Korean J. Math. **30** (2022), No. 1, pp. 11–23
http://dx.doi.org/10.11568/kjm.2022.30.1.11

PSEUDO - COMPLEMENTATION ON GENERALIZED ALMOST DISTRIBUTIVE FUZZY LATTICES

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Abstract. In this paper, the concept of pseudo - complementation on a generalized almost distributive fuzzy lattices (GADFLs) is introduced as a fuzzification of the crisp concept pseudo - complementation on a generalized almost distributive lattices. It is also established a one - to - one correspondence between the pseudo - complemented GADFL (R, A), R with 0 and the left identity element of R.

1. Introduction

The concept of generalized almost distributive lattices (GADLs) was introduced by Rao, Bandaru and Rafi [8] as a generalization of almost distributive lattices (ADLs) [11] which was a common abstraction of almost all the existing ring theoretic generalization of a boolean algebra on one hand and distributive lattice on the other. Swamy, Rao and Nanaji Rao [10] studied the concept of pseudo-complementation on almost distributive lattices. Later [9] Rao, Bandaru and Rafi studied the concept of pseudo-complementation on a generalized almost distributive lattices. On the other hand, L. A. Zadeh [12] in 1965 introduced the notion of fuzzy set. Again in 1971, Zadeh [13] defined a fuzzy ordering as a generalization of the concept of ordering, that is, a fuzzy ordering is a fuzzy relation that is transitive. In particular, a fuzzy partial ordering is a fuzzy ordering that is reflexive and antisymmetric. In 1994, Ajmal and Thomas [1] defined a fuzzy lattice as a fuzzy algebra and characterized fuzzy sublattices. In 2009, [4], considering the notion of fuzzy order of Zadeh [13], introduced a new notion of fuzzy lattice and studied the level sets of fuzzy lattices. He also introduced the notions of distributive and modular fuzzy lattices and considered some basic properties of fuzzy lattices. Again Chon, in 2015 and 2016 [5, 6] studied the properties of fuzzy lattices as fuzzy relations. In 2017, Berhanu *et al.* [2] introduce the concept of almost distributive fuzzy lattices (ADFLs) as a generalization of distributive fuzzy lattices and characterized some properties of an ADL using the fuzzy partial order relations and fuzzy lattices defined by Chon. Later Gerima [7] studied the concept of pseudo-supplemented almost distributive fuzzy lattices on the other hand Berhanu and Yohannes [3] introduce the concept of generalized almost distributive fuzzy lattices (GADFLs) as a generalization of almost distributive fuzzy lattices

Received March 8, 2021. Revised December 16, 2021. Accepted December 29, 2021.
2010 Mathematics Subject Classification: 06B75, 06D15, 06D72.
Key words and phrases: fuzzy lattices, generalized almost distributive fuzzy lattices, pseudo-complemented generalized almost distributive fuzzy lattices.

(ADFLs). As a continuation in this paper the concept of pseudo-complementation on generalized almost distributive fuzzy lattices is introduced.

2. Preliminaries

First we recall certain definitions and properties of a generalized almost distributive lattices.

DEFINITION 2.1. [8] An algebra $(L, \vee, \wedge)$ of type (2, 2) is called a generalized almost distributive lattice if it satisfies the following axioms:

(As $\wedge$) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
(LD $\wedge$) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(LD $\vee$) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
(A_1) $x \wedge (x \vee y) = x$
(A_2) $(x \vee y) \wedge x = x$
(A_3) $(x \wedge y) \vee y = y$
for all $x, y, z \in L$.

LEMMA 2.2. [8] *For any* $a \in L$,

(1) $a \vee a = a$
(2) $a \wedge a = a$.

In addition to the 3 absorption laws A_1, A_2, A_3 given in the above definition , we also get the following:

LEMMA 2.3. [8] *For any* $a, b \in L$,

(A_4) $a \vee (a \wedge b) = a$
(A_5) $a \vee (b \wedge a) = a$.

DEFINITION 2.4. [8] For any $a, b \in L$ we say that a is less than or equal to b and write $a \leq b$, if $a \wedge b = a$ or equivalently, $a \vee b = b$.

LEMMA 2.5. [8] *For any* $a, b, c \in R$, $a \wedge b \wedge c = b \wedge a \wedge c$

DEFINITION 2.6. [8] Let $(L, \vee, \wedge)$ be a GADL. An element $0 \in L$ is called a zero element of L if $0 \wedge a = 0$ for all $a \in L$.

LEMMA 2.7. [8] *Let* $(L, \vee, \wedge, 0)$ *be a GADL with 0. Then, for any* $a \in L$, *the following hold:*

(1) $a \vee 0 = a$
(2) $0 \vee a = a$
(3) $a \wedge 0 = 0$

DEFINITION 2.8. [9] Let $(L, \vee, \wedge)$ be a GADL with 0. Then a unary operation $a \to a^*$ on L is called a pseudo-complementation on L if, for any $a, b \in L$ it satisfies the following conditions

(1) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$
(2) $a \wedge a^* = 0$
(3) $(a \vee b)^* = a^* \wedge b^*$

DEFINITION 2.9. [9] For any nonempty subset A of a GADL L with 0, define

$$A^* = \{x \in L | x \wedge a = 0, \text{ for all } a \in A\}$$

This A^* is an ideal of L and is called the annihilator ideal of A.

Now, let us recall certain definitions and properties of fuzzy posets and fuzzy lattices.

