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WARPED PRODUCT SKEW SEMI-INVARIANT SUBMANIFOLDS OF LOCALLY GOLDEN RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we define and study warped product skew semi-invariant submanifolds of a locally golden Riemannian manifold. We investigate a necessary and sufficient condition for a skew semi-invariant submanifold of a locally golden Riemannian manifold to be a locally warped product. An equality between warping function and the squared normed second fundamental form of such submanifolds is established. We also construct an example of warped product skew semi-invariant submanifolds.

1. Introduction

Semi-invariant submanifolds were defined by A. Bejancu and N. Papaghuic [10] as an analogous to that of CR-submanifolds in an almost complex manifolds [9]. Semi-invariant submanifolds are generalization of holomorphic (invariant) and totally real (anti-invariant) submanifolds. A semi-invariant submanifold is called proper if it is neither holomorphic nor totally real submanifold.

In holomorphic submanifolds, the tangent space of the submanifolds is invariant under the action of the almost contact structure. On the other hand, in totally real submanifolds, the tangent space is anti-invariant, that is, it is mapped into the normal space. The geometry of semi-invariant submanifolds has been studied in several papers (see, [1], [2], [3], [4], [19], [34]).

Another generalization of holomorphic and totally real submanifolds is a slant submanifold. Slant submanifold were defined by Chen [15]. Since, then such submanifolds were investigated by many geometers ([8], [13], [16], [27]). If a slant submanifold is neither holomorphic nor totally real submanifolds, then it is said to be proper. We also notice that a proper semi-invariant submanifolds is never a slant submanifold. N. Papaghuic [29] defined semi-slant submanifolds

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as generalization of slant and CR-submanifolds. Carriazo [14] introduced bi-slant submanifolds as generalization of semi-slant submanifolds. Sahin [33] called these submanifolds as hemi-slant submanifolds.

Ronsse [31] introduced the notion of skew CR-submanifolds of Kaehler manifolds. Such submanifolds are generalization of bi-slant submanifolds. In fact, invariant, anti-invariant, CR, slant, semi-slant and hemi-slant submanifold are particular cases of skew CR-submanifolds. Skew CR-submanifolds are studied in [28]. We observe that skew CR-submanifolds in Kaehler manifolds corresponds to skew semi-invariant subamifolds in locally product golden Riemannian manifolds. Skew semi-invariant submanifolds are studied in [7], [35], [38].

Bishop and O' Neill [11] defined warped product. Let M_1 and M_2 be two Riemannian manifolds with Riemannian metric g_1 and g_2, respectively. Let f be a positive differentiable function on M_1. The warped product $M = M_1 \times_f M_2$ of M_1 and M_2 is the Riemannian manifold M, where

$$g = g_1 + f^2 g_2.$$

Thus, if $U \in T_pM$, then

$$\|U\|^2 = \|d\pi_1(U)\|^2 + (f^2 o \pi_1)\|d\pi_2\|^2,$$

where π_1 and π_2 are the canonical projections of $M_1 \times M_2$ onto M_1 and M_2, respectively. The function f is called the warping function of the warped product. If f is constant then M is said to be trivial. It is well known that M_1 is totally geodesic and M_2 is totally umbilical [12]. If U is a vector field on M_1 and V is a vector field on M_2, then it follows from [11, Lemma 7.3] that we have

$$\nabla_U V = \nabla_V U = U(lnf)V, \tag{1}$$

where ∇ is the Levi-Civita connection on $M_1 \times_f M_2$. Warped product submanifolds of several kind of structures have been studied in [26], [32], [36], [37].

Recently, C. E. Hretcanu and M. Crasmereanu [17] introduced and studied a golden Riemannian manifold by using the golden ratio. They also studied invariant submanifolds [18] in Riemannian manifold with golden structure. Gezer. A. et al [22] discussed the integrability conditions of golden Riemannian manifolds. Some properties of golden Riemannian manifolds have been studied in [5], [21], [23], [30]. M. Ahmad and M. A. Qayyoom [6], Hretcanu C.E. [25] studied submanifolds in Riemannian manifolds with golden structure. Semi-invariant submanifolds of golden Riemannian manifolds have been studied in [20], [24].

In this paper, we define and study warped product skew semi-invariant submanifolds of a locally golden Riemannian manifold.

