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LIFTINGS OF ABSOLUTELY SUMMING OPERATORS ON $\mathcal{L}_1^\lambda-$ SPACES

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Abstract. In this article, we prove that an absolutely summing operator on $\mathcal{L}_1^\lambda$ spaces has a lifting under the conditions that a target Banach space is a quotient of reflexive Banach subspaces.

1. Introduction

In this article, we prove some variation of the lifting of operators on $\mathcal{L}_1^\lambda-$spaces that was given by Lindenstrauss. In ([10]), Lindenstrauss proved a generalization of the lifting of operators on $\mathcal{L}_1^\lambda-$space that was used an idea of $weak*-$compactness in its second dual and the abundance of finite rank operators to verify the linearity of certain map by defining a composition with a nonlinear function (see [4, pp.1726]).

From this, main question arise under what conditions does there exist a lifting of an absolutely summing operator T if the restriction of T to all subspaces have lifting of operators. That is, if $X = \overline{\cup_{j\in\Lambda} X_j}$ and $T : X \to Y$ is an absolutely summing operator such that for each $j \in \Lambda$, $T|_{X_j}$ is liftable, then can we say that T has a lifting $\tilde{T}$ on whole space X? We will give a partial answer for an absolutely summing operator T from $\mathcal{L}_1^\lambda-$spaces into a Banach space with the quotient of a reflexive Banach subspace.

We begin with summary of well known results that are related with the lifting of bounded linear operators between Banach spaces. The basic question of the lifting property of a Banach space is given as following : "Suppose that X, Y, and Z are Banach spaces and π is a surjective linear map from Z onto Y which maps the closed unit ball in Z onto the closed unit ball in Y and that T is a bounded linear operator on X into Y. When does there exist a bounded linear operator $\tilde{T} : X \to Z$ such that $\|\tilde{T}\| = \|T\|$ and such that the following diagram commutes with $\pi \circ \tilde{T} = T$?"

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \tilde{T}\downarrow & \nearrow \pi & \\ Z & & \end{array} \tag{1.1}$$

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From the above fundamental question and the diagram (1.1), we can give the definition of the lifting property for Banach space.

DEFINITION 1.1. We say that a Banach space X has the *lifting property* if for a surjective operator π from a Banach space Z onto Y and for every $T \in B(X,Y)$, there is a $\tilde{T} \in B(X,Z)$ such that $T = \pi \circ \tilde{T}$.

For this question of the lifting property, Grothendieck [3], Pelczýnski [14] and Köthe [7] characterize the spaces ℓ_1 up to an isomorphism. We now state the well known fact that ℓ_1 space has the lifting property.

THEOREM 1.2. *For a Banach space ℓ_1, let Y and Z be Banach spaces such that there is a linear map π from Z onto Y. Then for every $T \in B(\ell_1, Y)$, there exists a $\tilde{T} \in B(\ell_1, Z)$ for which $\pi \circ \tilde{T} = T$. Moreover, if π is a quotient map, then for every $\epsilon > 0$, $\tilde{T}$ may be chosen so that $\|\tilde{T}\| \leq (1+\epsilon)\|T\|$.*

The property of ℓ_1 stated in Theorem 1.2 characterizes the space ℓ_1 of separable cases. Also the spaces having the lifting property have been characterized in the nonseparable case. In [7], Köthe extended Theorem 1.2 to the nonseparable $\ell_1(\Gamma)$ space, for given nonseparable index set Γ. Moreover he gave the converse of Theorem 1.2 for the ℓ_1 space.

THEOREM 1.3. [7] *For a Banach X, X has the lifting property if and only if X is isomorphic to $\ell_1(\Gamma)$, for some index set Γ.*

Also we need to introduce an important definition that goes to the work of our main question for this lifting.

DEFINITION 1.4. Let X and Y be Banach spaces. A bounded linear operator $T : X \to Y$ is said to be *absolutely (1-)summing* if there is a constant $K > 0$ such that for all finite choice of $(x_i)_{i=1}^n$ in X,

$$(\sum_{i=1}^{n} \|Tx_i\|) \leq K \sup\{(\sum_{i=1}^{n} |x^*(x_i)|) : x^* \in B_{X^*}\}. \tag{1.2}$$

The least such constant $K > 0$ is denoted by $\pi_1(T)$ and is called the *absolutely summing norm* of T. Moreover if $T : X \to Y$ is an absolutely summing operator, then T is bounded and $\|T\| \leq \pi_1(T)$ since, for each $x \in X$

$$\|T(x)\| \leq \pi_1(T) \sup\{|x^*(x)| : x^* \in B_{X^*}\} = \pi_1(T)\|x\|. \tag{1.3}$$

The following theorem gives some equivalent descriptions of absolutely 1-summing operators in [2].