DEFINITION 2.10. [4] Let X be a set. A function $A : X \times X \longrightarrow [0,\ 1]$ is called a fuzzy relation in X. The fuzzy relation A in X is reflexive iff $A(x, x) = 1$ for all $x \in X$, A is transitive iff $A(x, z) \geq \sup_{y \in X} \min(A(x, y), A(y, z))$, and A is antisymmetric iff $A(x, y) > 0$ and $A(y, x) > 0$ imply $x = y$. A fuzzy relation A is a fuzzy partial order relation if A is reflexive, antisymmetric and transitive. A fuzzy partial order relation A is a fuzzy total order relation iff $A(x, y) > 0$ or $A(y, x) > 0$ for all $x, y \in X$. If A is a fuzzy partial order relation in a set X, then (X, A) is called a fuzzy partially ordered set or a fuzzy poset. If B is a fuzzy total order relation in a set X, then (X, B) is called a fuzzy totally ordered set or a fuzzy chain.

DEFINITION 2.11. [4] Let (X, A) be a fuzzy poset and let $B \subseteq X$. An element $u \in X$ is said to be an upper bound for a subset B iff $A(b, u) > 0$ for all $b \in B$. An upper bound u_0 for B is the least upper bound of B iff $A(u_0,\ u) > 0$ for every upper bound u for B. An element $v \in X$ is said to be a lower bound for a subset B iff $A(v, b) > 0$ for all $b \in B$. A lower bound v_0 for B is the greatest lower bound of B iff $A(v,\ v_0) > 0$ for every lower bound v for B.

We denote the least upper bound of the set $\{x, y\}$ by $x \vee y$ and denote the greatest lower bound of the set $\{x, y\}$ by $x \wedge y$.

DEFINITION 2.12. [4] Let (X, A) be a fuzzy poset. (X, A) is a fuzzy lattice iff $x \vee y$ and $x \wedge y$ exist for all $x, y \in X$.

DEFINITION 2.13. [4] Let (X, A) be a fuzzy lattice. (X, A) is distributive if and only if $x \wedge (y \vee z)= (x \wedge y) \vee (x \wedge z)$ and $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$.

Finally, let us consider some definitions and properties of generalized almost distributive fuzzy lattices (GADFLs) as follow:

DEFINITION 2.14. [3] Let $(R,\ \vee,\ \wedge)$ be an algebra of type (2, 2) and (R, A) be a fuzzy poset. Then we call (R, A) is a Generalized almost distributive fuzzy lattice if it satisfies the following axioms:

(1) $A((a \wedge b) \wedge c, a \wedge (b \wedge c)) = A(a \wedge (b \wedge c), (a \wedge b) \wedge c) = 1$;
(2) $A(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)) = A((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) = 1$;
(3) $A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) = A((a \vee b) \wedge (a \vee c), a \vee (b \wedge c)) = 1$;
(4) $A(a \wedge (a \vee b), a) = A(a, a \wedge (a \vee b)) = 1$;
(5) $A((a \vee b) \wedge a, a) = A(a, (a \vee b) \wedge a) = 1$;
(6) $A((a \wedge b) \vee b, b) = A(b, (a \wedge b) \vee b) = 1$;
for all $a, b, c \in R$.

Now, we give some elementary properties of a GADFL.

THEOREM 2.15. [3] *Let (R, A) be a fuzzy poset . Then R is a GADL iff (R, A) is a GADFL.*

THEOREM 2.16. [3] *Let (R, A) be a GADFL . Then $a = b \Leftrightarrow A(a, b) = A(b, a) = 1$.*

Definition 2.17. [3] Let (R, A) be a GADFL . Then for any $a, b \in R$, $a \leq b$ if and only if $A(a, b) > 0$.

In view of the above definition, we have the following theorem.

Theorem 2.18. [3] *If (R, A) is a GADFL then $a \wedge b = a$ if and only if $A(a, b) > 0$.*

Lemma 2.19. [3] *Let (R, A) be a GADFL and $a, b \in R$such that $a \neq b$. If $A(a, b) > 0$ then $A(b, a) = 0$.*

3. Definition and properties

In this section, we give the definition of a pseudo-complementation on a generalized almost distributive fuzzy lattices (GADFLs), R with 0 and study some elementary properties of pseudo-complementation.

Definition 3.1. Let (R, A) be a GADFL, R with 0. Then a unary operation $a \to a^*$ on (R, A) $a^\star$ is called a pseudo-complementation on (R, A) if, for any $a, b \in R$ it satisfies the following conditions:

(1) $A(a \wedge b, 0) > 0 \Rightarrow A(a^* \wedge b, b) = A(b, a^* \wedge b) = 1$
(2) $A(a \wedge a^*, 0) > 0$
(3) $A((a \vee b)^*, a^* \wedge b^*) = A(a^* \wedge b^*, (a \vee b)^*) = 1$

If (R, A) is a GADFL, R with 0 and if $*$ is a pseudo-complementation on (R, A), then we say that (R, A) is a pseudo-complemented GADFL.
In the following, we give an example of pseudo-complemented GADFL.

Example 3.2. Let $R = \{0, a, b, c\}$ and define two binary operations $\vee$ and $\wedge$ in R as follows:

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	b	b
c	c	c	c	c

and

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	c
c	0	b	b	c

and define $A(x^*, 0) = A(0, x^*) = 1$ for all $x \neq 0$ and $A(0^*, a) = A(a, 0^*) = 1$.
Define a fuzzy relation $A : R \times R \longrightarrow [0, 1]$ as follows:
$A(0, 0) = A(a, a) = A(b, b) = A(c, c) = 1$,
$A(a, 0) = A(b, 0) = A(c, 0) = A(b, a) = A(b, c) = A(c, a) = A(c, b) = 0$
$A(0, a) = 0.3$, $A(0, b) = 0.5$, $A(0, c) = 0.8$, $A(a, b) = 0.2$ and $A(a, c) = 0.4$.
Then (R, A) is a GADFL, R with 0 and $*$ is a pseudo-complementation on (R, A).