2. Definition and preliminaries

Let $\overline{M}$ be an n-dimensional manifold endowed with a tensor field J of type $(1,1)$ such that

$$J^2 = J + I, \tag{2}$$

where I is the identity transformation on $\Gamma(T\overline{M})$. Then the structure J is called a golden structure. We say that the metric g is J-compatible if

$$g(JX, Y) = g(X, JY) \tag{3}$$

for all X,Y vector fields on $\Gamma(T\overline{M})$, and $(\overline{M}, g, J)$ is called golden Riemannian manifold. If we substitute JX into X in (3), then from (2) we have

$$g(JX, JY) = g(JX, Y) + g(X, Y). \tag{4}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Proposition 2.1 ([17]). (i) *The eigenvalues of a golden structure J are the golden ratio ϕ and $1 - \phi$.*

(ii) *A golden structure J is an isomorphism on the tangent space $T_x\overline{M}$ of the manifold $\overline{M}$ for every $x \in \overline{M}$.*

(iii) *It follows that J is invertible and its inverse $\widehat{J} = J^{-1}$ satisfies*

$$\widehat{\phi}^2 = -\widehat{\phi} + 1.$$

3. Skew semi-invariant submanifold

A submanifold M of a golden Riemannian manifold $\overline{M}$ is called a skew semi-invariant submanifold if there exist an integer k and constant functions α_i, $1 \leq i \leq k$, defined on M with values in $(0,1)$ such that

(i) each α_i, $1 \leq i \leq k$, is a distinct eigenvalue of P^2 with

$$T_pM = D_p^0 \bigoplus D_p^1 \bigoplus D_p^{\alpha_1} \bigoplus \cdots \bigoplus D_p^{\alpha_k}$$

for $p \in M$, and

(ii) the dimensions of D_p^0, D_p^1 and D_k^1, $1 \leq i \leq k$, are independent of $p \in M$.

Remark 3.1. *The above definition implies D_p^0, D_p^1, and $D_p^{\alpha_i}$, $1 \leq i \leq k$, defined P invariant, mutually orthogonal distribution which we denote by D_p^0, D_p^1, and $D_p^{\alpha_i}$, $1 \leq i \leq k$, respectively. The tangent bundle of M has the following decomposition*

$$TM = D^0 \bigoplus D^1 \bigoplus D^{\alpha_1} \bigoplus \cdots \bigoplus D^{\alpha_k}.$$

If $k = 0$, then M is a semi-invariant submanifold. Also, if $k = 0$ and $D_p^0(D_p^1)$ is trivial, then M is an invariant (anti-invariant) submanifold of $\overline{M}$.

We denote by $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}$ with respect to g. Let M be a Riemannian manifold isometrically immersed in $\overline{M}$ and let g be the Riemannian metric induced on M for $p \in M$ and tangent vector $X_p \in T_pM$. Then we write

$$JX_p = TX_p + NX_p, \tag{5}$$

where $TX_p \in T_pM$ is tangent to M and $NX_p \in T_p^{\perp}M$ is normal to M.

For any two vectors $X_p, Y_p \in T_pM$, we have

$$g(JX_p, Y_p) = g(TX_p, Y_p),$$

which implies that

$$g(JX_p, Y_p) = g(X_p, TY_p).$$

So, T and T^2 are all symmetric operators on the tangent space T_pM. Assume that $\alpha(p)$ is the eigenvalue of T^2 at $p \in M$. Since T^2 is a composition of an isometry and a projection, we have $\alpha(p) \in [0, 1]$.

For each $p \in M$, we set $D_p^{\alpha} = \ker(T^2 - \alpha(p)I)$, where I is the identity transformation on T_pM, and $\alpha(p)$ is an eigenvalue of T^2 at $p \in M$. Obviously, we have

$$D_p^0 = \ker T, \ D_p^1 = \ker N.$$

D_p^1 is the maximal J invariant subspace of T_pM and D_p^0 is the maximal J anti-invariant subspace of T_pM.

If $\alpha_1(p), \dots, \alpha_k(p)$ are all eigenvalues of T^2 at p, then T_pM can be decomposed as the direct sum of the mutually orthogonal eigenspaces, that is

$$T_pM = D_p^{\alpha_1} \bigoplus D_p^{\alpha_2} \bigoplus \cdots \bigoplus D_p^{\alpha_k}.$$

For $N \in T^{\perp}M$, we write

$$JN = tN + wN, \tag{6}$$

where $tN \in TM$, and $fN \in T^{\perp}M$.

If M is a submanifold in a golden Riemannian manifold $(\overline{M}, g, J)$, then it follows from [26] that for any $X \in \Gamma(TM)$ we get

$$\text{(i) } T^2X = TX + X - tNX \ \ \text{(ii) } NX = NTX + wNX, \tag{7}$$

and for any $V \in \Gamma(T^{\perp}M)$,

$$\text{(i) } w^2V = wV + V - NtV \ \ \text{(ii) } tT = TtV + twV. \tag{8}$$

We denote by ∇ the induced connection in M. Then we have formulas by Gauss and Weingarten

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{9}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{10}$$

for all vectors $X, Y \in TM$ and $N \in T^{\perp}M$. Also, we have

$$g(h(X, Y), N) = g(A_N X, Y). \tag{11}$$

If $(\overline{M}, g, J)$ is a golden Riemannian manifold and J is parallel with respect to Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$ (i.e. $\overline{\nabla}J = 0$), then $(\overline{M}, g, J)$ is called a locally golden Riemannian manifold.

