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ON SPIRALLIKE FUNCTIONS RELATED TO BOUNDED RADIUS ROTATION

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Dedicated to the memory of Professor Yaşar POLATOĞLU

Abstract. In the present paper, we prove the growth and distortion theorems for the spirallike functions class $\mathcal{S}_k(\lambda)$ related to boundary radius rotation, and by using the distortion result, we get an estimate for the Gaussian curvature of a minimal surface lifted by a harmonic function whose analytic part belongs to the class $\mathcal{S}_k(\lambda)$. Moreover, we determine and draw the minimal surface corresponding to the harmonic Koebe function.

1. Introduction

A complex-valued function f which is harmonic in a simply connected domain $\mathcal{D} \subset \mathbb{C}$ has the canonical representation $f = h + \overline{g}$, where h the analytic and g the co-analytic part of f in $\mathcal{D}$ with $g(z_0) = 0$ for some prescribed point $z_0 \in \mathcal{D}$. According to a theorem by Lewy [10], f is locally univalent if and only if its Jacobian $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 = |h^{'}(z)|^2 - |g^{'}(z)|^2$ does not vanish. The function f is said to be sense-preserving if its Jacobian is positive. In this case, then $h^{'}(z) \neq 0$ and the analytic function $w(z) = g'(z)/h'(z)$, called the second dilatation of f, has the property $|w(z)| < 1$ for all $z \in \mathcal{D}$. Throughout this paper we will assume that f is locally univalent, sense -preserving harmonic mapping, and $\mathcal{D} = \mathbb{D} \subset \mathbb{C}$, with $z_0 = 0$, where $\mathbb{D} := \{z : |z| < 1\}$ is the open unit disc on the complex plane.

Let $\mathcal{H}$ denote the family of continuous complex-valued sense-preserving harmonic functions $f = h + \overline{g}$ in the open unit disc $\mathbb{D}$. Clunie and Sheil-Small [3] introduced the class $\mathcal{S}_{\mathcal{H}}$ which is univalent and a subclass of $\mathcal{H}$ with $h(0) = g(0) = h'(0)-1 = 0$, and also introduced its subclass ${\mathcal{S}_{\mathcal{H}}}^0$ with $g'(0) = 0$.

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The function $f = h + \overline{g}$ in $\mathbb{D}$ is determined by the power series expansions

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad (z \in \mathbb{D})$$

where $a_n \in \mathbb{C}$ $(n = 0, 1, ...)$ and $b_n \in \mathbb{C}$ $(n = 1, 2...)$. Thus if $f \in \mathcal{S}_{\mathcal{H}}$, we have $a_0 = 0, a_1 = 1$. Moreover, we also have $g'(0) = b_1$ with $|b_1| = a$ $(0 \leq a < 1)$.

Let Ω be the family of functions ϕ which are analytic and satisfying the conditions $\phi(0) = 0$, and $|\phi(z)| < 1$ for every $z \in \mathbb{D}$; and let $\Omega(a)$, where $0 \leq a < 1$, be the class of functions w which are analytic in $\mathbb{D}$ and satisfy $w(0) = a$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$. We let $\Omega_{\cup}$ be the union of all classes $\Omega(a)$ where a ranges over $0 \leq a < 1$. In view of the relation $w = g'/h'$ with $|b_1| = a$, if $w \in \Omega_{\cup}$ then we have

$$\frac{|a - r|}{1 - ar} \leq |w(z)| \leq \frac{a + r}{1 + ar} \tag{1}$$

for all $z \in \mathbb{D}$.

