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STABILIZERS ON SHEFFER STROKE BL-ALGEBRAS

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Abstract. In this study, new properties of various filters on a Sheffer stroke BL-algebra are studied. Then some new results in filters of Sheffer stroke BL-algebras are given. Also, stabilizers of nonempty subsets of Sheffer stroke BL-algebras are defined and some properties are examined. Moreover, it is shown that the stabilizer of a filter with respect to a/n (ultra) filter of a Sheffer stroke BL-algebra is its (ultra) filter. It is proved that the stabilizer of the subset $\{0\}$ of a Sheffer stroke BL-algebra is $\{1\}$. Finally, it is stated that the stabilizer $St(P, Q)$ of P with respect to Q is an ultra filter of a Sheffer stroke BL-algebra when P is any filter and Q is an ultra filter of this algebra.

1. Introduction

The filter theory plays an important role in studying logical algebras. From logical point of view, they are typically used to prove the completeness of the non-classical logics. Various filters of logical algebras and residuated lattices such as ultra, prime, (positive) implicative, Boolean etc. have been comprehensively investigated [5]-[25].

The concept of BL-algebras is developed from the continuous t-norms as an algebraic structure of Hájek' s Basic Logic (BL) [7]. Hájek gave filters and prime filters on this algebraic structure and proved the completeness of basic logic by using these prime filters [7]. Also, Boolean, (positive) implicative, maximal, prime filters and deductive systems of BL-algebras are widely examined ([8], [11]). Especially, Turunen analysed filters and deductive systems which are implicative and Boolean filters of BL-algebras ([23], [24]). Recently, A. Borumand Saeid et al. studied on some types of filters of BL-algebras [1]-[3]. Indeed, Haveshki et al. defined a stabilizer on BL-algebras [9], and Borumand Saeid et al. described new types of stabilizers on residuated lattices and researched the relationships between these stabilizers and various filters such as obstinate, Boolean and fantastic filters [4].

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Sheffer operation (or Sheffer stroke) was introduced by H. M. Sheffer [22]. Sheffer stroke, which is a negation of a conjunction and is said to be NAND as well, is one of the two operators that can be used by itself, without any other logical operators, to construct a logical formal system. There is an interest in finding simple axiom systems for various algebras and logics, where simplicity is characterized by the number of axioms in a system. As a well-known example, the well-known Boolean algebra axioms can be written in a single axiom using the Sheffer stroke [12]. Since Boolean algebras are the basis of all modern programming languages and also Sheffer stroke has all diods on the chip forming processor in a computer, i.e., producing a single diod for this operation is simpler and cheaper than to produce different diods for other Boolean operations, this operation contributes to some new developments in computer science. It provides new and easily applicable axiom systems for many algebraic structures, and leads to various similarities and discrepancies among algebraic structures due to its commutative property. Particularly, Oner et al introduced a BL-algebra with the Sheffer operation called a Sheffer stroke BL-algebra, and examined various (fuzzy) filter and neutrosophic N-structures on this algebraic structure ([14], [10]). Recently, they studied filters and neutrosophic $N-$structures on strong Sheffer stroke non-associative MV-algebras ([13], [17]), fuzzy filters and neutrosophic $N-$structures on Sheffer stroke Hilbert algebras ([15], [16], [18]), Sheffer stroke BG-algebras [19] and their fuzzy implicative ideals [20] and Sheffer stroke BCK-algebras [21].

For better understanding this algebraic structure, we must get more results and study it in details and this motivate us to study various filters on a Sheffer stroke BL-algebra and stabilizers of nonempty subsets of Sheffer stroke BL-algebras. We analyze new properties and some results in filters of a Sheffer stroke BL-algebra. Then we define a stabilizer of a nonempty subset of a Sheffer stroke BL-algebra and prove that it is a filter of this algebraic structure. It is shown that a stabilizer of the subset $\{c\}$ of a Sheffer stroke BL-algebra is an ultra filter, which there exist no an element a of this algebraic structure such that $c < a < 1$. It is stated that the stabilizer of the subset $\{0\}$ of a Sheffer stroke BL-algebra is $\{1\}$ and it is involved via all filters of this algebraic structure. Moreover, a stabilizer of nonempty subsets of a Sheffer stroke BL-algebra with respect to each other is determined and examined. Also, it is demonstrated that the stabilizer $St(P, Q)$ of P with respect to Q is a filter of a Sheffer stroke BL-algebra if P and Q are two filters of this algebra. Finally, we state that $St(P, Q)$ is an ultra filter of a Sheffer stroke BL-algebra when P is any filter and Q is an ultra filter of this algebra.

2. Preliminaries

In this section, we give fundamental definitions and notions about Sheffer stroke BL-algebras and filters.

Definition 2.1. [6] *Let $\mathcal{L} = \langle L, | \rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke if it satisfies the following conditions:*
$(S1)$ $c_1|c_2 = c_2|c_1$,
$(S2)$ $(c_1|c_1)|(c_1|c_2) = c_1$,
$(S3)$ $c_1|((c_2|c_3)|(c_2|c_3)) = ((c_1|c_2)|(c_1|c_2))|c_3$,
$(S4)$ $(c_1|((c_1|c_1)|(c_2|c_2)))|(c_1|((c_1|c_1)|(c_2|c_2))) = c_1$,
for all $c_1, c_2, c_3 \in L$.

Definition 2.2. [14] *A Sheffer stroke BL-algebra is an algebra $(L, \vee, \wedge, |, 0, 1)$ of type $(2, 2, 2, 0, 0)$ satisfying the following conditions:*
$(sBL-1)$ $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice,*
$(sBL-2)$ $(L, |)$ *is a groupoid with the Sheffer stroke,*
$(sBL-3)$ $c_1 \wedge c_2 = (c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2)))$,
$(sBL-4)$ $(c_1|(c_2|c_2)) \vee (c_2|(c_1|c_1)) = 1$,
for all $c_1, c_2 \in L$.

Also, $1 = 0|0$ is the greatest element and $0 = 1|1$ is the least element of L.