THEOREM 1.5. [2] *Any one of the following statements about a bounded linear operator implies all others.*
1) T is an absolutely summing operator.
2) T maps unconditionally convergent series in X into absolutely convergent series in Y.
3) There exists a constant $K > 0$ such that for any finite elements $x_1, x_2, \cdots, x_n \in X$, the following inequality obtains;

$$\sum_{i=1}^{n} \|Tx_i\| \leq K \sup\{\sum_{i=1}^{n} | < x_i, x^* > | : x^* \in B_{X^*}\} \tag{1.4}$$

2. Main results

The main purpose of this article is to find some conditions under which bounded linear operators on Banach spaces have lifting of operators. It is well known fact that each bounded linear operator on ℓ_1 has the lifting of operator in Theorem 1.2. So our major questions are based on theorem 1.2, that is which bounded linear operator on Banach space X into a Banach space Y can have a lifting of operator whether it may preserve the norm or not. In this direction of research, we knew that by imposing more conditions, we can find a lifting on some Banach space and on $L_1(\mu)$. Here we can give a theorem that any absolutely summing operator on the Banach space X with a unconditional basis can have a lifting but resulting lifting of operator does not norm preserving in [6].

THEOREM 2.1. [6] *Let X be a Banach space with an unconditional basis $(e_i)_{i\in\Gamma}$. For any Banach spaces Y and Z, let $T : X \to Y$ be an absolutely 1-summing operator. Then for any surjective linear map $\pi : Z \to Y$, there is a lifting operator $\tilde{T} : X \to Z$ such that $\pi \circ \tilde{T} = T$, $\|\tilde{T}\| \le \lambda\pi_1(T)$ and the following diagram commutes;*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \tilde{T}\downarrow & \nearrow \pi & \\ Z & & \end{array} \tag{2.1}$$

Now we introduce the $\mathcal{L}_p$-spaces which provides a more general framework for our results. It mainly involved local results to stress the fact that they depend only on the finite dimensional structure of the Banach space.

DEFINITION 2.2. Let $\lambda \ge 1$ and $1 \le p \le \infty$. A Banach space X is said to be an $\mathcal{L}_p^\lambda$*-space* if there exists a directed net of finite dimensional subspaces $(X_j)_{j\in\Lambda}$ such that for all $j \in \Lambda$, the Banach-Mazur distances $d(X_j, \ell_p^{dim X_j}) \le \lambda$ and $X = \overline{\cup_{j\in\Lambda} X_j}$. Simply the space X is called an $\mathcal{L}_p$*-space* if it is an $\mathcal{L}_p^\lambda$-space for some $\lambda \ge 1$. We know that $L_\infty(\mu)$ and $C(K)$-spaces are $\mathcal{L}_\infty^\lambda$-spaces and $L_p(\mu)$ are $\mathcal{L}_p-$space.

In [10], Lindenstrauss proved the existence of the lifting operator on $\mathcal{L}_1$ by conditioning the kernel of quotient map by a complemented subspace in its second dual as following. Refer to the proof of the theorem in [4].

THEOREM 2.3. ([10] and [4, pp.1726]) *Let Y and Z be Banach spaces such that there is a surjective operator $\pi : Z \to Y$. Suppose that $kernel(\pi)$ is a complemented subspace in its second dual. Let X be any $\mathcal{L}_1-$space. Then every bounded linear operator $T : X \to Y$ has a lifting operator $\tilde{T} : X \to Z$ such that $\pi \circ \tilde{T} = T$.*

Here we can have a question for which kind of linear operator T does the lifting exist if we are giving the other conditions? In Theorem 2.3, if the $K = kernel(\pi)$ is complemented in its second dual, this implies that the unit ball B_K is $weak^*$ compact in K^{**}. From this, if $kernel(\pi)$ is reflexive subspace of Y, we can give same argument for weak compactness of unit ball. Now we can give an answer for this question about a lifting operator by changing the conditions of absolutely summing operators instead of $K = kernel(\pi)$ having been complemented in its second dual.