Lemma 3.3. *Let (R, A) be a GADFL, R with 0 and $*$ a pseudo-complementation on (R, A) and $a, b \in R$. Then we have the following:*

(1) 0^* *is left identity element*
(2) $A(0^{**}, 0) = A(0, 0^{**}) = 1$
(3) $A(a^{**} \wedge a, a) = A(a, a^{**} \wedge a) = 1$
(4) $A(a^{***}, a^*) = A(a^*, a^{***}) = 1$
(5) $A(a^* \wedge b^*, b^* \wedge a^*) = A(b^* \wedge a^*, a^* \wedge b^*) = 1$

(6) $A(a,b) > 0 \Rightarrow A(b^*,a^*) > 0$
(7) $A(a^*,(a\wedge b)^*) > 0$ *and* $A(b^*,(a\wedge b)^*) > 0$
(8) $A(a\wedge b,0) = A(0,a\wedge b) = 1 \Leftrightarrow A(a^{**}\wedge b,0) = A(0,a^{**}\wedge b) = 1$
(9) $A((a\wedge b)^{**},a^{**}\wedge b^{**}) = A(a^{**}\wedge b^{**},(a\wedge b)^{**}) = 1$

Proof. Let (R,A) be a pseudo-complemented GADFL and $a,b\in R$
(1) Since $A(0\wedge a,0) = A(0,0\wedge a) = 1$ then $A(0^*\wedge a,a) = A(a,0^*\wedge a) = 1$. Hence 0^* is left identity element.
(2) Since 0^* is left identity

$$\begin{aligned} A(0^*\wedge 0,0) = A(0,0^*\wedge 0) = 1 &\Rightarrow A(0^*\vee 0,0^*) = A(0^*,0^*\vee 0) = 1 \\ &\Rightarrow A((0^*\vee 0)^*,0^{**}) = A(0^{**},(0^*\vee 0)^*) = 1 \\ &\Rightarrow A(0^{**}\wedge 0^*,0^{**}) = A(0^{**},0^{**}\wedge 0^*) = 1 \\ &\Rightarrow A(0,0^{**}) = A(0^{**},0) = 1. \end{aligned}$$

(3) Since $a\in R$

$$\begin{aligned} A(a\wedge a^*,0) = A(0,a\wedge a^*) = 1 &\Rightarrow A(a^*\wedge a,0) = A(0,a^*\wedge a) = 1 \\ &\Rightarrow A(a^{**}\wedge a,a) = A(a,a^{**}\wedge a) = 1. \end{aligned}$$

(4)

$$\begin{aligned} A(a^{**}\vee a,a^{**}) &= A(a^{**}\vee(a^{**}\wedge a),a^{**}) \\ &= 1 \\ &= A(a^{**},a^{**}\vee a) \end{aligned}$$

Therefore

$$\begin{aligned} A(a^{***},a^*) &= A((a^{**}\vee a)^*,a^*) \\ &= A(a^{***}\wedge a^*,a^*) \\ &= A(a^*,a^*) \\ &= 1 \end{aligned}$$

Similarly, $A(a^*,a^{***}) = 1$
(5) We know that for any $a,b\in R$,
$A(a\vee 0,a) = A(a,a\vee 0) = 1$ and $A(b\vee 0,b) = A(b,b\vee 0) = 1$
Therefore $A(a^*\wedge 0^*,a^*) = A(a^*,a^*\wedge 0^*) = 1$ and $A(b^*\wedge 0^*,b^*) = A(b^*,b^*\wedge 0^*) = 1$
$\Rightarrow a^*\wedge 0^* = a^*$ and $b^*\wedge 0^* = b^*$
$\Rightarrow A(a^*,0^*) > 0$ and $A(b^*,0^*) > 0$
$\Rightarrow A(a^*\wedge b^*,b^*\wedge a^*) = A(b^*\wedge a^*,a^*\wedge b^*) = 1$
(6) Suppose $A(a,b) > 0$. Then $a\vee b = b$. So that

$$\begin{aligned} A(b^*,b^*\wedge a^*) &= A((a\vee b)^*,b^*\wedge a^*) \\ &= A(a^*\wedge b^*,b^*\wedge a^*) \\ &= A(b^*\wedge a^*,b^*\wedge a^*) \\ &= 1 \end{aligned}$$

Similarly, $A(b^*\wedge a^*) = 1$. Hence $b^*\wedge a^* = b^* \Rightarrow b^*\leq a^*$.
Therefore $A(b^*,a^*) > 0$.

(7) Since

$$\begin{aligned} A((a\wedge b)\wedge a^*,0) &= A(a\wedge b\wedge a^*,0)\\ &= A(b\wedge a\wedge a^*,0)\\ &= A(b\wedge 0,0)\\ &= A(0,0)\\ &= 1\end{aligned}$$

Similarly, $A(0,(a\wedge b)\wedge a^*)=1$. Hence form the above definition, (1) we have $A((a\wedge b)^*\wedge a^*,a^*)=A(a^*,(a\wedge b)^*\wedge a^*)=1$
$\Rightarrow A(a^*\wedge(a\wedge b)^*,a^*)=A(a^*,a^*\wedge(a\wedge b)^*)=1$ (by (5))
$\Rightarrow a^*\leq(a\wedge b)^*$. Hence $A(a^*,(a\wedge b)^*)>0$. Since $A(a\wedge b,b)>0$, by (6), we get $A(b^*,(a\wedge b)^*)>0$.
(8)($\Rightarrow$) Suppose $A(a\wedge b,0)=A(0,a\wedge b)=1$. Then $A(a^*\wedge b,b)=A(b,a^*\wedge b)=1$.
Now,

$$\begin{aligned} A(a^{**}\wedge b,0) &= A(a^{**}\wedge a^*\wedge b,0)\\ &= A(a^*\wedge a^{**}\wedge b,0)\\ &= A(0\wedge b,0)\\ &= A(0,0)\\ &= 1\end{aligned}$$