Example 2.3. [14] *For a set $L = \{0, u, v, 1\}$, the Sheffer stroke BL-algebra $(L, \vee, \wedge, |, 0, 1)$ has the Hasse diagram in Figure 1 and the binary operations $|$, $\vee$ and $\wedge$ on L have Cayley tables in Table 1:*

FIGURE 1. Hasse diagram of the Sheffer stroke BL-algebra in Example 2.3

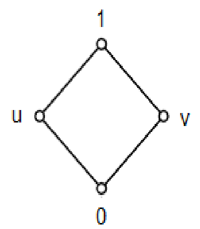

TABLE 1. Cayley tables of $|$, $\vee$ and $\wedge$ on L

$\|$	0	u	v	1
0	1	1	1	1
u	1	v	1	v
v	1	1	u	u
1	1	v	u	0

$\vee$	0	u	v	1
0	0	u	v	1
u	u	u	1	1
v	v	1	v	1
1	1	1	1	1

$\wedge$	0	u	v	1
0	0	0	0	0
u	0	u	0	u
v	0	0	v	v
1	0	u	v	1

Example 2.4. [14] *For a set $L = \{0, a, b, c, d, e, f, 1\}$, the Sheffer stroke BL-algebra $(L, \vee, \wedge, |, 0, 1)$ with the Hasse diagram in Figure 2 and the binary operations $|$, $\vee$ and $\wedge$ on L have Cayley tables in Table 2, 3 and 4, respectively:*

FIGURE 2. Hasse diagram of the Sheffer stroke BL-algebra in Example 2.4

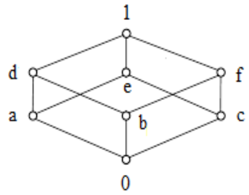

TABLE 2. Cayley table of $|$ on L

$\vert$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	1	f	1	1	f	f	1	f
b	1	1	e	1	e	1	e	e
c	1	1	1	d	1	d	d	d
d	1	f	e	1	c	f	e	c
e	1	f	1	d	f	b	d	b
f	1	1	e	d	e	d	a	a
1	1	f	e	d	c	b	a	0

TABLE 3. Cayley table of $\vee$ on L

$\vee$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	a	d	e	d	e	1	1
b	b	d	b	f	d	1	f	1
c	c	e	f	c	1	e	f	1
d	d	d	d	1	d	1	1	1
e	e	e	1	e	1	e	1	1
f	f	1	f	f	1	1	f	1
1	1	1	1	1	1	1	1	1

Unless otherwise specified, L is stated a Sheffer stroke BL-algebra.

Proposition 2.5. [14] *In any Sheffer stroke BL-algebra L, the following features hold, for all $c_1, c_2, c_3 \in L$:*

1. $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) = c_2|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$,
2. $c_1|(c_1|c_1) = 1$,
3. $1|(c_1|c_1) = c_1$,
4. $c_1|(1|1) = 1$,
5. $(c_1|1)|(c_1|1) = c_1$,

TABLE 4. Cayley table of $\wedge$ on L

$\wedge$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	0	a	a	0	a
b	0	0	b	0	b	0	b	b
c	0	0	0	c	0	c	c	c
d	0	a	b	0	d	a	b	d
e	0	a	0	c	a	e	c	e
f	0	0	b	c	b	c	f	f
1	0	a	b	c	d	e	f	1

6. $(c_1|c_2)|(c_1|c_2) \leq c_3 \Leftrightarrow c_1 \leq c_2|(c_3|c_3)$
7. $c_1 \leq c_2$ *iff* $c_1|(c_2|c_2) = 1$,
8. $c_1 \leq c_2|(c_1|c_1)$,
9. $c_1 \leq (c_1|c_2)|c_2$,
10. *(a)* $(c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) \leq c_1$,
    (b) $(c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) \leq c_2$.
11. *If* $c_1 \leq c_2$, *then*
    (i) $c_3|(c_1|c_1) \leq c_3|(c_2|c_2)$,
    (ii) $(c_1|c_3)|(c_1|c_3) \leq (c_2|c_3)|(c_2|c_3)$,
    (iii) $c_2|(c_3|c_3) \leq c_1|(c_3|c_3)$.
12. $c_1|(c_2|c_2) \leq (c_3|(c_1|c_1))|((c_3|(c_2|c_2))|(c_3|(c_2|c_2)))$,
13. $c_1|(c_2|c_2) \leq (c_2|(c_3|c_3))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$,
14. $((c_1 \vee c_2)|c_3)|((c_1 \vee c_2)|c_3) = ((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3))$,
15. $c_1 \vee c_2 = ((c_1|(c_2|c_2))|(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1))$.

Lemma 2.6. [14] *Let L be a Sheffer stroke BL-algebra. Then*

$$(c_1|(c_2|c_2))|(c_2|c_2) = (c_2|(c_1|c_1))|(c_1|c_1),$$

for all $c_1, c_2 \in L$.

Corollary 2.7. [14] *Let L be a Sheffer stroke BL-algebra. Then*

$$c_1 \vee c_2 = (c_1|(c_2|c_2))|(c_2|c_2),$$

for all $c_1, c_2 \in L$.

Lemma 2.8. [14] *Let L be a Sheffer stroke BL-algebra. Then*

$$((c_1|(c_2|c_2))|(c_2|c_2))|(c_2|c_2) = c_1|(c_2|c_2),$$

for all $c_1, c_2 \in L$.

Lemma 2.9. [14] *Let L be a Sheffer stroke BL-algebra. Then*

$$c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) = (c_1|(c_2|c_2))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3))),$$

for all $c_1, c_2, c_3 \in L$.