In 1971, Pinchuk [12] introduced and studied the classes $\mathcal{P}_k$ and $\mathcal{R}_k$. Here, $\mathcal{P}_k$ denotes the class of analytic functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ in $\mathbb{D}$ with $p(0) = 1$ and having the representation

$$p(z) = \frac{1}{2} \int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t),$$

where μ is a real-valued function of bounded variation on $[-\pi, \pi]$ such that for $k \geq 2$

$$\int_{-\pi}^{\pi} d\mu(t) = 2 \quad \text{and} \quad \int_{-\pi}^{\pi} |d\mu(t)| \leq k.$$

Clearly, $\mathcal{P}_2 \equiv \mathcal{P}$ where $\mathcal{P}$ is the class of analytic functions with positive real part. Then, $p \in \mathcal{P}_k$ if and only if there exists $p_1, p_2 \in \mathcal{P}$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad (z \in \mathbb{D}).$$

The class $\mathcal{R}_k$, defined by Pinchuk [12], consists of those functions h which satisfy the condition

$$\int_{-\pi}^{\pi} \left| \operatorname{Re} \left(re^{i\theta} \frac{h'(re^{i\theta})}{h(re^{i\theta})} \right) \right| d\theta \leq k\pi, \quad (k \geq 2, 0 < r < 1, z = re^{i\theta}).$$

Geometrically, the condition is that the total variation of angle between radius vector $h(re^{i\theta})$ makes with positive real axis is bounded by $k\pi$. Thus $\mathcal{R}_k$ is the class of functions of bounded radius rotation bounded by $k\pi$. Pinchuk [12] showed that

$$h \in \mathcal{R}_k \text{ if and only if } z\frac{h'(z)}{h(z)} \in \mathcal{P}_k.$$

Denote by $\mathcal{S}_k(\lambda)$ the class of spirallike functions related to bounded radius rotation is defined by

$$(2)\quad \mathcal{S}_k(\lambda)=\left\{h\in\mathcal{A}: e^{i\lambda}z\frac{h'(z)}{h(z)}=\cos\lambda p(z)+i\sin\lambda,\ p\in\mathcal{P}_k, k\geq 2, |\lambda|<\frac{\pi}{2}\right\},$$

where $\mathcal{A}$ is the class of functions of the form

$$(3)\qquad h(z)=z+\sum_{n=2}^{\infty}a_n z^n,$$

which are analytic in $\mathbb{D}$. It is noted that for $\lambda=0$, we get $\mathcal{S}_k(0)\equiv\mathcal{R}_k$ and for $k=2$, we have the class $\mathcal{S}_2(\lambda)\equiv\mathcal{S}^{\lambda}$, which was introduced and studied by Spacek in 1932 (see [14]). In fact, for $k=2$ and $\lambda=0$, we get the class $\mathcal{S}_2(0)\equiv\mathcal{S}^*$ of all starlike functions in $\mathbb{D}$; for further details, one may refer to [6].

In this paper, we introduce harmonic mappings for which the analytic part is the spirallike function with bounded radius rotation.

Definition 1.1. *Denote by $\mathcal{S}_k^{\mathcal{H}}(\lambda)$ the subclass of $\mathcal{S}_{\mathcal{H}}$ consisting of all harmonic mappings of the form $f=h+\overline{g}$ for which $h\in\mathcal{S}_k(\lambda)$ with normalization $h(0)=g(0)=h'(0)-1=0$ and $g'(0)=b_1$ with $|b_1|=a$.*

Minimal surfaces are most commonly known as those which have the minimum area amongst all other surfaces spanning a given closed curve in $\mathbb{R}^3$. Geometrically, the definition of a minimal surface is that the mean curvature H is zero at every point of the surface. If locally one can write the minimal surface in $\mathbb{R}^3$ as $(x,y,\Phi(x,y))$, then the minimal surface equation $H=0$ is equivalent to

$$(1+\Phi_y^2)\Phi_{xx}-2\Phi_x\Phi_y\Phi_{xy}+(1+\Phi_x^2)\Phi_{yy}=0.$$

There exists a choice of isothermal parameters $(u,v)\in\Omega\subset\mathbb{R}^2$ so that the surface $X(u,v)=(x(u,v),y(u,v),\Phi(u,v))\in\mathbb{R}^3$ satisfying the minimal surface equation is given by

$$E=|X_u|^2=|X_v|^2=G>0,\quad F=<X_u,X_v>=0,\quad \Delta_{(u,v)}X=0$$

where Δ denotes the Laplacian operator. The general solution of such an equation is called the local Weierstrass-Enneper representation [4].