Lemma 2.4. *Let X and Y be Banach spaces and let $(X_j)_{j\in\Lambda}$ be a net of subspaces of X directed by inclusion such that $X = \overline{\cup_{j\in\Lambda} X_j}$. Then if $T : X \to Y$ is an absolutely summing operator, then for any $j \in \Lambda$, $T|_{X_j}$ is absolutely summing on X_j where $T|_{X_j}$ is a restriction of T to X_j.*

Proof. Assume that $T : X \to Y$ is an absolutely summing operator and $X = \overline{\cup_{j\in\Lambda} X_j}$. Then for any finite elements $x_1, x_2, \cdots, x_n \in X_j$,

$$\sum_{i=1}^{n} \|T|_{X_j}(x_i)\| \leq \sum_{i=1}^{n} \|T(x_i)\|$$

$$\leq \pi_1(T) \sup\{\sum_{i=1}^{n} |x^*(x_i)| : x^* \in B_{X^*}\}. \tag{2.2}$$

Then we can say that $T|_{X_i}$ is absolutely summing and $\pi_1(T|_{X_j}) \leq \pi_1(T)$ for all $j \in \Lambda$. This proves the lemma. □

Let X be an $\mathcal{L}_1^\lambda-$space and $T : X \to Y$ be absolutely summing. By the definition of $\mathcal{L}_1^\lambda-$space, we can find a directed net of finite dimensional subspaces $(X_j)_{j\in\Lambda}$ such that for all $j \in \Lambda$, the Banach-Mazur distances $d(\ell_1^{dim X_j}, X_j) \leq \lambda$ and $X = \overline{\cup_{j\in\lambda} X_j}$. Then from Lemma 2.4, we can have for all $j \in \Lambda$, $T|_{X_j}$ is absolutely summing and $\pi_1(T|X_j) \leq \pi_1(T)$.

Lemma 2.5. *Let X be an $\mathcal{L}_1$-space such that $(X_j)_{j\in\Lambda}$ is a net of subspaces of X directed by inclusion and $X = \overline{\cup_{j\in\Lambda} X_j}$ with the Banach-Mazur distances $d(\ell_1^{dim X_j}, X_j) \leq \lambda$ and let Y be a Banach space. Then if $T : X \to Y$ is a bounded linear operator such that $T|_{X_j}$ are absolutely summing operators for all $j \in \Lambda$, then T is absolutely summing and $\pi_1(T) = \sup_{j\in\Lambda} \pi_1(T|_{X_j})$ where $T|_{X_j}$ is a restriction of T to X_j.*

Proof. Assume that for all $j \in \Lambda$, $T|_{X_j}$ are absolutely summing operators and let $C = \sup_{j\in\Lambda} \pi_1(T|_{X_j}) < \infty$.

Let $x_1, x_2, \cdots, x_n$ be finite elements of $\cup_{j\in\Lambda} X_j$ which is dense in X. Since (X_j) is directed by inclusion, $\{x_1, x_2, \cdots, x_n\}$ must be contained in X_j for some $j \in \Lambda$. Then

$$\sum_{i=1}^{n} \|T(x_i)\| = \sum_{i=1}^{n} \|T|_{X_j}(x_i)\|$$

$$\leq \pi_1(T|_{X_j}) \sup\{\sum_{i=1}^{n} |x^*(x_i)| : x^* \in B_{X^*}\}$$

$$\leq C \sup\{\sum_{i=1}^{n} |x^*(x_i)| : x^* \in B_{X^*}\}. \tag{2.3}$$

Since $\cup_{j\in\Lambda} X_j$ is dense in X, T can be extended continuously on whole space $X = \overline{\cup_{j\in\Lambda} X_j}$. Hence (2.3) is also true for any finite elements $x_1, x_2, \cdots, x_n \in X$. This implies T is an absolutely summing operator on X and $\pi_1(T) \leq \sup_{j\in\Lambda} \pi_1(T|_{X_j})$. On the other hand, since for each $j \in \Lambda$, $\pi_1(T|_{X_j}) \leq \pi_1(T)$, we have $\pi_1(T) \leq \sup_{j\in\Lambda} \pi_1(T|_{X_j})$. Hence we have $\sup_{j\in\Lambda} \pi_1(T|_{X_j}) = \pi_1(T)$. This proves the lemma. □

Theorem 2.6. *Let X, Y and Z be Banach spaces. Let $T : X \to Y$ be an absolutely summing operator and $q : Z \to Y$ be a surjective linear map from Z onto Y with* *norm 1. Then if $\tilde{T} : X \to Z$ is a lifting operator of T such that $q \circ \tilde{T} = T$, then $\tilde{T}$ is also an absolutely summing operator with $\pi_1(\tilde{T}) \le (1+\epsilon)\pi_1(T)$, for $\epsilon > 0$.*