Similarly, $A(0,a^{**}\wedge b)=1$.
($\Leftarrow$) Suppose $A(a^{**}\wedge b,0)=A(0,a^{**}\wedge b)=1$. Then,

$$\begin{aligned} A(a\wedge b,0) &= A(a^{**}\wedge a\wedge b,0)\\ &= A(a\wedge a^{**}\wedge b,0)\\ &= A(a\wedge 0,0)\\ &= 1\end{aligned}$$

Similarly, $A(0,a\wedge b)=1$
(9)

$$\begin{aligned} A(a^{**}\wedge b^{**}\wedge(a\wedge b)^{**},(a\wedge b)^{**}) &= A(a^{**}\wedge(b^*\vee(a\wedge b)^*)^*,(a\wedge b)^{**})\\ &= A(a^{**}\wedge(a\wedge b)^{**},(a\wedge b)^{**})\\ &= A((a^*\vee(a\wedge b)^*)^*,(a\wedge b)^{**})\\ &= A((a\wedge b)^{**},(a\wedge b)^{**})\\ &= 1\end{aligned}$$

Similarly, $A((a\wedge b)^{**},a^{**}\wedge b^{**}\wedge(a\wedge b)^{**})=1$
and $A(a\wedge b\wedge(a\wedge b)^*,0)=A(0,a\wedge b\wedge(a\wedge b)^*)=1$
$\Rightarrow A(a^{**}\wedge b\wedge(a\wedge b)^*,0)=A(0,a^{**}\wedge b\wedge(a\wedge b)^*)=1$ (by 8)
$\Rightarrow A(b\wedge a^{**}\wedge(a\wedge b)^*,0)=A(0,b\wedge a^{**}\wedge(a\wedge b)^*)=1$
$\Rightarrow A(b^{**}\wedge a^{**}\wedge(a\wedge b)^*,0)=A(0,b^{**}\wedge a^{**}\wedge(a\wedge b)^*)=1$ (by 8)
$\Rightarrow A(a^{**}\wedge b^{**}\wedge(a\wedge b)^*,0)=A(0,a^{**}\wedge b^{**}\wedge(a\wedge b)^*)=1$
$\Rightarrow A((a\wedge b)^*\wedge b^{**}\wedge a^{**},0)=A(0,(a\wedge b)^*\wedge b^{**}\wedge a^{**})=1$
$\Rightarrow A((a\wedge b)^*\wedge b^{**}\wedge a^{**},a^{**}\wedge b^{**})=A(a^{**}\wedge b^{**},(a\wedge b)^*\wedge b^{**}\wedge a^{**})=1$
Also, $A((a\wedge b)^{**}\wedge a^{**}\wedge b^{**},a^{**}\wedge b^{**}\wedge(a\wedge b)^{**})$
$=A(a^{**}\wedge(a\wedge b)^{**}\wedge b^{**},a^{**}\wedge b^{**}\wedge(a\wedge b)^{**})$

$= A(a^{**} \wedge ((a \wedge b)^* \vee b^*)^*, a^{**} \wedge b^{**} \wedge (a \wedge b)^{**})$
$= A(a^{**} \wedge (b^* \vee (a \wedge b)^*)^*, a^{**} \wedge b^{**} \wedge (a \wedge b)^{**})$ (Since $a \leq b \Rightarrow a \vee b = b \vee a$)
$= A(a^{**} \wedge (b^{**} \wedge (a \wedge b)^{**}), a^{**} \wedge b^{**} \wedge (a \wedge b)^{**})$
$= A(a^{**} \wedge b^{**} \wedge (a \wedge b)^{**}, a^{**} \wedge b^{**} \wedge (a \wedge b)^{**})$
$= 1$
Similarly, $A(a^{**} \wedge b^{**} \wedge (a \wedge b)^{**}, (a \wedge b)^{**} \wedge a^{**} \wedge b^{**}) = 1$
Therefore $A((a \wedge b)^{**}, a^{**} \wedge b^{**}) = A(a^{**} \wedge b^{**}, (a \wedge b)^{**}) = 1$ □

4. One-to-one Correspondence

In this section, we prove that there is a one-to-one correspondence between the pseudo-complemented GADFL (R, A), R with 0 and the left identity element of R . First we define the following:

DEFINITION 4.1. Let (R, A) be a GADFL. For any non-empty subset B of R with 0, define
$B^* = \{x \in R \mid A(x \wedge a, 0) > 0$, for all $a \in B\}$. This B^* is an ideal of (R, A) and is called the annihilator ideal of B. For any $a \in R$, we write $[a]^*$ for $\{a\}^*$ and is called annulet of (R, A).

It can be easily observed that, for any subset B of R, $B \cap B^* = \{0\}$. In the following we prove some properties of annihilator ideal.

LEMMA 4.2. *For any ideals I and J of a GADFL (R, A), R with 0, we have the following:*

(1) $I^* = \cap_{a \in I}[a]^*$
(2) If $I \subseteq J$ then $J^* \subseteq I^*$
(3) $I \subseteq I^{***}$
(4) $I^{***} = I^*$
(5) $I \cap J = (0] \Leftrightarrow I \subseteq J^*$