A harmonic mapping $f=h+\overline{g}$ can be lifted locally to a regular minimal surface given by conformal (or isothermal) parameters if and only if its dilatation is the square of an analytic function $w(z)=q^2(z)$ for some analytic function q with $|q(z)|<1$. Equivalently, the requirement is that any zero of w be of even order, unless $w\equiv 0$ on its domain, so that there is no loss of generality in supposing that z ranges over the unit disc $\mathbb{D}$, because any other isothermal representation can be precomposed with a conformal map from the unit disc $\mathbb{D}$ whose existence is guaranteed by the Riemann mapping theorem. For such a harmonic mapping $f = u+iv$, the minimal surface has the Weierstrass-Enneper representation with parameters (u, v, t) given by

$$\begin{aligned} u &= \operatorname{Re}\left\{\int_0^z \varphi_1(\zeta)d\zeta\right\} = \operatorname{Re}\{f(z)\}, \\ v &= \operatorname{Re}\left\{\int_0^z \varphi_2(\zeta)d\zeta\right\} = \operatorname{Im}\{f(z)\}, \\ t &= \operatorname{Re}\left\{\int_0^z \varphi_3(\zeta)d\zeta\right\} \end{aligned}$$

for $z \in \mathbb{D}$ with

$$\varphi_1 = h' + g' = p(1+q^2) = \frac{\partial u}{\partial z}, \tag{4}$$

$$\varphi_2 = -i(h' - g') = -ip(1-q^2) = \frac{\partial v}{\partial z}, \tag{5}$$

$$\varphi_3 = \sqrt{-4w(h')^2} = -2ipq = \frac{\partial t}{\partial z}. \tag{6}$$

By using (4)∼ (6), we get

$$w = \frac{g'}{h'} = q^2, \quad h' = p \quad \textit{and} \quad g' = pq^2 \ . \tag{7}$$

The metric of the surface has the form $ds = \lambda|dz|$, where $\lambda = \lambda(z) > 0$. Here, the function λ takes the form

$$\lambda = |h'| + |g'| = |h'|(1 + |w|) = |p|(1 + |q|^2) \ . \tag{8}$$

A classical theorem of differential geometry says that if a regular surface is represented by conformal parameters (or isothermal parameters) so that its metric has the form $ds = \lambda|dz|$ for some positive function λ, then the Gaussian curvature of the surface is $K = -\lambda^{-2}\Delta(\log\lambda)$. Here, if we use (8) in this formula, we obtain

$$K = -\frac{4|q'|^2}{|p|^2(1+|q|^2)^4} \ . \tag{9}$$

Next, using (7) in (9), we get an another expression of K as follows:

$$K = -\frac{|w'|^2}{|h'g'|(1+|w|)^4} \ . \tag{10}$$

For more details, we refer to the book by Duren [7].

2. Main Results

Lemma 2.1. [13] *If p is an element of $\mathcal{P}_k$, then*

$$\left|p(z) - \frac{1+r^2}{1-r^2}\right| \le \frac{kr}{1-r^2}.$$

Lemma 2.2. *If h of the form (3) is an element of $\mathcal{S}_k(\lambda)$, then*

$$\left| z\frac{h'(z)}{h(z)} - \frac{1+r^2e^{-2i\lambda}}{1-r^2}\right| \leq \frac{kr\cos\lambda}{1-r^2}. \tag{11}$$

Proof. Since $h \in \mathcal{S}_k(\lambda)$, from (2) we get

$$p(z) = \frac{1}{\cos\lambda}\left(e^{i\lambda}z\frac{h'(z)}{h(z)} - i\sin\lambda\right).$$

Using this relation in Lemma 2.1 and simplifying, we obtain (11). □

We next give the growth and distortion results for the class $\mathcal{S}_k(\lambda)$.

