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A REMARK ON p-CLASS NUMBERS IN CYCLOTOMIC $\mathbb{Z}_p$-EXTENSIONS

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Abstract. Let $\mathbb{B}_{p,n}$ be the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension over the rationals. In this note, we will prove that the p-class group of $\mathbb{B}_{p,n}$ is trivial for all n by elementary method.

Let $\mathbb{B}_{p,n}$ be the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension over the rationals, i.e., the unique real subfield of the cyclotomic field $\mathbb{Q}(\zeta_{p_{n+1}})$ that has degree p_n over $\mathbb{Q}$ for primes p. It has been expected that the class groups of these fields are all trivial, but very little is known. But for any fixed prime p, it was shown by Iwasawa for the p-class groups of these fields (See [1]). He generalized Weber's result on $p = 2$, which had been used in the proof of Kronecker-Weber's theorem. In fact, what Iwasawa proved covers more general result. In this note, we will prove that the p-class group of $\mathbb{B}_{p,n}$ is trivial for all n by elementary method. We will prove by the induction on n. It is well known that the class group of the rational number field $\mathbb{Q}$ is trivial, i.e., the p-class group of $\mathbb{Q}$ is also trivial. Suppose that the p-class group of $\mathbb{B}_{p,k}$ is trivial. The remaining task is to show that the p-class group of $\mathbb{B}_{p,k+1}$ is trivial. Let us assume that the p-part of the class group of $\mathbb{B}_{p,k+1}$ is nontrivial, i.e., $\mathrm{Cl}_p(\mathbb{B}_{p,k+1}) \simeq P$, for some abelian p-group P. By class field theory, $\mathbb{B}_{p,k+1}$ has nontrivial maximal p-unramified extension K, where $\mathrm{Gal}(K/\mathbb{B}_{p,k+1}) \simeq P$ (By the maximality, K is also Galois over $\mathbb{Q}$.) Since $K/\mathbb{Q}$ is a Galois extension, $K/\mathbb{B}_{p,k}$ is also a Galois extension. Let $|P|$ be p^m. Then $|\mathrm{Gal}(K/\mathbb{B}_{p,k})|$ is equal to p^{m+1} (One can easily check that every p-group P has a normal subgroup of order every divisor of $|P|$.). We know that $\mathrm{Gal}(K/\mathbb{B}_{p,k})$ has a normal subgroup N of order p^{m-1}. Let L be the fixed field on $K/\mathbb{B}_{p,k}$ by N. Then $|\mathrm{Gal}(L/\mathbb{B}_{p,k})|$ is p^2. Note that the ramification index of $\mathrm{Gal}(L/\mathbb{B}_{p,k})$ is at most p since $K/\mathbb{B}_{p,k+1}$ is unramified. Let I_p be the inertia subgroup of $\mathrm{Gal}(L/\mathbb{B}_{p,k})$ where P is the unique prime ideal of $\mathbb{B}_k$ above p. On the other hand, since $\mathrm{Gal}(L/\mathbb{B}_{p,k})$ is abelian, I_p is a normal subgroup of $\mathrm{Gal}(L/\mathbb{B}_{p,k})$. Let F be a subfield fixed by I_p. This means that $F/\mathbb{B}_{p,k}$ is an unramified p-extension. This contradicts to the triviality of p-class group of $\mathbb{B}_{p,k}$.

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