Proof. Suppose I and J are ideals of a GADFL (R, A), R with 0.
(1) $(\Rightarrow)$ Let $x \in I^* \Rightarrow A(x \wedge a, 0) = A(0, x \wedge a) = 1$ for $a \in I$.
$\Rightarrow x \in [a]^*$, for each $a \in I$
$\Rightarrow x \in \cap_{a \in I}[a]^*$. Hence $I^* \subseteq \cap_{a \in I}[a]^*$
$(\Leftarrow)$ Let $x \in \cap_{a \in I}[a]^* \Rightarrow x \in [a]^*$ for all $a \in I$.
$\Rightarrow A(x \wedge a, 0) = A(0, x \wedge a) = 1$
$\Rightarrow x \in I^*$, for $a \in I$
Hence $\cap_{a \in I}[a]^* \subseteq I^*$. Therefore $I^* = \cap_{a \in I}[a]^*$.
(2) Let $I \subseteq J$. Let $a \in J^*$ and $b \in I$. Then $b \in J$ and $A(a \wedge b, 0) = A(0, a \wedge b) = 1$. Hence $a \in I^*$. Therefore $J^* \subseteq I^*$.
(3) Let $x \in I$ and $a \in I^*$. Then $x \wedge a \in I$ and $x \wedge a \in I^*$. So that $x \wedge a \in I \cap I^* = \{0\}$. Hence $A(x \wedge a, 0) = A(0, x \wedge a) = 1$, for all $a \in I^*$. We get $x \in I^{**}$. Therefore $I \subseteq I^{**}$.
(4) Since $I \subseteq I^{**}$, we get by (2), $I^{***} \subseteq I^*$. Now, let $x \in I^*$ and $a \in I^{**}$. Then $x \wedge a \in I^* \cap I^{**} = \{0\}$. Hence $A(x \wedge a, 0) = A(0, x \wedge a) = 1$, for all $a \in I^{**}$. Therefore $x \in I^{***}$, and hence $I^* \subseteq I^{***}$. Thus $I^{***} = I^*$.
(5) Suppose $I \cap J = (0]$. Let $x \in I$ and $a \in J$. Then $x \wedge a \in I$ and $x \wedge a \in J$ and hence $x \wedge a \in I \cap J = (0]$. Therefore, $A(x \wedge a, 0) = A(0, x \wedge a) = 1$, we get $x \in J^*$. Thus $I \subseteq J^*$. Conversely, assume that $I \subseteq J^*$. Let $x \in I \cap J$. Then $x \in I$ and $x \in J$, since

$I \subseteq J^*$ then $x \in J^*$ and hence $A(x \wedge x, 0) = A(0, x \wedge x) = 1$. Hence $x = x \wedge x = 0$. Therefore $I \cap J = (0]$. □

LEMMA 4.3. *Let (R, A) be a GADFL, R with 0. For any $a, b \in R$, $[a \vee b]^* = [a]^* \cap b^*$.*

Proof. Let $x \in R$, Then,

$$\begin{aligned} x \in [a \vee b]^* &\Leftrightarrow A(x \wedge (a \vee b), 0) = A(0, x \wedge (a \vee b)) = 1 \\ &\Leftrightarrow A((x \wedge a) \vee (x \wedge b), 0) = A(0, (x \wedge a) \vee (x \wedge b)) = 1 \\ &\Leftrightarrow A(x \wedge a, 0) = A(0, x \wedge a) = 1 and A(x \wedge b, 0) = A(0, x \wedge b) = 1 \\ &\Leftrightarrow x \in [a]^* and x \in [b]^* \\ &\Leftrightarrow x \in [a]^* \cap [b]^* \end{aligned}$$

Hence $[a \vee b]^* = [a]^* \cap [b]^*$ □

LEMMA 4.4. *Let (R, A) be a GADFL, R with 0. For any $x, y \in R$, we have the following:*

(1) $[x \wedge y]^* = [y \wedge x]^*$
(2) $A(x, Y) > 0 \Rightarrow [y]^* \subseteq [x]^*$
(3) $[x \wedge y]^{**} = [x]^{**} \cap [y]^{**}$
(4) $[x]^{**} = [x]^*$
(5) $[x \vee y]^* = [x]^* \cap [y]^* = [y]^* \cap [x]^* = [y \vee x]^*$

Proof. Let (R, A) be a GADFL, R with 0 and $x, y \in R$,
(1) Let $a \in [x \wedge y]^*$. Then $A(a \wedge x \wedge y, 0) = A(0, a \wedge x \wedge y) = 1$. Now,

$$\begin{aligned} A(a \wedge y \wedge x, 0) &= A(a \wedge y \wedge x \wedge x, 0) \\ &= A(a \wedge x \wedge y \wedge x, 0) \\ &= A(0 \wedge x, 0) \\ &= A(0, 0) \\ &= 1 \end{aligned}$$

Similarly, $A(0, a \wedge y \wedge x) = 1$. Therefore $a \in [y \wedge x]^*$. Hence $[x \wedge y]^* \subseteq [y \wedge x]^*$. Also, $[y \wedge x]^* \subseteq [x \wedge y]^*$. Thus $[x \wedge y]^* = [y \wedge x]^*$.
(2) $A(x, y) > 0 \Rightarrow x \wedge y = x$.
Let $a \in [y]^*$. Then $A(a \wedge y, 0) = A(0, a \wedge y) = 1$. Now,

$$\begin{aligned} A(a \wedge x, 0) &= A(a \wedge x \wedge y, 0) \\ &= A(x \wedge a \wedge y, 0) \\ &= A(x \wedge 0, 0) \\ &= A(0, 0) \\ &= 1 \end{aligned}$$

Similarly, $A(0, a \wedge x) = 1$. Therefore $a \in [x]^*$. Hence $[y]^* \subseteq [x]^*$
(3) Since $A(x \wedge y, y) > 0$, we have by (2), $[y]^* \subseteq [x \wedge y]^*$.
Similarly $[x]^* \subseteq [y \wedge x]^* = [x \wedge y]^*$. Thus we get $[x \wedge y]^{**} \subseteq [x]^{**}, [y]^{**}$. Hence $[x \wedge y]^{**} \subseteq [x]^{**} \cap [y]^{**}$.