Theorem 2.3. *If $h \in \mathcal{S}_k(\lambda)$, then for $|z| = r < 1$, we have*
a) Growth theorem:

$$\frac{r}{(1-r^2)^{\cos^2\lambda}}\left(\frac{1-r}{1+r}\right)^{\frac{k}{2}\cos\lambda} \leq |h(z)| \leq \frac{r}{(1-r^2)^{\cos^2\lambda}}\left(\frac{1+r}{1-r}\right)^{\frac{k}{2}\cos\lambda}. \tag{12}$$

b) Distortion theorem:

$$\Gamma(\lambda, k, -r) \leq |h'(z)| \leq \Gamma(\lambda, k, r), \tag{13}$$

where

$$\Gamma(\lambda, k, r) = \frac{1 + r^2\cos 2\lambda + kr\cos\lambda}{(1-r^2)^{1+\cos^2\lambda}}\left(\frac{1+r}{1-r}\right)^{\frac{k}{2}\cos\lambda}. \tag{14}$$

These results are sharp.

Proof. (a): Since $h \in \mathcal{S}_k(\lambda)$, in view of Lemma 2.2 and applying some routine calculations, we obtain

$$\frac{1-(k\cos\lambda)r + (\cos 2\lambda)r^2}{1-r^2} \leq \operatorname{Re}\left(z\frac{h'(z)}{h(z)}\right) \leq \frac{1+(k\cos\lambda)r + (\cos 2\lambda)r^2}{1-r^2}. \tag{15}$$

On the other hand, we have

$$\operatorname{Re}\left(z\frac{h'(z)}{h(z)}\right) = r\frac{\partial}{\partial r}\log|h(z)|.$$

Therefore, (15) can be written in the form

$$\frac{1-(k\cos\lambda)r + (\cos 2\lambda)r^2}{r(1-r)(1+r)} \leq \frac{\partial}{\partial r}\log|h(z)| \leq \frac{1+(k\cos\lambda)r + (\cos 2\lambda)r^2}{r(1-r)(1+r)}. \tag{16}$$

Integrating both sides of (16) from 0 to r, we get (12). This completes the proof of the part (a).

(b): In order to prove the distortion theorem, we will apply Lemma 2.2. Hence, (11) may be written in the form

$$\frac{|1+r^2e^{-2i\lambda}| - kr\cos\lambda}{1-r^2} \leq \left|z\frac{h'(z)}{h(z)}\right| \leq \frac{|1+r^2e^{-2i\lambda}| + kr\cos\lambda}{1-r^2}. \tag{17}$$

By using the right sides of (12) and (17), we obtain

$$|h'(z)| \leq \frac{1+r^2\cos 2\lambda + +kr\cos\lambda}{(1-r^2)^{1+\cos^2\lambda}}\left(\frac{1+r}{1-r}\right)^{\frac{k}{2}\cos\lambda}.$$

Similarly, by using the left sides of (12) and (17), we get

$$|h'(z)| \geq \frac{1+r^2\cos 2\lambda - kr\cos\lambda}{(1-r^2)^{1+\cos^2\lambda}}\left(\frac{1-r}{1+r}\right)^{\frac{k}{2}\cos\lambda}.$$

Combining both of the above inequalities, we get (13) where $\Gamma(\lambda,k,r)$ is given by (14). This completes the proof of the part (b). □

The results given in Theorem 2.3 are sharp because extremal function is

$$\begin{aligned} h(z) &= z\left(\frac{(1+z)^{\frac{k}{2}-1}}{(1-z)^{\frac{k}{2}+1}}\right)^{e^{-i\lambda}\cos\lambda} \\ &= z + ke^{-i\lambda}\cos\lambda z^2 + \left(\frac{1}{2}k^2e^{-2i\lambda}\cos^2\lambda + e^{-i\lambda}\cos\lambda\right)z^3 + \ldots \;. \end{aligned}$$

Such functions are belong to the class $\mathcal{S}_k(\lambda)$. It is worth to note that for $k=2$, this function reduces to the $\lambda-$Spiral Koebe function

$$h(z) = \frac{z}{(1-z)^{2e^{-i\lambda}\cos\lambda}} \in \mathcal{S}^\lambda,$$

and for $\lambda = 0, k = 2$, the function reduces to the well-known Koebe function.

