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A TYPE OF WEAKLY SYMMETRIC STRUCTURE ON A RIEMANNIAN MANIFOLD

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Abstract. A new type of Riemannian manifold called semirecurrent manifold has been defined and some of its geometric properties are studied. Among others we show that the scalar curvature of semirecurrent manifold is constant and hence semirecurrent manifold is also concircularly recurrent. In addition, we show that the associated 1-form (resp. the associated vector field) of semirecurrent manifold is closed (resp. an eigenvector of its Ricci tensor). Furthermore, we prove that if a Riemannian product manifold is semirecurrent, then either one decomposition manifold is locally symmetric or the other decomposition manifold is a space of constant curvature.

1. Introduction

As a natural generalization of the notion of a space of constant curvature, the notion of a symmetric manifold was introduced by Cartan [5] who obtained a classification of such a manifold. The study on generalization of a symmetric manifold started in 1946 and continued to date in different directions. For instance the notions of recurrent manifold, conformally recurrent manifold, concircularly recurrent manifold and conharmonically recurrent manifold were introduced by Ruse [10] and Walker [11]; Adati and Miyazawa [1]; Maralabhavi and Rathnamma [8]; De, singh and Pandey [7], respectively. A Riemannian manifold (M^n, g) is said to be recurrent if its curvature tensor R satisfies the following relation

$$(\nabla_X R)(Y,Z,U,V) = A(X)R(Y,Z,U,V) \tag{1}$$

for any nonzero 1-form A, where ∇ denotes the Levi-Civita connection and $X, Y, Z, U, V \in TM^n$.

Conformal curvature tensor C, concircular curvature tensor C_1 and conharmonic curvature tensor C_2 are defined as follows:

$$C(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{n-2}[Ric(Y,Z)g(X,U) - Ric(X,Z)g(Y,U) + g(Y,Z)$$

(2)

$$Ric(X,U) - g(X,Z)Ric(Y,U)] + \frac{s}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)],$$

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$$C_1(X,Y,Z,U) = R(X,Y,Z,U) - \frac{s}{n(n-1)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] \tag{3}$$

and

$$\begin{aligned} C_2(X,Y,Z,U) &= R(X,Y,Z,U) - \frac{1}{n-2}[Ric(Y,Z)g(X,U) \\ &- Ric(X,Z)g(Y,U) + g(Y,Z)Ric(X,U) - g(X,Z)Ric(Y,U)], \end{aligned} \tag{4}$$

where Ric and s are the Ricci tensor and the scalar curvature tensor, respectively. A Riemannian manifold is said to be conformally recurrent, concircularly recurrent and conharmonically recurrent if

$$(\nabla_X H)(Y,Z,U,V) = A(X)H(Y,Z,U,V), \tag{5}$$

where A is a nonzero 1-form and H stands for C, C_1 and C_2 respectively.
It is easy to see that if (M^n, g) is recurrent, then the manifold is conformally recurrent, concircularly recurrent and conharmonically recurrent. In this paper, a type of Riemannian manifold (namely, semirecurrent manifold) is introduced. More precisely, a Riemannian manifold (M^n, g) is said to be semirecurrent if its curvature tensor R and concircular curvature C_1 fulfill the following condition:

$$(\nabla_X R)(Y,Z,U,V) = A(X)C_1(Y,Z,U,V), \tag{6}$$

for a nonzero 1-form A.
The purpose of this paper is to examine the various relationships that exist between semirecurrent manifold and the several recurrent manifolds mentioned in the above.

2. Some properties of semirecurrent manifold

First of all, the existence of semirecurrent manifold is ensured by a proper example as follows:

Example 2.1. Let (M^n, g_c) be a space of constant curvature. Then its curvature tensor R can be expressed as

$$R(X,Y,Z,U) = \frac{s}{n(n-1)}[g(X,U)g(Y,Z) - g(X,Z)g(Y,U)].$$

Hence it follows from (3) and the last relation that the concircular curvature tensor C_1 vanishes. On the other hand, it is well known [3] that a space of constant curvature is Einstein, and hence its scalar curvature is constant. Therefore by virtue of the last relation, the curvature tensor R is covariantly constant, that is, $\nabla R = 0$, which yields

$$(\nabla_X R)(Y,Z,U,V) = A(X)C_1(Y,Z,U,V).$$

Summing up the result above mentioned, we can conclude that the manifold is semirecurrent.