Definition 3.2. *A submanifold M of a locally golden Riemannian manifold $\overline{M}$ is called a skew semi-invariant submanifold of order 1 if M is a skew semi-invariant submanifold with $k = 1$.*

In this case, we have

$$TM = D^{\perp} \bigoplus D^{T} \bigoplus D^{\theta},$$

where $D^{\theta} = D^{\alpha_1}$ and α_1 is constant. A skew semi-invariant submanifold of order 1 is proper if $D^{\perp} \neq 0$ and $D^{T} \neq 0$.

A slant submanifold M of a locally golden Riemannian manifold $\overline{M}$ is characterized by

$$T^2 X = \alpha(T + I) \tag{12}$$

such that $\alpha \in [0, 1]$, where $X \in TM$. Moreover, if θ is the slant angle of M, then we have $\alpha = \cos^2 \theta$.

Lemma 3.3. *Let M be a proper skew semi-invariant submanifold of a locally golden Riemannian manifold $\overline{M}$. Then we have*

$$g(J(\nabla_V W), X) + g(\nabla_V W, X) = -g(A_{JW} V, JX), \tag{13}$$

$$g(\nabla_V TZ, X) + g(\nabla_V Z, X) = \csc^2 \theta [g(A_{NZ} V, JX) - g(A_{NTZ} V, X)], \tag{14}$$

$$g(J(\nabla_Z V), X) + g(\nabla_Z V, X) = -g(A_{JV} Z, JX). \tag{15}$$

Proof. For $V, W \in D^{\perp}$, $Z \in D^{\theta}$ and $X \in D^{T}$

$$g(\overline{\nabla} JW, JX) = g(\nabla_V JW + h(V, JW), JX),$$

$$g(\overline{\nabla} JW, JX) = g(\nabla_V JW, JX) + g(h(V, JW), JX),$$

$$g(\overline{\nabla} JW, JX) = g(J(\nabla_V W), JX). \tag{16}$$

Using (4) and (10), we have

$$g(-A_{JW} V + \nabla^{\perp}_V JW, JX) = g(J(\nabla_V W), X) + g(\nabla_V W, X),$$

$$g(J(\nabla_V W), X) + g(\nabla_V W, X) = g(-A_{JW} V, JX),$$

which is (13).

Now,

$$g(\overline{\nabla}_V JZ, JX) = g(\nabla_V JZ + h(V, JZ), JX),$$

$$g(\overline{\nabla}_V JZ, JX) = g(J(\nabla_V Z), JX).$$

Using (4), we get

$$g(\overline{\nabla}_V JZ, JX) = g(J(\nabla_V Z), X) + g(\nabla_V Z, X).$$

Using (5), we have

$$g(J(\nabla_V Z), X) + g(\nabla_V Z, X) = g(\overline{\nabla}_V (TZ + NZ), JX),$$

$$g(J(\nabla_V Z), X) + g(\nabla_V Z, X) = g(\overline{\nabla}_V TZ, JX) + g(\overline{\nabla}_V NZ, JX),$$

$$g(J(\nabla_V Z), X) + g(\nabla_V Z, X) = g(\overline{\nabla}_V JTZ, X) + g(\overline{\nabla}_V NZ, JX),$$

$$g(J(\nabla_V Z), X) + g(\nabla_V Z, X) = g(\overline{\nabla}_V T^2 Z, X) + g(\overline{\nabla}_V NTZ, X) + g(\overline{\nabla}_V QZ, JX).$$

Using (10), we have

$$\begin{aligned} &g(J(\nabla_V Z), X) + g(\nabla_V Z, X) \\ &= g(\overline{\nabla}_V T^2 Z, X) + g(\overline{\nabla}_V NTZ, X) - g(A_{NZ} V, JX) + g(\nabla^{\perp}_V NZ, JX). \end{aligned}$$

Using (12), (9), and (10), we get

$$\begin{aligned} &g((\nabla_V JZ), X) + g(\nabla_V Z, X) \\ &= \alpha g(\nabla_V TZ, X) + \alpha g(\nabla_V Z, X) - g(A_{NTZ} V, X) + g(A_{NZ} V, JX). \end{aligned}$$

Using (5), we have

$$\begin{aligned} &g((\nabla_V TZ), X) + g(\nabla_V Z, X) \\ &= \alpha g(\nabla_V TZ, X) + \alpha g(\nabla_V Z, X) - g(A_{NTZ} V, X) + g(A_{NZ} V, JX), \end{aligned}$$

$$(1 - \cos^2\theta) g((\nabla_V TZ), X) + g(\nabla_V Z, X) = -g(A_{NTZ} V, X) + g(A_{NZ} V, JX),$$

Since $\alpha = \cos^2\theta$, we have

$$g((\nabla_V TZ), X) + g(\nabla_V Z, X) = \csc^2\theta (g(A_{NZ} V, JX) - g(A_{NTZ} V, X))$$

which is (14).