Putting $k = 2$ in (12), we obtain the following result.

Remark 2.4. [11] *If $h \in \mathcal{S}^\lambda$, then the growth theorem is*

$$r\left(\frac{(1-r)^{1-\cos\lambda}}{(1+r)^{1+\cos\lambda}}\right)^{\cos\lambda} \leq |h(z)| \leq r\left(\frac{(1+r)^{1-\cos\lambda}}{(1-r)^{1+\cos\lambda}}\right)^{\cos\lambda}.$$

This result is sharp for the $\lambda-$Spiral Koebe function.

Setting $\lambda = 0, k = 2$ in (12) and (13), respectively, we obtain the following known growth and distortion theorems.

Remark 2.5. [9] *If $h \in \mathcal{S}^*$, then*

$$\frac{r}{(1+r)^2} \leq |h(z)| \leq \frac{r}{(1-r)^2},$$

$$\frac{1-r}{(1+r)^3} \leq |h'(z)| \leq \frac{1+r}{(1-r)^3}.$$

These results are sharp for the Koebe function.

The rest of this section includes several results for the harmonic function class $\mathcal{S}_k^{\mathcal{H}}(\lambda)$. Hence, before move forward, we need to present the following lemma which was proved by Taştan and Polatoğlu in [15].

Lemma 2.6. *If* $w \in \Omega_{\cup}$, *then*

$$\frac{(1-a)(1-r)}{1+ar} \le (1-|w(z)|) \le \frac{1-ar-|a-r|}{1-ar} \quad , \tag{18}$$

$$\frac{1-ar+|a-r|}{1-ar} \le 1+|w(z)| \le \frac{(1+a)(1+r)}{1+ar} \quad , \tag{19}$$

$$\frac{(1+a)(1-r)}{1-ar} \le |1+w(z)| \le \frac{(1+a)(1+r)}{1+ar}, \tag{20}$$

and

$$\frac{(1-a)(1-r)}{1+ar} \le |1-w(z)| \le \frac{(1-a)(1+r)}{1-ar} \quad . \tag{21}$$

Theorem 2.7. *Let* $f = h + \overline{g}$ *be an element of* $\mathcal{S}_k^{\mathcal{H}}(\lambda)$, *then*

$$\frac{|a-r|\Gamma(\lambda,k,-r)}{1-ar} \le |g'(z)| \le \frac{(a+r)\Gamma(\lambda,k,r)}{1+ar}, \tag{22}$$

and

$$\int_0^r \frac{|a-\rho|\Gamma(\lambda,k,-\rho)}{1-a\rho} d\rho \le |g(z)| \le \int_0^r \frac{(a+\rho)\Gamma(\lambda,k,\rho)}{1+a\rho} d\rho. \tag{23}$$

Proof. Since the second dilatation is $w(z) = g'(z)/h'(z)$, then by using (1) and (13) we get the assertion (22).

If g is univalent and $m'(r) \le |g'(z)| \le M'(r)$ $(0 \le |z| = r < 1)$, then $\int_0^r m'(r)dr \le |g(z)| \le \int_0^r M'(r)dr$. Applying this together with the assertion (22), we get (23). □

Corollary 2.8. *Let* $f = h + \overline{g}$ *be an element of* $\mathcal{S}_k^{\mathcal{H}}(\lambda)$, *then*

$$(\Gamma(\lambda,k,-r))^2\left(1-\left(\frac{|a-r|}{1-ar}\right)^2\right) \le J_f(z) \le (\Gamma(\lambda,k,r))^2\left(1-\left(\frac{a+r}{1+ar}\right)^2\right).$$

Proof. Since Jacobian is

$$J_f(z) = |h'(z)|^2 - |g^{'}(z)|^2 = |h'(z)|^2(1-|w(z)|^2),$$

then using (1) and (13) we get the result. □

Corollary 2.9. *Let* $f = h + \overline{g}$ *be an element of* $\mathcal{S}_k^{\mathcal{H}}(\lambda)$, *then*

$$\int_0^r \Gamma(\lambda,k,-\rho)\left(1-\frac{|a-\rho|}{1-a\rho}\right)d\rho \le |f(z)| \le \int_0^r \Gamma(\lambda,k,\rho)\left(1+\frac{a+\rho}{1+a\rho}\right)d\rho.$$

