d\varphi|\big)-L'_\varphi\rho^2|\operatorname{grad}^{M^m}|d\varphi||^2\\ &\quad -2L'_\varphi\rho|d\varphi|g(\operatorname{grad}^{M^m}\rho,\operatorname{grad}^{M^m}|d\varphi|).\end{aligned} \tag{18}$$

From (17), (18) and the Stokes theorem, we deduce

$$\begin{aligned}0 &\leq -\int_{M^m}L'_\varphi\rho^2|\operatorname{grad}^{M^m}|d\varphi||^2v^g\\ &\quad -2\int_{M^m}L'_\varphi\rho|d\varphi|g(\operatorname{grad}^{M^m}\rho,\operatorname{grad}^{M^m}|d\varphi|)v^g.\end{aligned} \tag{19}$$

Using the Young inequality [14], we have

$$
\begin{aligned}
-2g(|d\varphi| \operatorname{grad}^{M^m} \rho, \rho \operatorname{grad}^{M^m} |d\varphi|) \leq & \frac{1}{\epsilon}|d\varphi|^2 |\operatorname{grad}^{M^m} \rho|^2 \\
& + \epsilon\rho^2 |\operatorname{grad}^{M^m} |d\varphi||^2
\end{aligned} \tag{20}
$$

for any $\epsilon > 0$. Substituting (20) in (19), we obtain

$$
\begin{aligned}
0 \leq & -\int_{M^m} L'_\varphi \rho^2 |\operatorname{grad}^{M^m} |d\varphi||^2 v^g + \frac{1}{\epsilon}\int_{M^m} L'_\varphi |d\varphi|^2 |\operatorname{grad}^{M^m} \rho|^2 v^g \\
& + \epsilon \int_{M^m} L'_\varphi \rho^2 |\operatorname{grad}^{M^m} |d\varphi||^2 v^g,
\end{aligned}
$$

that is,

$$
(1-\epsilon)\int_{M^m} L'_\varphi \rho^2 |\operatorname{grad}^{M^m} |d\varphi||^2 v^g \leq \frac{1}{\epsilon}\int_{M^m} L'_\varphi |d\varphi|^2 |\operatorname{grad}^{M^m} \rho|^2 v^g. \tag{21}
$$

Choose the smooth cut-off $\rho = \rho_R$ i.e., $\rho \leq 1$ on M^m, $\rho = 1$ on the ball $B(0,R)$, $\rho = 0$ on $M^m \backslash B(0,2R)$ and $|\operatorname{grad}^{M^m} \rho| \leq \frac{2}{R}$. Let $0 < \epsilon < 1$. Replacing $\rho = \rho_R$ in (21), we obtain

$$
0 \leq (1-\epsilon)\int_{M^m} L'_\varphi \rho^2 |\operatorname{grad}^{M^m} |d\varphi||^2 v^g \leq \frac{4}{\epsilon R^2}\int_{M^m} L'_\varphi |d\varphi|^2 v^g. \tag{22}
$$

Since $\int_{M^m} L'_\varphi \, |d\varphi|^2 \, v^g < \infty$, when $R \longrightarrow \infty$, we have

$$
\frac{4}{\epsilon R^2}\int_{M^m} L'_\varphi |d\varphi|^2 v^g \longrightarrow 0.
$$

Thus, by (22), we have $|\operatorname{grad}^{M^m} |d\varphi|| = 0$ i.e., $|d\varphi| = c$ constant. By the assumptions,

$$
\frac{c^2}{2}\int_{M^m} L'_\varphi v^g < \infty \quad \text{and} \quad \int_{M^m} L'_\varphi v^g = \infty.
$$

Hence $c = 0$, that is, φ is constant map. □

If $L(x,y,r) = r$ for all $(x,y,r) \in M^m \times N^n \times \mathbb{R}_+$, then we recover the following classical result.

Corollary 3.4. [7] *Let (M^m, g) be a complete noncompact Riemannian manifold of infinite volume with positive Ricci curvature and (N^n, h) a Riemannian manifold with non-positive sectional curvature. Then a harmonic map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ with finite energy, i.e.,*

$$
E(\varphi) = \frac{1}{2}\int_{M^m} |d\varphi|^2 \, v^g < \infty,
$$

is constant.

Let $f : M^m \longrightarrow (0,\infty)$ be a smooth function. If $L(x,y,r) = f(x)r$ for all $(x,y,r) \in M^m \times N^n \times \mathbb{R}_+$. We recover the following result.

Corollary 3.5. [11,13] *Let (M^m, g) be a complete noncompact Riemannian manifold with positive Ricci curvature and (N^n, h) a Riemannian manifold with non-positive sectional curvature. Let f be a smooth positive function on M^m with non-positive Hessian $\operatorname{Hess}^{M^m} f \leq 0$. If $\operatorname{Vol}_f(M^m)$ is infinite, then an f-harmonic map*

$\varphi : (M^m, g) \longrightarrow (N^n, h)$ with finite f-energy, i.e.,

$$E_f(\varphi) = \frac{1}{2}\int_{M^m} f\,|d\varphi|^2\, v^g < \infty,$$

is constant.

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