Definition 2.10. [14] *A filter P of L is a nonempty subset $P \subseteq L$ satisfying*
$(SF-1)$ *if $c_1, c_2 \in P$, then $(c_1|c_2)|(c_1|c_2) \in P$,*
$(SF-2)$ *if $c_1 \in P$ and $c_1 \leq c_2$, then $c_2 \in P$.*

Proposition 2.11. [14] *Let P be a nonempty subset of L. Then P is a filter of L if and only if the following hold:*
$(SF-3)$ $1 \in P$,
$(SF-4)$ *$c_1 \in P$ and $c_1|(c_2|c_2) \in P$ imply $c_2 \in P$.*

Lemma 2.12. [14] *Let P be a filter of L. Then*

(a) *$c_3|((c_2|(c_1|c_1))|(c_2|(c_1|c_1))) \in P$ and $c_3 \in P$ imply $((c_1|(c_2|c_2))|(c_2|c_2))|(c_1|c_1) \in P$,*
(b) *$c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) \in P$ and $c_1|(c_2|c_2) \in P$ imply $c_1|(c_3|c_3) \in P$,*
(c) *$c_1|(((c_2|(c_3|c_3))|(c_2|c_2))|((c_2|(c_3|c_3))|(c_2|c_2))) \in P$ and $c_1 \in P$ imply $c_2 \in P$,*

for any $c_1, c_2, c_3 \in L$.

Definition 2.13. [14] *Let P be a filter of L. Then P is called an ultra filter of L if it satisfies $c \in P$ or $c|c \in P$, for all $c \in L$.*

Lemma 2.14. [14] *A filter P of L is an ultra filter of L if and only if $c_1 \vee c_2 \in P$ implies $c_1 \in P$ or $c_2 \in P$, for all $c_1, c_2 \in L$.*

3. Characterization by filters

In this section, we present characterizations of Sheffer stroke BL-algebras via filters.

Define a subset $C(c_1, c_2)$ of L by

$$C(c_1, c_2) = \{z \in L : c_2 \leq c_1|(z|z)\},$$

for any $c_1, c_2 \in L$.

Proposition 3.1. *Let P be a nonempty subset of L. Then the following conditions are equivalent:*

1. *P is a filter of L.*
2. *$C(c_1, c_2) \subseteq P$, for any $c_1, c_2 \in P$.*
3. *$c_2|((c_1|(c_3|c_3))|(c_1|(c_3|c_3))) = 1$ implies $c_3 \in P$, for any $c_1, c_2 \in P$ and $c_3 \in L$.*

Proof. (1)⇒(2) Let P be a filter of L and $c_1, c_2 \in P$. Assume that $z \in C(c_1, c_2)$. Then $c_2 \leq c_1|(z|z)$. Thus, $c_1|(z|z) \in P$ from (SF-2), and so, $z \in P$ from (SF-4). It means that $C(c_1, c_2) \subseteq P$, for any $c_1, c_2 \in P$.

(2)⇒(3) It is clear.

(3)⇒(1) Let P be a nonempty subset of L such that $c_2|((c_1|(c_3|c_3))|(c_1|(c_3|c_3))) = 1$ implies $c_3 \in P$, for any $c_1, c_2 \in P$ and $c_3 \in L$.

• Let c_1 and c_2 be any elements of P. Since $c_2|((c_1|(((c_1|c_2)|(c_1|c_2))|((c_1|c_2)|(c_1|$

$c_2))))|(c_1|(((c_1|c_2)|(c_1|c_2))|((c_1|c_2)|(c_1|c_2))))) = c_2|((c_1|(c_1|c_2))|(c_1|(c_1|c_2))) = (c_1|c_2)|((c_1|c_2)|(c_1|c_2)) = 1$ from (S1)-(S3) and Proposition 2.5 (2), it is obtained that $(c_1|c_2)|(c_1|c_2) \in P$.

• Let $c_1 \in P$ and $c_1 \leq c_2$. Since $c_1|((c_1|(c_2|c_2))|(c_1|(c_2|c_2))) = ((c_1|c_1)|(c_1|c_1))|(c_2|c_2) = c_1|(c_2|c_2) = 1$ from (S3), (S2) and Proposition 2.5 (7), respectively, it follows that $c_2 \in P$. □

Lemma 3.2. *Let L be a Sheffer stroke BL-algebra. Then*

1. $C(c_1, c_2) = C(c_2, c_1)$,
2. $C(c, 0) = C(0, c) = L$,
3. $C(c, 1) = C(1, c) = \{z \in L : c \leq z\}$,
4. $C(0, 0) = L$,
5. $C(1, 1) = \{1\}$,
6. $1 \in C(c_1, c_2)$, *for all* $c_1, c_2 \in L$,
7. *if* $c_1 \leq c_2$, *then*
   (a) $C(x, c_2) \subseteq C(x, c_1)$,
   (b) $C(c_2, x) \subseteq C(c_1, x)$,

for all $x, c, c_1, c_2 \in L$.

Proof. 1. Since we have from Proposition 2.5 (1) and (7) that

$$\begin{aligned} z \in C(c_1, c_2) &\Leftrightarrow c_2 \leq c_1|(z|z) \\ &\Leftrightarrow c_2|((c_1|(z|z))|(c_1|(z|z))) = 1 \\ &\Leftrightarrow c_1|((c_2|(z|z))|(c_1|(z|z))) = 1 \\ &\Leftrightarrow c_1 \leq c_2|(z|z) \\ &\Leftrightarrow z \in C(c_2, c_1), \end{aligned}$$

for all $c_1, c_2, z \in L$, it follows that $C(c_1, c_2) = C(c_2, c_1)$.
2. $C(c, 0) = C(0, c) = L$ is obtained from (1), (S1) and Proposition 2.5 (4).
3. $C(c, 1) = C(1, c) = \{z \in L : c \leq z\}$ follows from (1) and Proposition 2.5 (3).
4. It is obvious from (2).
5. $C(1, 1) = \{z \in L : 1 \leq z\} = \{1\}$ by (3) and the fact that 1 is the greatest element of L.
6. Since $c_2 \leq c_1|(1|1) = 1$ from Proposition 2.5 (4) and 1 is the greatest element of L, it follows that $1 \in C(c_1, c_2)$, for all $c_1, c_2 \in L$.
7. (a) Let $c_1 \leq c_2$ and $z \in C(x, c_2)$. Then $c_2 \leq x|(z|z)$. Since $c_1 \leq c_2 \leq x|(z|z)$, it is obtained that $z \in C(x, c_1)$. Thus, $C(x, c_2) \subseteq C(x, c_1)$.
   (b) $C(c_2, x) \subseteq C(c_1, x)$ proved from (a) and (1).