Proof. Since total differential of f is

$$(|h'(z)| - |g'(z)|)|dz| \le |df(z)| \le (|h'(z)| + |g'(z)|)|dz| \Rightarrow$$

$$|h'(z)|(1-|w(z)|)|dz| \le |df(z)| \le |h'(z)|(1+|w(z)|)|dz|,$$

then setting (1) and (13) into the last inequality, and integrating both sides, we get the result. □

By (4)∼(6) and (18)∼(21), we prove the following result.

Theorem 2.10. *Let the functions φ_m $(m = 1, 2, 3)$ be the Weierstrass-Enneper parameters of a regular minimal surface $f = h + \overline{g} \in \mathcal{S}_k^{\mathcal{H}}(\lambda)$. Then*

$$\frac{(1+a)(1-r)\Gamma(\lambda,k,-r)}{1-ar} \leq |\varphi_1| \leq \frac{(1+a)(1+r)\Gamma(\lambda,k,r)}{1+ar}, \tag{24}$$

$$\frac{(1-a)(1-r)\Gamma(\lambda,k,-r)}{1+ar} \leq |\varphi_2| \leq \frac{(1-a)(1+r)\Gamma(\lambda,k,r)}{1-ar}, \tag{25}$$

and

$$\frac{4|a-r|(\Gamma(\lambda,k,-r))^2}{1-ar} \leq |\varphi_3|^2 \leq \frac{4(a+r)(\Gamma(\lambda,k,r))^2}{1+ar} \ . \tag{26}$$

Next, we give a growth theorem for the Gaussian curvature K.

Theorem 2.11. *Let K be the Gaussian curvature of a minimal surface with isothermal parameters lifted by a harmonic function $f = h + \overline{g} \in \mathcal{S}_k^{\mathcal{H}}(\lambda)$. Then we have*

$$\begin{aligned} |K| &\geq \frac{|w'|^2(1+ar)^5}{(\Gamma(\lambda,k,r))^2(a+r)(1+a)^4(1+r)^4} \\ |K| &\leq \frac{(1-ar-|a-r|)^2(1-ar)^3}{(\Gamma(\lambda,k,-r))^2|a-r|(1+ar)^2(1+a)^2(1-r)^6}. \end{aligned} \tag{27}$$

Proof. By using (13), (20) and (22), we have

$$\frac{|\omega'|^2(1+ar)^5}{(\Gamma(\lambda,k,r))^2(a+r)(1+a)^4(1+r)^4} \leq |K| \leq \frac{|\omega'|^2(1-ar)^5}{(\Gamma(\lambda,k,-r))^2|a-r|(1+a)^4(1-r)^4} \tag{28}$$

from (10). Here, if we use the Schwarz-Pick's Lemma for the function

$$\frac{w(z)-w(0)}{1-\overline{w(0)}w(z)},$$

we obtain

$$|w'|^2 \leq \frac{(1-|w(z)|^2)^2}{(1-r^2)^2} \ . \tag{29}$$

Using the inequality (29) in the inequality (28), we arrive

$$\frac{|w'|^2(1+ar)^5}{(\Gamma(\lambda,k,r))^2(a+r)(1+a)^4(1+r)^4} \leq |K| \leq \frac{(1-|w(z)|^2)^2(1-ar)^5}{(\Gamma(\lambda,k,-r))^2|a-r|(1+a)^4(1-r)^6(1+r)^2}.$$

Applying the inequalities (18) and (19) in the last expression, we get (27). □

From Theorem 2.11, we deduce that:

Corollary 2.12. *Let $f = h + \overline{g} \in \mathcal{S}_k^{\mathcal{H}}(\lambda)$ lift a minimal surface with isothermal parameters. Then we have*

$$\frac{(1+ar)^5}{(\Gamma(\lambda,k,r))^2(a+r)(1+r)^4} \leq \frac{(1-ar)^5}{(\Gamma(\lambda,k,-r))^2|a-r|(1-r)^4}.$$

In a special case, we have the following.