Using (4) and (16), we get

$$g(J(\nabla_Z V, X) + g(\nabla_Z V, X) = g(\overline{\nabla}_Z JV, JX).$$

Using (10) in above equation we can obtain (15). □

Lemma 3.4. *Let M be a proper skew semi-invariant submanifold of a locally golden Riemannian manifold $\overline{M}$. Then we have*

$$g(\nabla_U TZ, X) + g(\nabla_U Z, X) = -\csc^2[g(A_{NTZ} U, X) + g(A_{NT} U, JX), \tag{17}$$

$$g(\nabla_X Y, TZ) + g(\nabla_X Y, Z) = \csc^2[g(h(X, Y), NTZ) + g(h(X, JY), NZ)], \tag{18}$$

$$g(A_{JV} X, JY) = 0. \tag{19}$$

Proof. For $V, W \in D^{\perp}, Z \in D^{\theta}$ and $X \in D^T$, by using (4), (5), and (16), we get

$$g(J(\nabla_U Z), X) + g(\nabla_U Z, X) = g(\overline{\nabla} JTZ, X) + g(\overline{\nabla}_U NZ, JX).$$

Using (10) and (5), we obtain

$$g((\nabla_U TZ), X) + g(\nabla_U Z, X) = g(\overline{\nabla}_U T^2 Z, X) + g(\overline{\nabla}_U NTZ, X) - g(A_{NZ} U, JX).$$

Using (9), (10), and (12), we get

$$g((\nabla_U TZ), X) + g(\nabla_U Z, X)$$
$$= \alpha g(\nabla_U TZ, X) + \alpha g(\nabla_U Z, X) - g(A_{NTZ}U, X) - g(A_{NT}U, JX).$$

Thus we get

$$g((\nabla_U TZ), X) + g(\nabla_U Z, X) = -\csc^2[g(A_{NTZ}U, X) + g(A_{NT}U, JX)],$$

which is (17).

Using (5), (9), and (16), we can obtain

$$g(J(\nabla_X Y), Z) + g(\nabla_X Y, Z) = g(\overline{\nabla}_X Y, JTZ) + g(h(X, JY), NZ).$$

Using (5), (9), and (12), we get

$$g((\nabla_X Y), TZ) + g(\nabla_X Y, Z)$$
$$= \alpha g(\nabla_X Y, TZ) + \alpha g(\nabla_X Y, Z) + g(h(X, Y), NTZ)$$
$$+ g(h(X, JY), NZ).$$

Since $\alpha = \cos^2\theta$, we have

$$g((\nabla_X Y), TZ) + g(\nabla_X Y, Z) = \csc^2[g(h(X, Y), NTZ) + g(h(X, JY), NZ)],$$

which is (18).

From (16) and (9), we get

$$g(J(\nabla_X Y), V) + g(\nabla_X Y, V) = g(J(\nabla_X Y), JV) + g(h(X, JY), JV).$$

Using (4) and (11) together with the above equation, we obtain (19). □

Lemma 3.5. *Let M be a proper skew semi-invariant submanifold of a golden Riemannian manifold $\overline{M}$. Then we have*

$$g(\nabla_U Z, V) + g(J(\nabla_U Z), V) = \sec^2\theta[g(A_{JV}U, TZ) - g(A_{NZ}U, V) + g(A_{NTZ}U, V)], \tag{20}$$

$$g(\nabla_X V, TZ) + g(\nabla_X V, Z) = \sec^2\theta[-g(A_{JV}X, TZ) + g(A_{NZ}X, V) - g(A_{NTZ}X, V)]. \tag{21}$$

Proof. For any $U, Z \in D^\theta$ and $V \in D^\perp$

$$g(J(\overline{\nabla}_U Z), JV) = g(\overline{\nabla}_U JZ, JV).$$

Using (4), (5), and (9), we obtain

$$g(J(\overline{\nabla}_U Z), V) + g(\overline{\nabla}_U Z, V) = g(h(U, TZ), JV) + g(\overline{\nabla}_U NZ, JV).$$

From (11) and (6), we get

$$g(J(\overline{\nabla}_U Z), V) + g(\overline{\nabla}_U Z, V)$$
$$= g(A_{JV}U, TZ) + g(\overline{\nabla}_U tNZ, V) + g(\overline{\nabla}_U wNZ, V).$$

Using (7), (8), and (12), we obtain

$$\begin{aligned} g(J(\overline{\nabla}_U Z), V) + g(\overline{\nabla}_U Z, V) &= g(A_{JV}U, TZ) + \sin^2\theta g(\overline{\nabla}_U (T+I)Z, V) \\ &\quad + g(\overline{\nabla}_U NZ, V) - g(\overline{\nabla}_U NTZ, V). \end{aligned}$$

Using (5), (9), and (10) together with the above equation, we get

$$\begin{aligned} &g(\nabla_U Z, V) + g(T(\nabla_U Z), V) \\ &= \sec^2\theta[g(A_{JV}U, TZ) - g(A_{NZ}U, V) + g(A_{NTZ}U, V)], \end{aligned}$$

which is (20).