□

Lemma 3.3. *Let L be a Sheffer stroke BL-algebra. Then $c_1 \leq c_2 \Leftrightarrow c_2|c_2 \leq c_1|c_1$, for all $c_1, c_2 \in L$.*

Proof. It follows from Proposition 2.5 (7), (S1) and (S2). □

Lemma 3.4. *Let L be a Sheffer stroke BL-algebra. Then $C(c_1 \vee c_2, c_3) = C(c_1, c_3) \cup C(c_2, c_3)$, for all $c_1, c_2, c_3 \in L$.*

Proof. Let $z \in C(c_1, c_3) \cup C(c_2, c_3)$. Then $z \in C(c_1, c_3)$ or $z \in C(c_2, c_3)$. Thus, $c_3 \leq c_1|(z|z)$ or $c_3 \leq c_2|(z|z)$, and so, $(c_1|(z|z))|(c_1|(z|z)) \leq c_3|c_3$ or $(c_2|(z|z))|(c_2|(z|z)) \leq c_3|c_3$ from Lemma 3.3. Hence, it is obtained from Proposition 2.5 (14) that $((c_1 \vee c_2)|(z|z))|((c_1 \vee c_2)|(z|z)) = ((c_1|(z|z))|(c_1|(z|z))) \vee ((c_2|(z|z))|(c_2|(z|z))) \leq c_3|c_3$. So, it follows from Lemma 3.3 and (S2) that $c_3 \leq (c_1 \vee c_2)|(z|z)$ which implies $z \in C(c_1 \vee c_2, c_3)$. Therefore, $C(c_1, c_3) \cup C(c_2, c_3) \subseteq C(c_1 \vee c_2, c_3)$. Also, $C(c_1 \vee c_2, c_3) \subseteq C(c_1, c_3) \cup C(c_2, c_3)$ follows from Lemma 3.2 7(b). □

Lemma 3.5. *Let L be a Sheffer stroke BL-algebra. Then $C(c_1, c_3) \cap C(c_2, c_3) \subseteq C(c_1 \wedge c_2, c_3)$, for all $c_1, c_2, c_3 \in L$.*

Proof. By Lemma 3.2 7(b), $C(c_1, c_3) \cap C(c_2, c_3) \subseteq C(c_1 \wedge c_2, c_3)$, for all $c_1, c_2, c_3 \in L$. □

Example 3.6. *Consider the Sheffer stroke BL-algebra L in Example 2.4. Then $\{e, 1\} = \{a, d, e, 1\} \cap \{c, e, f, 1\} = C(a, e) \cap C(f, e) \subseteq C(a \wedge f, e) = C(0, e) = L$.*

Lemma 3.7. *Let L be a Sheffer stroke BL-algebra. Then*

1. *$c_1|c_1 \leq c_1|c_2$ and $c_2|c_2 \leq c_1|c_2$,*
2. *$c_1 \leq c_2$ and $d_1 \leq d_2$ imply $c_2|d_2 \leq c_1|d_1$,*

for all $c_1, c_2, d_1, d_2 \in L$.

Proof. 1. It is proved from Proposition 2.5 (8), (S1) and (S2).
2. Let $c_1 \leq c_2$ and $d_1 \leq d_2$. Since $c_2|d_2 = c_2|((d_2|d_2)|(d_2|d_2)) \leq c_1|((d_2|d_2)|(d_2|d_2)) = c_1|d_2$ and $c_1|d_2 = d_2|((c_1|c_1)|(c_1|c_1)) \leq d_1|((c_1|c_1)|(c_1|c_1)) = c_1|d_1$ from Proposition 2.5 11(iii), (S1) and (S2), we have $c_2|d_2 \leq c_1|d_1$. □

Lemma 3.8. *Let L be a Sheffer stroke BL-algebra. Then $C(c_1|c_2, c_3) = C(c_1|c_1, c_3) \cap C(c_2|c_2, c_3)$, for all $c_1, c_2, c_3 \in L$.*

Proof. Since $c_1|c_1 \leq c_1|c_2$ and $c_2|c_2 \leq c_1|c_2$ from Lemma 3.7 (1), it follows from Lemma 3.2 7(b) that $C(c_1|c_2, c_3) \subseteq C(c_1|c_1, c_3)$ and $C(c_1|c_2, c_3) \subseteq C(c_2|c_2, c_3)$, and so, $C(c_1|c_2, c_3) \subseteq C(c_1|c_1, c_3) \cap C(c_2|c_2, c_3)$.

Conversely, let $z \in C(c_1|c_1, c_3) \cap C(c_2|c_2, c_3)$. Then $z \in C(c_1|c_1, c_3)$ and $z \in C(c_2|c_2, c_3)$, and so, $c_3 \leq (c_1|c_1)|(z|z)$ and $c_3 \leq (c_2|c_2)|(z|z)$. Thus, $(c_3|(z|z))|(c_3|(z|z)) \leq c_1$ and $(c_3|(z|z))|(c_3|(z|z)) \leq c_2$ from Proposition 2.5 (6) and (S1). Hence, it follows from Lemma 3.7 (2) and (S2) that $c_1|c_2 \leq c_3|(z|z)$. So, we have from Lemma 3.2 (1) that $z \in C(c_3, c_1|c_2) = C(c_1|c_2, c_3)$ which means that $C(c_1|c_1, c_3) \cap C(c_2|c_2, c_3) \subseteq C(c_1|c_2, c_3)$. □

Lemma 3.9. *Let P be a nonempty subset of L. Then P is a filter of L if and only if*
$(SF-5)$ $1 \in P$ *and*
$(SF-6)$ $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) \in P$ *and* $c_1|(c_2|c_2) \in P$ *imply* $c_1|(c_3|c_3) \in P$, *for all* $c_1, c_2, c_3 \in L$.