Proof. Assume that $T : X \to Y$ is an absolutely summing operator with 1-summing norm $\pi_1(T) < \infty$. Then for any $x_1, x_2, \cdots, x_n \in X$, we have

$$(\sum_{i=1}^{n} \|T(x_i)\|) \le \pi_1(T) \sup\{(\sum_{i=1}^{n} |x^*(x_i)|) : x^* \in B_{X^*}\}. \tag{2.4}$$

Since $q : Z \to Y$ is a surjective map, for each $x_i \in X$ we can choose $z_i \in Z$ such that $q(z_i) = T(x_i)$ and $\|z_i\| \le (1+\epsilon)\|Tx_i\|$, for $\epsilon > 0$. Then since $\tilde{T}$ is a lifting of T and $q \circ \tilde{T} = T$, we can have $\tilde{T}(x_i) = z_i$ for all $i = 1, 2, \cdots, n$. Therefore for $\epsilon > 0$, we can see

$$\begin{aligned}
(\sum_{i=1}^{n} \|\tilde{T}(x_i)\|) &= (\sum_{i=1}^{n} \|z_i\|), \qquad (\|z_i\| \le (1+\epsilon)\|Tx_i\|) \\
&\le (1+\epsilon)(\sum_{i=1}^{n} \|Tx_i\|), \qquad (T \text{ is absolutely summing}) \\
&\le (1+\epsilon)\pi_1(T) \sup\{(\sum_{i=1}^{n} |x^*(x_i)|) : x^* \in B_{X^*}\}. \qquad (2.5)
\end{aligned}$$

Hence $\tilde{T} : X \to Z$ is an absolutely summing operator with $1-$summing norm $\pi_1(\tilde{T}) \le (1+\epsilon)\pi_1(T)$. This proves the theorem. □

Now by using idea of Lindenstrauss's proof in [10] and in [4], we can prove our main result for the variation of Lindenstrauss theorem by changing a bounded linear operator into an absolutely summing operator and replacing the quotient map $q : Y \to Y/Z$ where Z is a reflexive subspace of Y.

Theorem 2.7. *Let X be an $\mathcal{L}_1^\lambda-$space and Y be any Banach spaces such that Z is a reflexive subspace of Y. Assume that $T : X \to Y/Z$ is an absolutely summing operator and $q : Y \to Y/Z$ is a quotient map from Y onto Y/Z. Then there exists an absolutely summing operator $\tilde{T} : X \to Y$ that is a lifting operator of T such that $q \circ \tilde{T} = T$ with $\pi_1(\tilde{T}) \le (2+\lambda)(1+\epsilon)\pi_1(T)$, for $\epsilon > 0$.*

Proof. Let X be an $\mathcal{L}_1^\lambda-$space. Then $X = \overline{\cup_{j\in\Lambda} X_j}$ where X_j is a finite dimensional subspace of X and for all $j \in \Lambda$, the Banach-Mazur distances $d(X_j, \ell_1^{dim(X_j)}) \le \lambda$. Let $T|_{X_j} = T_j$ be the restriction of T on X_j, for each $j \in \Lambda$. Then by Lemma 2.4, for all $j \in \Lambda$, T_j is an absolutely summing operator on X_j with summing norm $\pi_1(T_j) \le \pi_1(T)$.

By the definition of $\mathcal{L}_1^\lambda-$space, let $S_j : \ell_1^{dim(X_j)} \to X_j$ be an isomorphism such that $\|S_j\|\|S_j^{-1}\| \le \lambda$ with $\|S_j\| = 1$ and $\|S_j^{-1}\| \le \lambda$. Then we can have the following diagram.

$$\begin{array}{ccccc}
\ell_1^{dim(X_j)} & \overset{S_j}{\rightleftarrows} & X_j & \overset{T_j}{\longrightarrow} & Y/Z \\
 & \searrow & \downarrow & \nearrow q & \\
 & & Y & &
\end{array} \tag{2.6}$$

where $T_j : X_j \to Y/Z$ is an absolutely summing operator and $q : Y \to Y/Z$ is a quotient map.