For any $X \in D^T, Z \in D^\theta$, and $V \in D^\theta$, we get

$$g(\overline{\nabla}_X JV, JZ) = g(J(\overline{\nabla}_X V), Z) + g(\overline{\nabla}_X V, Z).$$

From (5), (9) and (6), we get

$$g(\nabla_X V, TZ) + g(\nabla_X V, Z) = -g(A_{JV}X, TZ) + g(h(X,V), tNZ + wNZ).$$

By using (7), (8), and (11), we can obtain

$$\begin{aligned} &g(\nabla_X V, TZ) + g(\nabla_X V, Z) \\ &= \sec^2\theta[-g(A_{JV}X, TZ) + g(A_{NZ}X, V) - g(A_{NTZ}X, V)], \end{aligned}$$

which is (21). □

4. Warped product skew semi-invariant submanifolds of a locally golden Riemannian manifold

We consider a warped product submanifold of type $M = M_1 \times_f M_T$ in a locally golden Riemannian manifold $\overline{M}$, where M_1 is a hemi-slant submanifold and M_T is an invariant submanifold. Then it is clear that M is a proper skew semi-invariant submanifold of $\overline{M}$. Thus, by definition of hemi-slant submanifold and skew semi-invariant submanifold, we have

$$TM = D^\theta \bigoplus D^\perp \bigoplus D^T. \tag{22}$$

In particular, if $D^\theta = 0$, then M is a warped product semi-invariant submanifold. If $D^T = 0$, then M is a warped product semi-slant submanifold.

Since M_1 is a hemi-slant submanifold, then normal bundle of $T^\perp M_1$ of M_1 is decomposed as

$$T^\perp M_1 = J(D^\perp) \bigoplus N(D^\theta) \bigoplus \mu.$$

Thus, we have

$$T^\perp M = J(D^\perp) \bigoplus N(D^\theta) \bigoplus \mu.$$

Since D^T is an invariant distribution, where μ is the orthogonal complementary distribution of $J(D^\perp) \bigoplus N(D^\theta)$ in $T^\perp M$, it is an invariant subbundle of $T^\perp M$ with respect of J.

Proposition 4.1. *Let $M = M_1 \times_f M_T$ be a (D^θ, D^T)-mixed totally geodesic proper skew semi-invariant submanifold with integrable distribution $D^\perp$ of a locally golden Riemannian manifold $\overline{M}$. Then M is a locally warped product submanifold if*

$$A_{JV}JX = -V(lnf)(TX - X), \tag{23}$$

$$g(A_{NTZ}X, Y) + g(A_{NZ}X, JZ) = sin^2\theta X(lnf)[g(Y, Z) + g(Y, TZ)]. \tag{24}$$

Proof. Suppose that $M = M_1 \times_f M_T$ is a (D^θ, D^T)-mixed totally geodesic warped product proper skew semi-invariant submanifold with integrable distribution D^T of a locally golden Riemannian manifold $\overline{M}$.

Since M is a $(D^\theta, D^T)-$ mixed totally geodesic, then $h(Z, JX) = 0$. From (13) we get $g(A_{JW}Z, JX) = 0$. Similarly, $g(h(Z, JX), JV) = 0$, then from (15) we get $g(A_{JV}Z, JX) = 0$ for any $V, W \in D^\perp, Z \in D^\theta$ and $X \in D^T$. Since A is self adjoint, then $g(Z, A_{JV}JX) = 0$ and $g(W, A_{JV}JX) = 0$. Hence $A_{JV}JX$ has no component in TM_1.

Using (4), (9), and (10), we obtain

$$g(A_{JV}JX, Y) = -g(J(\nabla_Y V)X) - g(\nabla_Y V, X).$$

Using (5) and (1), we get

$$A_{JV}JX = -V'(\ln f)(TX - X),$$

which is (23).

Since M is $(D^\theta, D^T)-$ mixed totally geodesic for any $Z \in D^\theta$ and $X \in D^T$, we have

$$g(A_{NTZ}X, Z) = g(h(X, Z), NTZ).$$

Since $h(X, Z) = 0$, we have $g(A_{NTZ}X, Z) = 0$. Hence $A_{NTZ}X$ has no component in D^θ.

Using (1) and (21), we get

$$\begin{aligned} &g(V(lnf)X, TZ) + g(V(lnf)X, Z) \\ &= sec^2\theta[-g(h(X, TZ), JV) + g(A_{NZ}X, V) - g(A_{NTZ}X, V))]. \end{aligned}$$

Since M is (D^θ, D^T)-mixed geodesic, we have

$$g(A_{NZ}X, V) - g(A_{NTZ}X, V) = 0,$$

$$g(A_{NZ}X - A_{NTZ}X, V) = 0.$$

Thus $A_{NZ}X$ and $A_{NTZ}X$ have no component in $D^\perp$. From (22), we can obtain $A_{NTZ}X \in D^T$ and $A_{NZ}X \in D^T$ for $X, Y \in D^T$ and $Z \in D^\theta$.