Proof. It is obvious from $(SF-3)$and Lemma 2.12 (b).

Conversely, let P be a nonempty subset of L satisfying $(SF-5)$ and $(SF-6)$. Suppose that $c_1 \in P$ and $c_1|(c_2|c_2) \in P$. Since $1|((c_1|(c_2|c_2))|(c_1|(c_2|c_2))) = c_1|(c_2|c_2) \in P$ and $1|(c_1|c_1) = c_1 \in P$ from Proposition 2.5 (3), it follows from $(SF-6)$ that $c_2 = 1|(c_2|c_2) \in P$. Thus, P is a filter of L by Proposition 2.11. □

Lemma 3.10. *Let P be a nonempty subset of L. Then P is a filter of L if and only if $P_a = \{c \in L : a|(c|c) \in P\}$ is a filter of L, for any $a \in L$.*

Proof. Let P be a filter of L. Since $a|(1|1) = 1 \in P$ from Proposition 2.5 (4), we have $1 \in P_a$. Let $c_1 \in P_a$ and $c_1|(c_2|c_2) \in P_a$. Then $a|(c_1|c_1) \in P$ and $a|((c_1|(c_2|c_2))|(c_1|(c_2|c_2))) \in P$. Thus, it is obtained from Lemma 2.12 (b) that $a|(c_2|c_2) \in P$ which implies $c_2 \in P_a$. Hence, P_a is a filter of L from Proposition 2.11.

Conversely, let P_a be a filter of L. Since $1 \in P_a$, it follows from Proposition 2.5 (4) that $1 = a|(1|1) \in P$. Let $a|((c_1|(c_2|c_2))|(c_1|(c_2|c_2))) \in P$ and $a|(c_1|c_1) \in P$. Then $c_1|(c_2|c_2) \in P_a$ and $c_1 \in P_a$. Thus, $c_2 \in P_a$ which means that $a|(c_2|c_2) \in P$. Hence, P is a filter of L by Lemma 3.9. □

Example 3.11. *Consider the Sheffer stroke BL-algebra L in Example 2.4. For the filter $P = \{f, 1\}$ of L, $P_b = \{b, d, f, 1\}$ is a filter of L.*

Lemma 3.12. *Let P be a nonempty subset of L. Then P is a filter of L if and only if*
$(SF-7)$ $1 \in P$ *and*
$(SF-8)$ $c_1|(c_2|c_2) \in P$ *and* $c_2|(c_3|c_3) \in P$ *imply* $c_1|(c_3|c_3) \in P$, *for all* $c_1, c_2, c_3 \in L$.

Proof. Let P be a filter of L, $c_1|(c_2|c_2) \in P$ and $c_2|(c_3|c_3) \in P$. Since $c_1|(c_2|c_2) \leq (c_2|(c_3|c_3))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$ from Proposition 2.5 (13), we have from $(SF-2)$ and $(SF-4)$ that $c_1|(c_3|c_3) \in P$. Also, $1 \in P$ from $(SF-3)$.

Conversely, let P be a nonempty subset of L satisfying $(SF-7)$ and $(SF-8)$, $c_1 \in P$ and $c_1|(c_2|c_2) \in P$. Since $1|(c_1|c_1) = c_1 \in P$ from Proposition 2.5 (3) and $c_1|(c_2|c_2) \in P$, it follows from $(SF-8)$ and Proposition 2.5 (3) that $c_2 = 1|(c_2|c_2) \in P$. Hence, P is a filter of L by Proposition 2.11. □

Proposition 3.13. *Let P be a filter of L. Then P_a is the minimal filter of L containing P and a.*

Proof. Let P be a filter of L. Then P_a is a filter of L from Lemma 3.10. Assume that $c \in P$. Since $c \leq a|(c|c)$ from Proposition 2.5 (8) it follows from $(SF-2)$ that $a|(c|c) \in P$ which implies $c \in P_a$. Thus, $P \subseteq P_a$. Also, we get $a \in P_a$ since $a|(a|a) = 1 \in P$ from Proposition 2.5 (2) and $(SF-3)$. Let Q be a filter of L containing P and a. So, $a|(c|c) \in P \subseteq Q$, for any $c \in P_a$. Since $a \in Q$ and $a|(c|c) \in Q$, it is obtained from $(SF-4)$ that $c \in Q$. Therefore, $P_a \subseteq Q$. □

Remark 3.14. *Let P and Q be two filters of L. Then $P \cap Q$ is always a filter of L. However, $P \cup Q$ is generally not a filter of L.*

Example 3.15. *Consider the Sheffer stroke BL-algebra L in Example 2.3. For the filters $\{u, 1\}$ and $\{v, 1\}$ of L, $\{u, 1\} \cup \{v, 1\} = \{u, v, 1\}$ is not a filter of L since $(u|v)|(u|v) = 1|1 = 0 \notin \{u, v, 1\}$ when $u, v \in \{u, v, 1\}$.*

Corollary 3.16. *Let P and Q be two filters of L. If $L = \{0, 1\}$, then $P \cup Q$ is a filter of L.*

Proposition 3.17. *Let P and Q be two filters of L. Then*

1. *$P_a = P$ if and only if $a \in P$,*
2. *$a \leq b$ implies $P_b \subseteq P_a$,*
3. *$P \subseteq Q$ implies $P_a \subseteq Q_a$,*
4. *$(P \cap Q)_a = P_a \cap Q_a$,*
5. *$P_{(a|b)|(a|b)} = (P_a)_b$,*
6. *$(P_a)_b = (P_b)_a$,*
7. *$(P_a)_a = P_a$,*
8. *$P_1 = P$ and $P_0 = L$,*

for any $a, b \in L$.