Concerning the scalar curvature of semirecurrent manifold, we have

Theorem 2.2. *Let (M^n, g) be a semirecurrent manifold. Then its scalar curvature is constant.*

Proof. Contracting (6) on Y and V, and then contracting the relation obtained thus on Z and U, we obtain

$$\nabla_X s = 0,$$

showing that the scalar curvature s is constant. This completes the proof. □

Concerning the associated 1-form A in (6), we get

THEOREM 2.3. *Let (M^n, g) be a semirecurrent manifold. Then either the manifold is a space of constant curvature or the associated 1-form A in (6) is closed.*

Proof. From (3), (6) and Theorem 2.2, it follows that

$$(\nabla_X \nabla_Y R)(Z,U,V,W) - (\nabla_Y \nabla_X R)(Z,U,V,W)$$

$$= dA(X,Y)C_1(Z,U,V,W). \tag{7}$$

By virtue of (7), Walker's Lemma 1 [11], namely

$$\begin{aligned}(\nabla_X \nabla_Y R)(Z,U,V,W) &- (\nabla_Y \nabla_X R)(Z,U,V,W)\\ +(\nabla_V \nabla_W R)(X,Y,Z,U) &- (\nabla_W \nabla_V R)(X,Y,Z,U)\\ +(\nabla_Z \nabla_U R)(V,W,X,Y) &- (\nabla_U \nabla_Z R)(V,W,X,Y) = 0\end{aligned}$$

reduces to

$$dA(X,Y)C_1(Z,U,V,W) + dA(V,W)C_1(X,Y,Z,U) + dA(Z,U)C_1(V,W,X,Y) = 0. \tag{8}$$

Since

$$C_1(X,Y,Z,U) = C_1(Z,U,X,Y),$$

it follows from Walker's Lemma 2 [11] and (8) that either $dA = 0$ or $C_1 = 0$, showing that either the associated 1-form A in (6) is closed or the manifold is a space of constant curvature. This completes the proof. □

As an immediate consequence of Theorem 2.3, we obtain

COROLLARY 2.4. *Let (M^n, g) be a semirecurrent manifold with non closed 1-form A in (6). Then the manifold is a space of constant curvature.*

THEOREM 2.5. *Let (M^n, g) be a semirecurrent manifold. Then the vector field $A^\sharp$ defined by $g(X, A^\sharp) = A(X)$ is an eigenvector of Ricci tensor Ric corresponding to eigenvalue $\frac{s}{n}$,i.e., $Ric(X, A^\sharp) = \frac{s}{n} g(X, A^\sharp)$.*

Proof. From the second Bianchi identity and (6), it follows that

$$A(X)C_1(Y,Z,U,V)+A(Y)C_1(Z,X,U,V)+A(Z)C_1(X,Y,U,V))=0. \tag{9}$$

Contracting (9) on Z and V, and then contracting the relation obtained thus on Y and U, we have

$$2[Ric(X,A^\sharp)-\frac{s}{n}g(X,A^\sharp)]=0,$$

showing that the associated vector field $A^\sharp$ is an eigenvector of Ricci tensor Ric corresponding to eigenvalue $\frac{s}{n}$. This completes the proof. □

Let (M^n,g) be a Riemannian product manifold $(M^p\times M^{n-p},\widehat{g}+\widetilde{g})$. In local coordinates, we adopt the Latin indices (resp. the Greek indices) for tensor components which are constructed on $(M^p,\widehat{g})$ (resp. $(M^{n-p},\widetilde{g})$). Therefore, the Latin indices take the values from $1,...,p$ whereas the Greek indices run over the range $p+1,...,n$. Now we can state the following.