From (18) and (11), we obtain

$$g(A_{NTZ}X, Y) + g(A_{NZ}X, JZ) = \sin^2\theta[g(\nabla_X Y, TZ) + g(\nabla_X Y, Z)].$$

Using (1), we get

$$g(A_{NTZ}X, Y) + g(A_{NZ}X, JZ) = \sin^2\theta X(lnf)[g(Y, Z) + g(Y, TZ)],$$

which is (24). □

Lemma 4.2. *Let $M = M_1 \times_f M_T$ be a warped product proper skew semi-invariant submanifold of a locally golden Riemannian manifold. Then we have*

$$g(h(X,V),JW) = 0, \tag{25}$$

$$g(h(X,V),NZ) = 0. \tag{26}$$

Proof. For any $V, W \in D^{\perp}$ and $X \in D^T$, by using (9) we get

$$g(h(X,V),JW) = g(\overline{\nabla}_V X, JW),$$
$$g(h(X,V),JW) = g(\nabla_V JX, W),$$
$$g(h(X,V),JW) = g(V(lnf)JX, W),$$
$$g(h(X,V),JW) = 0,$$

which is (25).

Similarly,

$$g(h(X,V),NZ) = g(\overline{\nabla}_V X, NZ),$$
$$g(h(X,V),NZ) = V(lnf)[g(JX,Z) - g(\nabla_V X, TZ)],$$
$$g(h(X,V),NZ) = 0,$$

which is (26). □

Lemma 4.3. *Let $M = M_1 \times_f M_T$ be a warped product proper skew semi-invariant submanifold M of a locally golden Riemannian manifold. Then*

$$g(h(X,JY),JV) = -V(lnf)[g(JY,X) + g(Y,X)]. \tag{27}$$

Proof. From (9) and (4), we get

$$g(h(X,JY),JV) = g(\overline{\nabla}_X JY, JV),$$
$$g(h(X,JY),JV) = g(\nabla_X JY, V) + g(\nabla_X V, Y),$$
$$g(h(X,JY),JV) = -g(\nabla_X V, JY) - g(\nabla_X V, Y),$$
$$g(h(X,JY),JV) = -V(lnf)[g(JY,X) + g(Y,X)],$$

which is (27). □

Theorem 4.4. *Let $M = M_1 \times_f M_T$ be a $(p+q+r)$-dimensional warped product proper skew semi-invariant of a $(2p+2q+r)$-dimensional locally golden Riemannian manifold $\overline{M}$. Then we have the following statements:*

(1) *The squared norm of the second fundamental form of M satisfies*

$$\|h\|^2 \geq r\{2\|\nabla^{\perp}(lnf)\|^2 + 2\cos^2\theta \|\nabla^{\theta}(lnf)\|^2\}, \tag{28}$$

where $r = \dim(M_T)$, and $\nabla^{\perp}(lnf)$ and $\nabla^{\theta}(lnf)$ are gradients of (lnf) on $D^{\perp}$ and D^{θ}, respectively.

(2) *Assume that the equality sign holds identically. Then M_1 is a totally geodesic submanifold of $\overline{M}$ and M is a mixed totally geodesic. Moreover, M_T can never be a minimal submanifold of $\overline{M}$.*

Proof. Let

$$\{e_1,\ldots,e_r,\overline{e_1},\ldots,\overline{e_p},\widetilde{e_1},\ldots,\widetilde{e_q},e_1^*,\ldots,e_p^*,\ldots,e_1',\ldots,e_q'\}$$

be an orthonormal frame of a locally golden Riemannian manifold $\overline{M}$ such that $\{e_1,\ldots,e_r\}$ is an orthonormal basis of D^T, $\{\overline{e}_1,\ldots,\overline{e}_p\}$ is an orthonormal basis of D^θ, $\{\widetilde{e}_1,\ldots,\widetilde{e}_q\}$ is an orthonormal basis of $D^\perp$, $\{e_1^*=N\overline{e}_1,\ldots,e_p^*=N\overline{e}_p\}$ is an orthonormal basis of ND^θ, and $\{e_1'=J(\widetilde{e}_1),\ldots,e_q'=J(\widetilde{e}_q)\}$ is an orthonormal basis of $JD^\perp$.