Define $R_j = T_j \circ S_j$ for each $j \in \Lambda$. Then by the ideal properties of absolutely summing operators, each R_j is also an absolutely summing operator. Since $\ell_1^{dim(X_j)}$ has the lifting property, we can find a lifting $\tilde{R_j}$ of R_j such that $q \circ \tilde{R_j} = R_j$ for all $j \in \Lambda$. Then we have $q \circ \tilde{R_j} = R_j = T_j \circ S_j$ and by Theorem 2.6,

$$\begin{aligned} \pi_1(\tilde{R_j}) &\le (1+\epsilon)\pi_1(R_j) \\ &\le (1+\epsilon)\pi_1(T_j)\|S_j\| \\ &= (1+\epsilon)\pi_1(T_j). \end{aligned} \tag{2.7}$$

Now define $\tilde{T_j} = \tilde{R_j} \circ S_j^{-1}$ as a lifting of T_j. Then we have $\tilde{T_j} : X_j \to Y$ and

$$\begin{aligned} \pi_1(\tilde{T_j}) &= \pi_1(\tilde{R} \circ S_j^{-1}) \\ &\le \pi_1(\tilde{R})\|S^{-1}\|, \qquad (\because \|S^{-1}\|) \le \lambda) \\ &\le (1+\epsilon)\lambda\pi_1(T_j) \\ &\le (1+\epsilon)\lambda\pi_1(T). \end{aligned} \tag{2.8}$$

Then for all $j \in \Lambda$, $\tilde{T_j} : X_j \to Y$ are absolutely summing operators with $\pi_1(\tilde{T_j}) \le (1+\epsilon)\lambda\pi_1(T)$.

Now we find a lifting $\tilde{T}$ of T such that $q \circ \tilde{T} = T$. From the definition of a quotient map q, for each $x \in X$, let $\phi : X \to Y$ be a map defined by $\phi(x) = y$ such that

i) $q(y) = T(x)$,
ii) $\|y\| \le (1+\epsilon)\|T(x)\| \le (1+\epsilon)\pi_1(T)\|x\|$, by open mapping theorem.
iii) $\phi(kx) = k\phi(x)$, for all $x \in X$, $k \in \mathbf{R}$.

For $j \in \Lambda$, define ψ_j by

$$\begin{aligned} \psi_j(x) &= \tilde{T_j}(x) - \phi(x) \quad \text{if } x \in X_j \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{2.9}$$

Then

$$\begin{aligned} q \circ \psi_j(x) &= q \circ (\tilde{T_j}(x) - \phi(x)) \\ &= q \circ \tilde{T_j}(x) - q \circ \phi(x) \\ &= T_j(x) - q(y) \quad (\text{where } y = \phi(x)) \\ &= T_j(x) - T(x) \quad \text{if } x \in X_j \\ &= 0. \end{aligned} \tag{2.10}$$

Hence this implies an element is $\tilde{T_j}(x) - \phi(x) \in Z$ and moreover

$$\begin{aligned} \|\tilde{T_j}(x) - \phi(x)\| &\le \|\tilde{T_j}\|\|x\| + \phi(x) \\ &\le (1+\epsilon)\lambda\pi_1(T)\|x\| + (1+\epsilon)\pi_1(T)\|x\| \\ &= (1+\epsilon)(1+\lambda)\pi_1(T)\|x\|. \end{aligned} \tag{2.11}$$

Let $K_x = (1+\epsilon)(1+\lambda)\pi_1(T)\|x\|$. Then $\psi_j(x) \in K_x \cdot B_Z$ where B_Z is the unit ball of Z. Now consider the following product space

$$\prod_{x\in X} K_x \cdot B_Z \tag{2.12}$$

Since Z is a reflexive Banach subspace of Y, B_Z is compact for the weak topology. Hence by the Tychonoff's theorem $\prod_{x\in X} K_x \cdot B_Z$ is compact for the weak topology. Then $(\psi_j)_{j\in\Lambda}$ has a convergent subnet. Let (ψ_i) be a such convergent subnet of $(\psi_j)_{j\in\Lambda}$ in the topology of pointwise convergent on X and taking the weak topology on Y. Let Ψ be the limit point of (ψ_i). Since $\cup_{j\in\Lambda}X_j$ is dense in X and the net $(X_j)_{j\in\Lambda}$ is directed by inclusion. Then if for every $x_1, x_2 \in X_{i_0}$ for some i_0, then for the inclusion $X_i \supset X_{i_0}$, we have

$$\begin{aligned}\psi_i(x_1+x_2) - \psi_i(x_1) - \psi_i(x_2) &= \tilde{T}_i(x_1+x_2) - \phi(x_1+x_2) - \\ &\qquad \tilde{T}_i(x_1) + \phi(x_1) - \tilde{T}_i(x_2) + \phi(x_2) \\ &= -\phi(x_1+x_2) + \phi(x_1) + \phi(x_2) \quad \text{by linearity of } \tilde{T}_i.\end{aligned} \tag{2.13}$$

Hence in the weak limit, we can have $\Psi(x_1+x_2) - \Psi(x_1) - \Psi(x_2) = -\phi(x_1+x_2) + \phi(x_1) + \phi(x_2)$.