Proof.
1. Let $P_a = P$. Since $a|(a|a) = 1 \in P$ from Proposition 2.5 (2) and $(SF-3)$, it follows that $a \in P_a = P$. Conversely, let $a \in P$. Since it is known from Proposition 2.5 (8) that $c \leq a|(c|c)$, for any $c \in P$, we have from $(SF-2)$ that $a|(c|c) \in P$ which implies $c \in P_a$. Thus, $P \subseteq P_a$. Moreover, since $a|(c|c) \in P$, for any $c \in P_a$, and $a \in P$, it is obtained from $(SF-4)$ that $c \in P$ which implies $P_a \subseteq P$. Hence, $P_a = P$.
2. Let $a \leq b$ and $c \in P_b$. Then $b|(c|c) \in P$. Since $b|(c|c) \leq a|(c|c)$ from Proposition 2.5 11(iii), it follows from $(SF-2)$ that $a|(c|c) \in P$ which implies $c \in P_a$. Thus, $P_b \subseteq P_a$.
3. Let $P \subseteq Q$ and $c \in P_a$. Then $a|(c|c) \in P$, and $a|(c|c) \in Q$. Thus, $c \in Q_a$ which means that $P_a \subseteq Q_a$.
4. Since $P \cap Q \subseteq P$ and $P \cap Q \subseteq Q$, $(P \cap Q)_a \subseteq P_a$ and $(P \cap Q)_a \subseteq Q_a$ from (3). Then $(P \cap Q)_a \subseteq P_a \cap Q_a$. Let $c \in P_a \cap Q_a$. Thus, $c \in P_a$ and $c \in Q_a$, and so, $a|(c|c) \in P$ and $a|(c|c) \in Q$. Hence, $a|(c|c) \in P \cap Q$ which means that $c \in (P \cap Q)_a$. Therefore, $P_a \cap Q_a \subseteq (P \cap Q)_a$. Consequently, $(P \cap Q)_a = P_a \cap Q_a$.

5. Since

$$\begin{aligned} c \in P_{(a|b)|(a|b)} &\Leftrightarrow ((a|b)|(a|b))|(c|c) \in P \\ &\Leftrightarrow a|((b|(c|c))|(b|(c|c))) = ((a|b)|(a|b))|(c|c) \in P \\ &\Leftrightarrow b|(c|c) \in P_a \\ &\Leftrightarrow c \in (P_a)_b \end{aligned}$$

from (S3), it follows that $P_{(a|b)|(a|b)} = (P_a)_b$.
6. $(P_a)_b = P_{(a|b)|(a|b)} = P_{(b|a)|(b|a)} = (P_b)_a$ from (5) and (S1).
7. By substituting $[b := a]$ in (5), it follows from (S2) that

$$(P_a)_a = P_{(a|a)|(a|a)} = P_a.$$

8. $P_1 = \{c \in L : c = 1|(c|c) \in P\} = P$ from Proposition 2.5 (3), and $P_0 = \{c \in L : 1 = (c|c)|(1|1) = 0|(c|c) \in P\} = L$ from Proposition 2.5 (4) and (S1).

□

It is not necessary that $a \leq b$ when $P_b \subseteq P_a$, and $P_a \subseteq Q_a$ does not imply $P \subseteq Q$.

Example 3.18. *Consider the Sheffer stroke BL-algebra L in Example 2.4. For the filter $P = \{e, 1\}$ of L, $d \nleq e$ when $P_e = P = \{e, 1\} \subseteq P_d = \{a, d, e, 1\}$. Besides, $P = \{e, 1\} \nsubseteq Q = \{f, 1\}$ when $P_a = \{a, d, e, 1\} \subseteq L = Q_a$.*

Lemma 3.19. *Let P be a filter of L. Then*

1. $P_{a \vee b} \subseteq P_a \cup P_b$,
2. $P_a \cap P_b \subseteq P_{a \wedge b}$,

for any $a, b \in L$.

Proof. The proof is completed from Proposition 3.17 (2). □

Example 3.20. *Consider the Sheffer stroke BL-algebra L in Example 2.4. For the filter $P = \{d, 1\}$ of L, $P_{a \vee c} = P_e = \{a, d, e, 1\} \subseteq L = \{a, d, e, 1\} \cup L = P_a \cup P_c$. Moreover, $P_d \cap P_e = \{b, d, f, 1\} \cap \{c, e, f, 1\} = \{f, 1\} \subseteq L = P_a = P_{d \wedge e}$, for the filter $P = \{f, 1\}$ of L.*

Theorem 3.21. *Let P be a filter of L. Then*

1. $\bigcap_{a \in L} P_a = P$ *and*
2. $\bigcup_{a \in L} P_a = L$,

for any $a \in L$.

Proof. It is proved from Proposition 3.13 and Proposition 3.17 (8). □

Proposition 3.22. *Let L be a Sheffer stroke BL-algebra. Then $C(a) = \{c \in L : a|(c|c) = 1\}$ is a filter of L.*

Proof. Since $a|(1|1) = 1$ from Proposition 2.5 (4), we get $1 \in C(a)$. Let $c_1 \in C(a)$ and $c_1|(c_2|c_2) \in C(a)$. Then $a|(c_1|c_1) = 1$ and $a|((c_1|(c_2|c_2))|(c_1|(c_2|c_2))) = 1$. Since

$$\begin{aligned} a|(c_2|c_2) &= 1|((a|(c_2|c_2))|(a|(c_2|c_2))) \\ &= (a|(c_1|c_1))|((a|(c_2|c_2))|(a|(c_2|c_2))) \\ &= a|((c_1|(c_2|c_2))|(c_1|(c_2|c_2))) \\ &= 1 \end{aligned}$$

from Proposition 2.5 (3) and Lemma 2.9, it follows that $c_2 \in C(a)$. Thus, $C(a)$ is a filter of L by Proposition 2.11. □

Lemma 3.23. *Let L be a Sheffer stroke BL-algebra. Then*

1. $C(0) = L$ *and* $C(1) = \{1\}$,
2. $a \leq b$ *if and only if* $C(b) \subseteq C(a)$, *and*
3. $C(a|b) = C(a|a) \bigcap C(b|b)$,

for any $a, b \in L$.