Since

$$TM=D^\theta\bigoplus D^\perp\bigoplus D^T,$$

the squared norm of the second fundamental form h can be decomposed as

$$\|h\|^2=\|h(D^T,D^T)\|^2+\|h(D^\theta,D^\theta)\|^2+\|h(D^\perp,D^\perp)\|^2+2\|h(D^T,D^\perp)\|^2+$$
$$\|h(D^T,D^\theta)\|^2+2\|h(D^\perp,D^\theta)\|^2.$$

Note that M is $(D^\perp,D^T)$-mixed totally geodesic, so we have

$$\|h\|^2=\|h(D^T,D^T)\|^2+\|h(D^\theta,D^\theta)\|^2+\|h(D^\perp,D^\perp)\|^2+$$
$$\|h(D^T,D^\theta)\|^2+2\|h(D^\perp,D^\theta)\|^2,$$

$$\begin{aligned}\|h\|^2 \;=\;& \sum_{i,j=1}^{r}\sum_{a=1}^{q}g(h(e_i,e_j),e_a')^2+\sum_{i,j=1}^{r}\sum_{m=1}^{p}g(h(e_i,e_j),e_m^*)^2+\\ &\sum_{a,b,c=1}^{q}g(h(\widetilde{e}_a,\widetilde{e}_j),e_c')^2+\sum_{a,b=1}^{q}\sum_{m=1}^{p}g(h(\widetilde{e}_a,\widetilde{e}_b),e_m^*)^2+\\ &\sum_{m,n=1}^{p}\sum_{a=1}^{q}g(h(\overline{e}_a,\overline{e}_b),e_a')^2+\sum_{m,n,k=1}^{p}g(h(\overline{e}_m,\overline{e}_n),e_k^*)^2+\\ &2\sum_{i=1}^{r}\sum_{m=1}^{p}\sum_{a=1}^{q}g(h(e_i,\overline{e}_m),e_a')^2+2\sum_{i=1}^{r}\sum_{m,n=1}^{p}g(h(e_i,\overline{e}_m),e_n^*)^2+\\ &2\sum_{m=1}^{p}\sum_{a,b=1}^{q}g(h(\widetilde{e}_m,\widetilde{e}_a),e_b')^2+2\sum_{m,n=1}^{p}\sum_{a=1}^{q}g(h(\overline{e}_m,\widetilde{e}_a),e_n^*)^2,\end{aligned}\tag{29}$$

$$\begin{aligned}\|h\|^2=&\sum_{i,j=1}^{r}\sum_{a=1}^{q}g(h(e_i,e_j),J\widetilde{e}_a)^2+\sum_{i,j=1}^{r}\sum_{m=1}^{p}g(h(e_i,e_j),N\overline{e}_m)^2+\\ &\sum_{a,b,c=1}^{q}g(h(\widetilde{e}_a,\widetilde{e}_j),J\widetilde{e'}_c)^2+\sum_{a,b=1}^{q}\sum_{m=1}^{p}g(h(\widetilde{e}_a,\widetilde{e}_b),N\overline{e}_m)^2\\ &\sum_{m,n=1}^{p}\sum_{a=1}^{q}g(h(\overline{e}_a,\overline{e}_b),J\widetilde{e'}_a)^2+\sum_{m,n,k=1}^{p}g(h(\overline{e}_m,\overline{e}_n),N\overline{e}_k)^2+\end{aligned}$$

$$2\sum_{i=1}^{r}\sum_{m=1}^{p}\sum_{a=1}^{q} g(h(e_i,\overline{e}_m),J(\overline{e}_a))^2+2\sum_{i=1}^{r}\sum_{m,n=1}^{p} g(h(e_i,\overline{e}_m),N\overline{e}_n)^2$$

$$2\sum_{m=1}^{p}\sum_{a,b=1}^{q} g(h(\widetilde{e}_m,\widetilde{e}_a),J(\widetilde{e}_b))^2+2\sum_{m,n=1}^{p}\sum_{a=1}^{q} g(h(\overline{e}_m,\widetilde{e}_a),N\overline{e}_n)^2.$$

Using (25), (26), and (27), we have

$$\sum_{i,j=1}^{r}\sum_{a=1}^{q} g(h(e_i,e_j),J\widetilde{e}_a)^2=\sum_{i,j=1}^{r}\sum_{a=1}^{q}(-\widetilde{e}_a(lnf)[g(Je_j,e_i)+g(e_j,e_i)])^2. \tag{30}$$

Using Lemma 2 of [26, p. 9], we have

$$\begin{aligned}\sum_{i,j=1}^{r}\sum_{m=1}^{p} g(h(e_i,e_j),N\overline{e}_m)^2=\sum_{i,j=1}^{r}\sum_{m=1}^{p}&(T\overline{e}_m(lnf)g(e_i,e_j)\\ &-\overline{e}_m(lnf)g(e_i,Te_j))^2.\end{aligned} \tag{31}$$