Let $a \in [x]^{**} \cap [y]^{**}$ and $t \in [x \wedge y]^{*}$. Now,

$$\begin{aligned} A(t\wedge x\wedge y,0)=A(0,t\wedge x\wedge y)=1 &\Rightarrow t\wedge x\in [y]^{*} \\ &\Rightarrow A(a\wedge t\wedge x,0)=A(0,a\wedge t\wedge x)=1 \\ &\Rightarrow A(t\wedge a\wedge x,0)=A(0,t\wedge a\wedge x)=1 \\ &\Rightarrow t\wedge a\in [x]^{*} \\ &\Rightarrow A(a\wedge t\wedge a,0)=A(0,a\wedge t\wedge a)=1 \end{aligned}$$

Therefore $A(a\wedge t,0)=A(0,a\wedge t)=1$ for all $t\in[x\wedge y]^{*}$.
$\Rightarrow a\in[x\wedge y]^{**}$.
Therefore $[x]^{**}\cap[y]^{**}\subseteq[x\wedge y]^{**}$ and hence $[x\wedge y]^{**}=[x]^{**}\cap[x]^{**}$.
(4) If $a\in[x]^{*}$ then $A(a\wedge b,0)=A(0,a\wedge b)=1$ for all $b\in[x]^{**}$.
Therefore $a\in x^{***}$. Thus $[x]^{*}\subseteq x^{***}$.
For any $t\in[x]^{*}$, $A(x\wedge t,0)=A(0,x\wedge t)=1\Rightarrow x\in x^{**}$. So that $s\in[x]^{***}\Rightarrow A(s\wedge x,0)=A(0,s\wedge x)=1\Rightarrow s\in[x]^{*}$. Therefore $[x]^{***}\subseteq[x]^{*}$. Hence $[x]^{***}=[x]^{*}$.
(5) Follows from the above two lemmas. □

LEMMA 4.5. *Let (R,A) be a GADFL, R with 0 and $a\in R$. Then $(a]=R$ if and only if a is left identity element.*

Proof. Suppose $(a]=R$. Then for any $x\in R$, we have $x\in(a]$ and since 4.1.5,(i), $A(x,a\wedge x)=A(a\wedge x,x)=1$. Therefore a is a left identity element. Conversely suppose a is a left identity element. We have $(a]\subseteq R$. Let $x\in R$. Then $A(a\wedge x,x)=A(x,a\wedge x)=1$. Therefore $x\in(a]$. Hence $(a]=R$ □

In the following we give a necessary and sufficient conditions for a GADFL to have a pseudo-complementation.

THEOREM 4.6. *Let (R,A) be a GADFL, R with 0. Then for any $a\in R$, the annulate $[a]^{*}$ is a principal ideal if and only if (R,A) has a pseudo-complementation.*

Proof. ($\Rightarrow$) Assume that for any $a\in R$, $[a]^{*}$ is a principal ideal. Since $0\in R$, we have $R=[0]^{*}=(m]$ for some $m\in R$. Hence by lemma 4.4, m is left identity element in R. Let $a\in R$, then $[a]^{*}=(x]$ for some $x\in R$. Define $A(a^{\perp},x\wedge m)=A(x\wedge m,a^{\perp})=1$. Now, we prove that $\perp$ is a pseudo-complementation on (R,A).
First we prove that $\perp$ is well defined.
Let $a\in R$ and suppose $[a]^{*}=(c]=(y]$ for some $x,y\in R$. Then $A(x\wedge y,y)=A(y,x\wedge y)=1$ and $A(y\wedge x,x)=A(x,y\wedge x)=1$. Thus,

$$\begin{aligned} A(x\wedge m,y\wedge m) &= A(y\wedge x\wedge m,y\wedge m) \\ &= A(x\wedge y\wedge m,y\wedge m) \\ &= A(y\wedge m,y\wedge m) \\ &= 1 \end{aligned}$$

Similarly, $A(y\wedge m,x\wedge m)=1$. Therefore $\perp$ is well defined.
Now,

$$\begin{aligned} A(a\wedge a^{\perp},0) &= A(a\wedge x\wedge m,0) \\ &= A(0,0) \\ &= 1 \end{aligned}$$

Similarly, $A(0, a \wedge a^{\perp}) = 1$.
Suppose $b \in R$ and $A(a \wedge b, 0) = A(0, a \wedge b) = 1$. Then $b \in [a]^* = (x]$. Therefore

$$\begin{aligned} A(a^{\perp} \wedge b, b) &= A(a^{\perp} \wedge b, x \wedge b) \\ &= A(a^{\perp} \wedge b, x \wedge m \wedge b) \\ &= A(a^{\perp} \wedge b, a^{\perp} \wedge b) \\ &= 1 \end{aligned}$$

Similarly, $A(b, a^{\perp} \wedge b) = 1$.
Finally, Suppose $a, b \in R$, $[a]^* = (x]$ and $[b]^* = (y]$ for some $x, y \in R$. Then $[a \vee b]^* = [a]^* \cap [b]^* = (x] \cap (y] = (x \wedge y]$. Therefore

$$\begin{aligned} A((a \vee b)^{\perp}, a^{\perp} \wedge b^{\perp}) &= A(x \wedge y \wedge m, a^{\perp} \wedge b^{\perp}) \\ &= A(x \wedge m \wedge y \wedge m, a^{\perp} \wedge b^{\perp}) \\ &= A(a^{\perp} \wedge b^{\perp}, a^{\perp} \wedge b^{\perp}) \\ &= 1 \end{aligned}$$

Similarly, $A(a^{\perp} \wedge b^{\perp}, (a \vee b)^{\perp}) = 1$. Thus $\perp$ is a pseudo-complementation on (R, A).
Conversely, suppose $*$ is a pseudo-complementation on (R, A) and $a \in R$. Now,

$$\begin{aligned} x \in [a]^* &\Leftrightarrow A(x \wedge a, 0) = A(0, x \wedge a) = 1 \\ &\Leftrightarrow A(a \wedge x, 0) = A(0, a \wedge x) = 1 \\ &\Leftrightarrow A(a^* \wedge x, x) = A(x, a^* \wedge x) = 1 \\ &\Leftrightarrow x \in (a]^* \end{aligned}$$

So that $[a]^* = (a]^*$. Hence every annihilator ideal is a principal ideal. □

Let (R, A) be a GADFL with pseudo-complementation. In the following we establish a one-to-one correspondence between the set of all left identity elements in R and the set of all pseudo-complementations on (R, A). For this we need the following lemma.