Proof.
1. $C(0) = \{c \in L : 0|(c|c) = 1\} = \{c \in L : (c|c)|(1|1) = 1, for\ all\ c \in L\} = L$ and $C(1) = \{c \in L : c = 1|(c|c) = 1\} = \{1\}$ from (S1) and Proposition 2.5 (3)-(4).
2. Let $a \leq b$ and $c \in C(b)$. Then $b|(c|c) = 1$. Since $1 = b|(c|c) \leq a|(c|c)$ from Proposition 2.5 11(iii), we get $a|(c|c) = 1$ which means that $c \in C(a)$. Thus, $C(b) \subseteq C(a)$.

   Conversely, let $C(b) \subseteq C(a)$. Since $b|(b|b) = 1$ from Proposition 2.5 (2), we have $b \in C(b)$. Then $b \in C(a)$, and so, $a|(b|b) = 1$. Hence, $a \leq b$ from Proposition 2.5 (7).
3. Since $a|a \leq a|b$ and $b|b \leq a|b$ from Lemma 3.7 (1), it is obtained from (2) that $C(a|b) \subseteq C(a|a)$ and $C(a|b) \subseteq C(b|b)$. Then $C(a|b) \subseteq C(a|a) \bigcap C(b|b)$.

   Conversely, let $c \in C(a|a) \bigcap C(b|b)$. Then $c \in C(a|a)$ and $c \in C(b|b)$, and so, $(a|a)|(c|c) = 1$ and $(b|b)|(c|c) = 1$. Thus, $a|a \leq c$ and $b|b \leq c$ from Proposition 2.5 (7). Since $c|c \leq a$ and $c|c \leq b$ from Lemma 3.3 and (S2), it follows from Lemma 3.7 (2) and (S2) that $a|b \leq c$. Hence, $(a|b)|(c|c) = 1$ from Proposition 2.5 (7). It means that $c \in C(a|b)$, i.e., $C(a|a) \bigcap C(b|b) \subseteq C(a|b)$.

□

Theorem 3.24. *Let L be a Sheffer stroke BL-algebra. Then*

1. $C(a) \cap C(b) \subseteq C(a \wedge b)$,
2. $C(a \vee b) \subseteq C(a) \cup C(b)$,

for any $a, b \in L$.

Proof. The proof follows from Lemma 3.23 (2). □

Example 3.25. *Consider the Sheffer stroke BL-algebra L in Example 2.4. Then* $C(e) \cap C(f) = \{e, 1\} \bigcap \{f, 1\} = \{1\} \subseteq \{c, e, f, 1\} = C(c) = C(e \wedge f)$. *Also,* $C(a \vee b) = C(d) = \{d, 1\} \subseteq \{a, b, d, e, f, 1\} = \{a, d, e, 1\} \cup \{b, d, f, 1\} = C(a) \cup C(b)$.

Lemma 3.26. *Let P be a filter of L. Then $c_1, c_2 \in P$ implies $c_1 \wedge c_2 \in P$, for any $c_1, c_2 \in L$.*

Proof. Let P be a filter of L and $c_1, c_2 \in P$. Then $(c_1|c_2)|(c_1|c_2) \in P$ from $(SF-1)$. Since $(c_1|c_2)|(c_1|c_2) \leq c_1$ and $(c_1|c_2)|(c_1|c_2) \leq c_2$ from Proposition 2.5 (6), (8) and (S1), we have $(c_1|c_2)|(c_1|c_2) \leq c_1 \wedge c_2$. Thus, $c_1 \wedge c_2 \in P$ from $(SF-2)$. □

Lemma 3.27. *Let P be a filter of L such that $P \neq L$. Then P is an ultra filter of L if and only if there exist no a filter Q of L such that $P \subset Q \subset L$.*

Proof. Let P be an ultra filter of L such that $P \neq L$. Suppose that Q is a filter of L such that $P \subset Q \subset L$ and $c \in Q$ such that $c \notin P$. Then $c|c \in P$, and so, $c|c \in Q$. Thus,

$$\begin{aligned} 0 &= 1|1 \\ &= (c|(c|c))|(c|(c|c)) \\ &= (c|(c|((c|c)|(c|c))))|(c|(c|((c|c)|(c|c)))) \\ &= c \wedge (c|c) \in Q \end{aligned}$$

from Proposition 2.5 (2), (S2), $(sBL-3)$ and Lemma 3.26. Since $0 \in Q$ and 0 is the least element of C, it follows from $(SF-2)$ that $c \in Q$, for all $c \in L$. Hence, $Q = L$ which is a contradiction. Therefore, there exist no a filter Q of L such that $P \subset Q \subset L$.

Conversely, let there exist no a filter Q of L such that $P \subset Q \subset L$. Assume that $c_1 \vee c_2 \in P$ but $c_1, c_2 \notin P$. Then there exists a filter Q of L such that $c_1 \in Q$ or $c_2 \in Q$. Since $c_1 \leq c_1 \vee c_2 \in P$ and $c_2 \leq c_1 \vee c_2 \in P$, it is obtained from $(SF-2)$ that $c_1 \vee c_2 \in Q$. Thus, $P \subseteq Q$ which is a contradiction. Hence, $c_1 \vee c_2 \in P$ implies $c_1 \in P$ or $c_2 \in P$ which means that P is an ultra filter of L by Lemma 2.14. □

4. Stabilizers

In this section, we introduce stabilizers in a Sheffer stroke BL-algebra.

Definition 4.1. *Let L be a Sheffer stroke BL-algebra and S be a nonempty subset of L. Then a stabilizer of S is defined as follows:*

$$St(S) = \{c \in L : c|(x|x) = x \ (or\ x|(c|c) = c),\ \forall x \in S\}.$$

Example 4.2. *Consider the Sheffer stroke BL-algebra L in Example 2.4. For the subsets $S_1 = \{c, e\}$ and $S_2 = \{d, 1\}$ of L, the stabilizer of S_1 is $St(S_1) = \{d, 1\}$ and the stabilizer of S_2 is $St(S_2) = \{c, e, f, 1\}$, resp.*

Lemma 4.3. *Let S, T and S_i $(i \in I)$ be nonempty subsets of L. Then*

1. $S \subseteq T$ *implies* $St(T) \subseteq St(S)$,
2. $St(L) = \{1\}$ *and* $St(\{1\}) = L$,
3. $St(S) = \bigcap\{St(\{x\}) : x \in S\}$,
4. $\bigcap_{i\in I} St(S_i) = St(\bigcap_{i\in I} S_i)$ *and*
5. $\bigcup_{i\in I} St(S_i) = St(\bigcup_{i\in I} S_i)$.