Using (12), (30), (31), and (29), we get

$$\|h\|^2\geq\sum_{i,j=1}^{r}\sum_{a=1}^{q}(-\widetilde{e}_a(lnf)[g(Je_j,e_i)+g(e_j,e_i)])^2+\sum_{i,j=1}^{r}\sum_{m=1}^{p}\cos^2\theta(T+I)$$

$$(\overline{e}_m(lnf)g(e_i,e_j))^2+(\overline{e}_m(lnf)g(e_i,Te_j))^2.$$

Thus we obtain

$$\|h\|^2\geq r[\|\nabla^{\perp}(lnf)\|^2+2\cos^2\theta\|\nabla^{\theta}(lnf)\|^2],$$

which is (28). □

Example 4.5. *Let us consider the golden Riemannian manifold $\mathbb{R}^9=\mathbb{R}^5\times\mathbb{R}^4$ with usual metric g and golden structure J defined by*

$$J(\frac{\partial}{\partial x_i},\frac{\partial}{\partial y_i})=(\phi\frac{\partial}{\partial x_1},\overline{\phi}\frac{\partial}{\partial x_2},\overline{\phi}\frac{\partial}{\partial x_3},\overline{\phi}\frac{\partial}{\partial x_4},\phi\frac{\partial}{\partial x_5},\phi\frac{\partial}{\partial y_1},\phi\frac{\partial}{\partial y_2},\overline{\phi}\frac{\partial}{\partial y_3},\overline{\phi}\frac{\partial}{\partial y_4}),$$

where $i, j \in \{1, 2, 3, 4\}$. *Then we have*

$$J^2(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i})$$
$$= (\phi^2 \frac{\partial}{\partial x_1}, \overline{\phi}^2 \frac{\partial}{\partial x_2}, \overline{\phi}^2 \frac{\partial}{\partial x_3}, \phi^2 \frac{\partial}{\partial x_4}, \overline{\phi}^2 \frac{\partial}{\partial x_4}, \phi^2 \frac{\partial}{\partial y_1}, \phi^2 \frac{\partial}{\partial y_2}, \overline{\phi}^2 \frac{\partial}{\partial y_3}, \overline{\phi}^2 \frac{\partial}{\partial y_4}),$$

$$J^2(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}) = ((\phi+1)\frac{\partial}{\partial x_1}, (\overline{\phi}+1)\frac{\partial}{\partial x_2}, (\overline{\phi}+1)\frac{\partial}{\partial x_3}, (\overline{\phi}+1)\frac{\partial}{\partial x_4}, (\overline{\phi}+1)\frac{\partial}{\partial x_5},$$
$$(\phi+1)\frac{\partial}{\partial y_1}, (\phi+1)\frac{\partial}{\partial y_2}, (\overline{\phi}+1)\frac{\partial}{\partial y_3}, (\overline{\phi}+1)\frac{\partial}{\partial y_4}),$$

$$J^2(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}) = J(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_4}, \frac{\partial}{\partial y_5})$$
$$+ (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_4}),$$
$$J^2 = J + I.$$

Consider a submanifold $M = (\mathbb{R}^9, g, J)$ *given by*

$$f(\gamma, \beta, u, v) = (u+v, u-v, u\cos\beta, u\sin\beta, u, v, u\cos\gamma, u\sin\gamma, \sqrt{\phi}).$$

The tangent bundle of M *is spanned by*

$$Z_1 = -u\sin\gamma\frac{\partial}{\partial x_1} + u\cos\gamma\frac{\partial}{\partial x_4},$$
$$Z_2 = -u\sin\beta\frac{\partial}{\partial x_2} + u\cos\beta\frac{\partial}{\partial y_3},$$
$$Z_3 = \cos\gamma\frac{\partial}{\partial x_2} + \sin\gamma\frac{\partial}{\partial y_3} + u\frac{\partial}{\partial y_2} - v\frac{\partial}{\partial y_4} + \cos\beta\frac{\partial}{\partial x_1} + \sin\beta\frac{\partial}{\partial x_4},$$
$$Z_4 = \cos\beta\frac{\partial}{\partial y_1} + \cos\gamma\frac{\partial}{\partial x_3} + \sqrt{\phi}\frac{\partial}{\partial x_5} + \sin\beta\frac{\partial}{\partial y_2} + \sin\gamma\frac{\partial}{\partial y_4}.$$

By direct calculations, we see that $D^T = span\{Z_1, Z_2\}$ *is an invariant distribution,* $D^\alpha = span\{Z_3\}$ *is a slant distribution with slant angle*

$$\alpha = \arccos\left(\frac{\phi(1+u^2) + \overline{\phi}(1+v^2)}{\sqrt{(2+u^2+v^2)(\phi^2(1+v^2) + \overline{\phi}(1+u^2))}}\right),$$

and $D^\perp = \{Z_4\}$ *is an anti-invariant, since* $J(Z_4)$ *is orthogonal to* TM. *Thus, we can conclude that* M *is a proper skew semi-invariant submanifold of* $\overline{M}$.

If we denote the integral submanifolds of D^α, $D^\perp$ *and* D^T *by* M_α, $M_\perp$, *and* M_T, *respectively, then induced metric tensor of* M *is*

$$ds^2 = u^2(d\gamma + d\beta) + (2+u^2+v^2)du + (2+\phi)dv.$$

Thus $M = M_\alpha \times M_\perp \times M_T$ *is a warped product skew semi-invariant submanifold of* M *with warping function* $f = u$.

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