Lemma 4.7. *Let (R, A) be a GADFL with two pseudo-complementations $*$ and $\perp$. Then, for any $a, b \in R$, we have the following:*

(1) $A(a^* \wedge a^{\perp}, a^{\perp}) = A(a^{\perp}, a^* \wedge a^{\perp}) = 1$, $A(a^{\perp} \wedge a^*, a^*) = A(a^*, a^{\perp} \wedge a^*) = 1$, $A(a^* \vee a^{\perp}, a^*) = A(a^*, a^* \vee a^{\perp}) = 1$ *and* $A(a^{\perp} \vee a^*, a^{\perp}) = A(a^{\perp}, a^{\perp} \vee a^*) = 1$
(2) $A(a^{*\perp}, a^{\perp\perp}) = A(a^{\perp\perp}, a^{*\perp}) = 1$
(3) $A(a^*, b^*) = A(b^*, a^*) = 1 \Leftrightarrow A(a^{\perp}, b^{\perp}) = A(b^{\perp}, a^{\perp}) = 1$
(4) $A(a^*, 0) = A(0, a^*) = 1 \Leftrightarrow A(a^{\perp}, 0) = A(0, a^{\perp}) = 1 \Leftrightarrow (A(a \wedge b, 0) = A(0, a \wedge b) = 1 \Rightarrow A(b, 0) = A(0, b) = 1)$
(5) $A(a^{\perp}, a^* \wedge 0^{\perp}) = A(a^* \wedge 0^{\perp}, a^{\perp}) = 1$

Proof. Let $a, b \in R$, $*$ and $\perp$ are pseudo-complementations in (R, A).
(1) Since $A(a \wedge a^*, 0) = A(0, a \wedge a^*) = 1$ then we have $A(a^{\perp} \wedge a^*, a^{\perp}) = A(a^{\perp}, a^{\perp} \wedge a^*) = 1$. Also, $A(a \wedge a^{\perp}, 0) = A(0, a \wedge a^{\perp}) = 1$ then we have $A(a^* \wedge a^{\perp}, a^{\perp}) = A(a^{\perp}, a^* \wedge a^{\perp}) = 1$. Now,

$$\begin{aligned} A(a^* \vee a^{\perp}, a^*) &= A(a^* \vee (a^* \wedge a^{\perp}), a^*) \\ &= A(a^*, a^*) \\ &= 1 \end{aligned}$$

Similarly, $A(a^*, a^* \vee a^{\perp}) = 1$. Again,

$$\begin{aligned} A(a^{\perp} \vee a^*, a^{\perp}) &= A(a^{\perp} \vee (a^{\perp} \wedge a^*), a^{\perp}) \\ &= A(a^{\perp}, a^{\perp}) \\ &= 1 \end{aligned}$$

Similarly, $A(a^{\perp}, a^{\perp} \vee a^*) = 1$
(2) From (1) and lemma 3.2, (5) we have

$$\begin{aligned} A(a^{*\perp}, a^{\perp\perp}) &= A((a^* \vee a^{\perp})^{\perp}, a^{\perp\perp}) \\ &= A((a^{\perp} \vee a^*)^{\perp}, a^{\perp\perp}) \\ &= A(a^{\perp\perp}, a^{\perp\perp}) \\ &= 1 \end{aligned}$$

Similarly, $A(a^{\perp\perp}, a^{*\perp}) = 1$.
(3) Assume $A(a^*, b^*) = A(b^*, a^*) = 1$ and from (2) $a^{*\perp} = a^{\perp\perp}$. Then,

$$\begin{aligned} A(a^{\perp}, b^{\perp}) &= A(a^{\perp\perp\perp}, b^{\perp}) \\ &= A(a^{*\perp\perp}, b^{\perp}) \\ &= A(b^{*\perp\perp}, b^{\perp}) \\ &= A(b^{\perp\perp\perp}, b^{\perp}) \\ &= A(b^{\perp}, b^{\perp}) \\ &= 1 \end{aligned}$$

Similarly, $A(b^{\perp}, a^{\perp}) = 1$.
Conversely assume $A(a^{\perp}, b^{\perp}) = A(b^{\perp}, a^{\perp}) = 1$ and from (2) $a^{\perp *} = a^{**}$. Then,

$$\begin{aligned} A(a^*, b^*) &= A(a^{***}, b^*) \\ &= A(a^{\perp **}, b^*) \\ &= A(b^{\perp **}, b^*) \\ &= A(b^{***}, b^*) \\ &= A(b^*, b^*) \\ &= 1 \end{aligned}$$

Similarly, $A(b^*, a^*) = 1$.
(4) Let $A(a^*, 0) = A(0, a^*) = 1$. Then,

$$\begin{aligned} A(a^{\perp}, 0) &= A(a^* \wedge a^{\perp}, 0) \\ &= A(0, 0) \\ &= 1 \end{aligned}$$

Similarly, $A(0, a^{\perp}) = 1$.
Let $A(a^{\perp}, 0) = A(0, a^{\perp}) = 1$ and $A(a \wedge b, 0) = A(0, a \wedge b) = 1$. Then,

$$\begin{aligned} A(b, 0) &= A(a^{\perp} \wedge b, 0) \\ &= A(0 \wedge b, 0) \\ &= A(0, 0) \\ &= 1 \end{aligned}$$