THEOREM 2.6. *Let a Riemannian product manifold $(M^p\times M^{n-p},\widehat{g}+\widetilde{g})$ be a semirecurrent manifold. Then either one decomposition manifold $(M^p,\widehat{g})$ is locally symmetric or the other decomposition manifold $(M^{n-p},\widetilde{g})$ is a space of constant curvature.*

Proof. Since any tensor components of R and its covariant derivatives with both Latin and Greek indices together should be zero, we have from (6)

$$0=R_{\alpha\beta\gamma\delta;p}=A_pC_{1\alpha\beta\gamma\delta},$$

which leads to either

$$A_p=0 \tag{10}$$

or

$$C_{1\alpha\beta\gamma\delta}=0. \tag{11}$$

Here semicolon ";" indicates covariant differentiation.
In case of $A_p=0$, we have from (6)

$$R_{ijkl;p}=0,$$

showing that the manifold $(M^p,\widehat{g})$ is locally symmetric.
On the other hand, if we assume that $A_p\neq 0$, then we have from (3) and (11)

$$R_{\alpha\beta\gamma\delta}=\frac{s}{n(n-1)}(g_{\beta\gamma}g_{\alpha\delta}-g_{\alpha\gamma}g_{\beta\delta}), \tag{12}$$

showing that the manifold $(M^{n-p},\widetilde{g})$ is a space of constant curvature. Therefore either one decomposition manifold $(M^p,\widehat{g})$ is locally symmetric or the other decomposition manifold $(M^{n-p},\widetilde{g})$ is a space of constant curvature. This completes the proof. □

3. Relationships between semirecurrent manifolds and various recurrent manifolds

First, as a consequence of Theorem 2.2 we have

LEMMA 3.1. *Let (M^n, g) be a semirecurrent manifold. Then the manifold is a concircularly recurrent manifold with the same recurrence form.*

Proof. Since the scalar curvature of semirecurrent manifold is constant, we have from (3)

$$(\nabla_X C_1)(Y, Z, U, V) = (\nabla_X R)(Y, Z, U, V). \tag{13}$$

Therefore it follows from (6) and (13) that

$$(\nabla_X C_1)(Y, Z, U, V) = A(X) C_1(Y, Z, U, V), \tag{14}$$

showing that the manifold is concircularly recurrent with the same recurrent 1-form. This completes the proof. □

Now we can state the following.

THEOREM 3.2. *Let (M^n, g) be a semirecurrent manifold. Then the manifold is conformally recurrent with the same recurrence form if and only if the manifold is conharmonically recurrent with the same recurrence form.*

Proof. It is well known [9] that every concircularly recurrent manifold is a recurrent manifold with the same recurrence form. Since the relation

$$C(X, Y, Z, U) = C_2(X, Y, Z, U) + \frac{n}{(n-2)}[R(X, Y, Z, U) - C_1(X, Y, Z, U)] \tag{15}$$

holds, it follows from (14), (15) and Lemma 3.1 that a semirecurrent manifold is conformally recurrent with the same recurrence form if and only if the manifold is conharmonically recurrent with the same recurrence form. This completes the proof. □

A Riemannian manifold (M^n, g) is said to be Einstein [3] if its Ricci tensor Ric is proportional to the metric tensor g, i.e.,

$$Ric(X, Y) = \frac{s}{n} g(X, Y). \tag{16}$$

On the other hand, a Riemannian manifold (M^n, g) is said to be an Einstein and semirecurrent manifold if the manifold is both Einstein and semirecurrent. Concerning an Einstein and semirecurrent manifold, we obtain

THEOREM 3.3. *Let (M^n, g) be an Einstein and semirecurrent manifold. Then the manifold is a conharmonically recurrent manifold with the same recurrence form.*

Proof. Taking account of (2), (3) and (16), it is easy to see that a conformal curvature tensor C is equal to a concircular curvature tensor C_1, i.e.,

$$C(X,Y,Z,U) = C_1(X,Y,Z,U). \tag{17}$$

Since every concircularly recurrent manifold is recurrent with the same recurrence form [9], it follows from (15), (17) and Lemma 3.1 that the manifold is conharmonically recurrent with the same recurrent 1-form. □

As an immediate consequence of Theorem 3.2 and Theorem 3.3, we get

Theorem 3.4. *Let (M^n, g) be an Einstein and semirecurrent manifold. Then the manifold is a conformally recurrent manifold with the same recurrence form.*

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