Corollary 2.13. *Let $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ lift a minimal surface with isothermal parameters. Then we have*

$$\frac{(1-r)^2(1+ar)^5}{(1+r)^6(a+r)(1+r)^4} \leq \frac{(1+r)^2(1-ar)^5}{(1-r)^6|a-r|(1-r)^4}.$$

3. Minimal Surface For Harmonic Koebe Function

We will examine minimal surface for the harmonic Koebe function which gives Weierstrass representation connects the harmonic function theory to minimal surface.

Example 3.1. *For $k = 2$ and $\lambda = 0$, the class $\mathcal{S}_k^{\mathcal{H}}(\lambda)$ reduces to the harmonic function class $\mathcal{S}_{\mathcal{H}}$. For such functions, sharp function is the harmonic Koebe function [5, p.15] given by*

$$f(z) = h(z) + \overline{g(z)} = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \overline{\left(\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}\right)}. \tag{30}$$

The function f maps $\mathbb{D}$ conformally onto complex plane except the real slit $(-\infty, -1/6]$. The harmonic Koebe function is generated via shearing technique

$$h(z) - g(z) = \frac{z}{(1-z)^2}$$

with the dilatation $w(z) = z$ and is univalent. Harmonic functions of the form $f = h + \overline{g}$ can be written

$$f(z) = \operatorname{Re}\big(h(z) + g(z)\big) + i \operatorname{Im}\big(h(z) - g(z)\big),$$

thus the function f, equivalently, can also be written by

$$f(z) = \operatorname{Re}\left(\frac{z + \frac{1}{3}z^3}{(1-z)^3}\right) + i \operatorname{Im}\left(\frac{z}{(1-z)^2}\right).$$

Here, derivatives of the functions h and g given by (30) are

$$h'(z) = \frac{z+1}{(z-1)^4} \quad \text{and} \quad g'(z) = \frac{z(z+1)}{(z-1)^4},$$

then

$$p = h'(z) = \frac{z+1}{(z-1)^4},$$

and

$$q^2 = \frac{g'(z)}{h'(z)} = z \Rightarrow q = \sqrt{z}.$$

Thus using p and q into (4)∼(6), we get

$$\varphi_1 = \frac{(z+1)^2}{(z-1)^4},$$
$$\varphi_2 = -i\frac{(z+1)(1-z)}{(z-1)^4},$$
$$\varphi_3 = -2i\frac{\sqrt{z}(z+1)}{(z-1)^4}.$$

By direct computations, we observe that $(\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 = 0$ *and* $|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 > 0$.

Hence, by [Theorem 1 [7]*, p. 166] the triple*

$$\left(\operatorname{Re}\int_0^z \varphi_1(\xi)d\xi,\ \operatorname{Re}\int_0^z \varphi_2(\xi)d\xi,\ \operatorname{Re}\int_0^z \varphi_3(\xi)d\xi \right)$$
$$= \left(\operatorname{Re}\big(\frac{z+\frac{1}{3}z^3}{(1-z)^3}\big), \operatorname{Im}\big(\frac{z}{(1-z)^2}\big), \operatorname{Im}\big(\frac{4z^{3/2}}{3(1-z)^3}\big) \right)$$

defines a regular minimal surface with isothermal parameters.

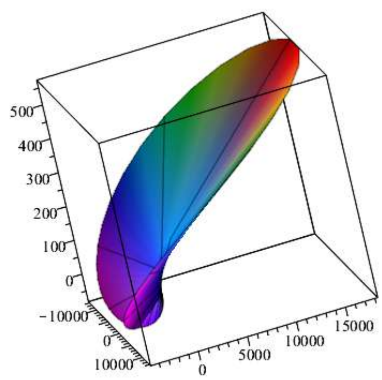

Figure 1. Minimal surface

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