Similarly, $A(0,b)=1$.
Let $A(a\wedge b,0)=A(0,a\wedge b)=1\Rightarrow A(b,0)=A(0,b)=1$
Since $A(a\wedge a^*,0)=A(0,a\wedge a^*)=1$ implies $A(a^*,0)=A(0,a^*)=1$.
(5) From (1) and associativity of $\wedge$ we have

$$\begin{aligned} A(a^*0^\perp,a^\perp) &= A(a^\perp\wedge a^*\wedge 0^\perp,a^\perp)\\ &= A(a^*\wedge a^\perp\wedge 0^\perp,a^\perp)\\ &= A(a^*\wedge(a^\perp\wedge 0^\perp),a^\perp)\\ &= A(a^*\wedge a^\perp,a^\perp)......\lceil 0\leq a\Rightarrow a^\perp\leq 0^\perp\rfloor\\ &= A(a^\perp,a^\perp)\\ &= 1\end{aligned}$$

Similarly, $A(a^\perp,a^*\wedge 0^\perp)=1$. □

Let (R,A) be a GADFL, R with 0, (R,A) with a pseudo-complementation $*$ and m a left identity element. If we define $*_m:R\to R$ by $a^{*_m}=a^*\wedge m$ then $*_m$ is again a pseudo-complementation on (R,A). In fact, we prove that this correspondence between the left identity elements of R and pseudo-complementation on R is a one-to-one correspondence in the following theorem.

THEOREM 4.8. *Let (R,A) be a GADFL, R with 0 and $*$ a pseudo-complementation on (R,A). Let M be the set of all left identity elements in (R,A) and L_P the set of all pseudo-complementation on (R,A). For any $m\in M$, define $*_m:R\to R$ by $a^{*_m}=a^*\wedge m$ for all $a\in R$. Then $m\to *_m$ is a bijection of M onto L_P.*

Proof. Define $f:M\to L_P$ by $A(f(m),*_m)=A(*_m,f(m))=1$
Claim: f is one-to-one correspondence.
(i) Suppose for $m,n\in M$, $A(f(m),f(n))=A(f(n),f(m))=1$. Now,

$$\begin{aligned} A(f(m),f(n))=A(f(n),f(m))=1 &\Rightarrow A(*_m,*_n)=A(*_n,*_m)=1\\ &\Rightarrow A(0^{*_m},0^{*_n})=A(0^{*_n},0^{*_m})=1\\ &\Rightarrow A(0^*\wedge m,0^*\wedge n)=A(0^*\wedge n,0^*\wedge m)=1\\ &\Rightarrow A(m,n)=A(n,m)=1\end{aligned}$$

Hence f is one-to-one.
(ii) Let $\perp\in L_P$. Then $m=0^\perp\in M$ (as $0^\perp$ is left identity element).
Now,

$$\begin{aligned} A(a^{*_m},a^\perp) &= A(a^*\wedge m,a^\perp)\\ &= A(a^*\wedge 0^\perp,a^\perp)\\ &= A(a^\perp\wedge a^*\wedge 0^\perp,a^\perp)\\ &= A(a^*\wedge a^\perp\wedge 0^\perp,a^\perp)\\ &= A(a^*\wedge a^\perp,a^\perp)\\ &= A(a^\perp,a^\perp)\\ &= 1\end{aligned}$$

Hence $A(a^{*_m},a^\perp)=A(a^\perp,a^{*_m})=1\Rightarrow A(*_m,\perp)=A(\perp,*_m)=1$.
Therefore for any $*_m\in L_P$, $\exists m\in M$ such that $A(f(m),*_m)=A(*_m,f(m))=1$. Hence f is onto. Thus f is one-to-one correspondence. □

5. Conclusion

The concept of pseudo - complementation on a generalized almost distributive fuzzy lattices is introduced as a fuzzification of the crisp concept pseudo - complementation on a generalized almost distributive lattices by using the approach of I. Chon. Some lemmas and theorems are also proved.

References

[1] N. Ajmal, K. V. Thomas, *Fuzzy lattices*, Information sciences, **79** (1994), 271–291.

[2] A. Berhanu, G. Yohannes, T. Bekalu, *Almost Distributive Fuzzy Lattices*, Int. J. of Math. and its App., **5** (1-C) (2017), 307–316.

[3] A. Berhanu, G. Yohannes, *Generalized Almost Distributive Fuzzy Lattices*, Int. J. Comput. Sci. Appl. Math., **3** (2) (2017), 46–49.

[4] I. Chon, *Fuzzy partial order relations and fuzzy lattices*, Korean J. Math., **17** (4) (2009), 361–374.

[5] I. Chon, *Fuzzy lattices as fuzzy relations*, Korean J. Math., **23** (4) (2015), 557–569. http://dx.doi.org/10.11568/kjm.2015.23.4.557

[6] I. Chon, *Some properties of fuzzy lattices as fuzzy relations*, Korean J. Math., **24** (3) (2016), 525–535. http://dx.doi.org/10.11568/kjm.2016.24.3.525

[7] Gerima Tefera Dejen, *Pseudo-Supplemented Almost Distributive Fuzzy Lattices*, Journal of Applied Mathematics and Computation **5** (2) (2021), 119–123.

[8] G.C. Rao, R. K. Bandaru, N. Rafi, *Generalized Almost Distributive Lattice-I*, Southeast Asian Bulletin of Mathematics, **33** (2009), 1175 – 1188.

[9] G. C. Rao, Ravi Kumar Bandaru and N. Rafi, *Pseudo-complementation on generalized almost distributive lattices*, Asian-European Journal of Mathematics, **3** (2) (2010), 335–346. https://doi.org/10.1142/S1793557110000246.

[10] U.M. Swamy, G.C. Rao and G. Nanaji Rao *Psedo- complementation on Almost Distributive Lattices*, Southeast Asian Bulletin of Mathematics , **24** (2000), 95–104.

[11] U.M. Swamy, G.C. Rao, *Almost distributive lattices*, Journ. Aust. Math.Soc., **31** (Series-A) (1981), 77–91.

[12] L. A. Zadeh, *Fuzzy sets*, Inform. and Control, **8** (1965), 338–353.

[13] L. A. Zadeh, *Similarity relations and fuzzy orderings*, Information Sciences, **3** (1971), 177–200.

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