Finally, if we define $\tilde{T}(x) = \phi(x) + \Psi(x)$, then $\tilde{T}$ is additive. On the other hand, we can show that for each scalar $k \in \mathbf{R}$, $\tilde{T}(kx) = k\tilde{T}(x)$ as same as for additivity. This proves the $\tilde{T} : X \to Y$ is a linear map. For the boundedness of $\tilde{T}$, we can see

$$\begin{aligned}\|\tilde{T}(x)\| &\le \|\phi(x)\| + \|\Psi(x)\| \\ &\le (1+\epsilon)\pi_1(T)\|\|x\| + K_x \qquad (K_x = (1+\epsilon)(1+\lambda)\pi_1(T)\|x\|) \\ &= ((1+\epsilon)(2+\lambda))\pi_1(T)\|x\| \\ &= M \cdot \|x\| \quad \text{where } M = ((1+\epsilon)(2+\lambda))\pi_1(T).\end{aligned} \tag{2.14}$$

Hence $\tilde{T}$ is a bounded linear operator from X into Y. Then by Theorem 2.6, $\tilde{T} : X \to Y$ is an absolutely summing operator and since $q : Y \to Y/Z$ is a quotient map,

$$\begin{aligned}\pi_1(\tilde{T}) &\le (1+\epsilon)(2+\lambda)\sup_{j\in\Lambda}(T_j) \\ &= (1+\epsilon)(2+\lambda)\pi_1(T).\end{aligned} \tag{2.15}$$

Again we can easily see that $q \circ \tilde{T} = T$ by the density of $\cup_{j\in\Lambda}X_j$ and a lifting $\tilde{T}_j$ on X_j. This proves the theorem. □

References

[1] J. Diestel and J. J. Uhl. Jr. *Vector Measure*, Math. Surveys Monographs 15, AMS, 1977, Providence RI.

[2] J. Diestel, H. Jarchow and A. Tonge *Absolutely summing operators*, Cambridge Univ. Press (1995).

[3] A. Grothendieck *Une caraterisation vectorielle-métrique* L_1, Canad J. Math. **7** (1955), 552–562 MR17.

[4] W.B. Johnson and J. Lindenstrauss, *Handbook of the Geometry of Banach Spaces*, Vol. 2, Elsevier Science B.V.(2001).

[5] W.B. Johnson and M. Zippin, *Extension of operators from subspaces of* $c_0(\Gamma)$ *into* $C(\mathbf{K})$ *space*, Proc. Amer. Math. Soc. **107** (1989), 751–754.
[6] J.H. Kang, *Lifting on G.T. Banach Spaces with unconditional basis*, International J. Math. Analysis **10** (14) (2016), 677–686.
[7] G. Köthe *Hebbare lokakonvex Raumes* , Math. Ann., 4651. **vol 165** (1993), 188–195.
[8] D. R. Lewis and C. Stegall, *Banach spaces whose duals are isomorphic to* $\ell^1(\Gamma)$, J. Functional Analysis, **12** (1973), 177–187, Mr49.
[9] J. Lindenstrauss, *A remark on* ℓ_1 *spaces*, Israel J. Math. **8** (1970), 80–82.
[10] J. Lindensrauss, *On a certain subspace of* ℓ_1, Bull. Pol.Sci. **12** (9) (1964), 539–542.
[11] J. Lindenstrauss and A. Pelczyński, *Contributions to the theory of classical Banach spaces*, J. Funct. Anal. **8** (1971), 225–249.
[12] J. Lindenstrauss and A. Pelczyński, *Absolutely summing operators in* $\mathcal{L}_p-$*spaces and their applications*, Studia Math. **29** (1968), 275–326.
[13] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I,*, Springer-Verlag, Berlin and New York 1977.
[14] A. Pelczynski, *Projections in certain Banach spaces*, Studia Math. **19** (1960), 209–228.
[15] G. Pisier *Factorizations of linear operators and geometry of Banach spaces*, C.B.M.S. Amer. Math.Soc. **60** (1985).
[16] M. Zippin, *Applications of E. Michael's continuous selection theorem to operator extension problems.* Proc. Amer. Soc. **127** (1999).

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