Proof. 1. Let $S \subseteq T$ and $c \in St(T)$. Then $c|(x|x) = x$, for all $x \in T$. Since $S \subseteq T$, $c|(y|y) = y$, for all $y \in S$. Thus, $c \in St(S)$ which implies $St(T) \subseteq St(S)$.

2. Since it is known from Proposition 2.5 (3) that $1|(x|x) = x$, for all $x \in L$, it follows that $\{1\} \subseteq St(L)$. Let $c \in St(L)$. Then $c|(x|x) = x$, for all $x \in L$. Since $1 = c|(c|c) = c$ from Proposition 2.5 (2), we get $St(L) \subseteq \{1\}$. Thus, $St(L) = \{1\}$. Moreover, since it is known from Proposition 2.5 (4) that $c|(1|1) = 1$, for all $c \in L$, it is obtained that $St(\{1\}) = L$.
3. Let $c \in \bigcap\{St(\{x\}) : x \in S\}$. Then $c \in St(\{x\})$, for all $x \in S$. Thus, $c|(x|x) = x$, for all $x \in S$, and so, $c \in St(S)$. Hence, $\bigcap\{St(\{x\}) : x \in S\} \subseteq St(S)$. Conversely, since $\{x\} \subseteq S$, for all $x \in S$, it follows from (1) that $St(S) \subseteq St(\{x\})$, for all $x \in S$. So, $St(S) \subseteq \bigcap\{St(\{x\}) : x \in S\}$. Therefore, $St(S) = \bigcap\{St(\{x\}) : x \in S\}$.
4. Since $\bigcap_{i\in I} S_i \subseteq S_i$, it follows from (1) that $St(S_i) \subseteq St(\bigcap_{i\in I} S_i)$, and so, $\bigcap_{i\in I} St(S_i) \subseteq St(\bigcap_{i\in I} S_i)$. Conversely, let $c \in St(\bigcap_{i\in I} S_i)$. Then $c|(x|x) = x$, for all $x \in \bigcap_{i\in I} S_i$. Since $c|(x|x) = x$, for all $x \in S_i$ and $i \in I$, we have $c \in St(S_i)$, for all $i \in I$ which means that $c \in \bigcap_{i\in I} St(S_i)$. Thus, $St(\bigcap_{i\in I} S_i) \subseteq \bigcap_{i\in I} St(S_i)$. Therefore,
$$\bigcap_{i\in I} St(S_i) = St(\bigcap_{i\in I} S_i).$$
5. Since $S_i \subseteq \bigcup_{i\in I} S_i$, it is obtained from (1) that $St(\bigcup_{i\in I} S_i) \subseteq St(S_i)$, and so, $St(\bigcup_{i\in I} S_i) \subseteq \bigcup_{i\in I} St(S_i)$. Conversely, let $c \in \bigcup_{i\in I} St(S_i)$. Then $c \in St(S_{i_0})$, for some $i_0 \in I$. Since $c|(x|x) = x$, for all $x \in S_{i_0}$ and some $i_0 \in I$, we get $c|(x|x) = x$, for all $x \in \bigcup_{i\in I} S_i$. So, $c \in St(\bigcup_{i\in I} S_i)$ which means that $\bigcup_{i\in I} St(S_i) \subseteq St(\bigcup_{i\in I} S_i)$. Hence,
$$\bigcup_{i\in I} St(S_i) = St(\bigcup_{i\in I} S_i).$$

□

Theorem 4.4. *Let L be a Sheffer stroke BL-algebra and S be a nonempty subset of L. Then $St(S)$ is a filter of L.*

Proof. Let $c_1, c_2 \in St(S)$. Then $c_1|(x|x) = x$ and $c_2|(x|x) = x$, for all $x \in S$. Since it is obtained from (S3) that $((c_1|c_2)|(c_1|c_2))|(x|x) = c_1|((c_2|(x|x))|(c_2|(x|x))) = c_1|(x|x) = x$, for all $x \in S$, we have $(c_1|c_2)|(c_1|c_2) \in St(S)$.

Let $c_1 \in St(S)$ and $c_1 \leq c_2$. Then $c_1|(x|x) = x$, for all $x \in S$, and so, $c_2|(x|x) \leq c_1|(x|x) = x$ from Proposition 2.5 11(iii). Since $x \leq c_2|(x|x)$ from

Proposition 2.5 (8), it follows that $c_2|(x|x) = x$, for all $x \in S$, which means that $c_2 \in St(S)$. □

However, S is generally not a filter of L if $St(S)$ is a filter of L.

Example 4.5. *Consider the Sheffer stroke BL-algebra L in Example 2.3. Then $St(\{0,u\}) = \{v,1\}$ is a filter of L but $\{0,u\}$ is not a filter of L.*

Theorem 4.6. *Let L be a Sheffer stroke BL-algebra and c be an element of L which there exist no an element $a \in L$ such that $c < a < 1$. Then $St(\{c\})$ is an ultra filter of L.*

Proof. Let L be a Sheffer stroke BL-algebra and c be an element of L which there exist no an element $a \in L$ such that $c < a < 1$. By 4.4, $St(\{c\})$ is a filter of L. Assume that $u \notin St(\{c\})$ and $u|u \notin St(\{c\})$. Then $u|(c|c) \neq c$ and $(u|u)|(c|c) \neq c$. Since $c \leq u|(c|c)$ and $c \leq (u|u)|(c|c)$ from Proposition 2.5 (8), it follows that $u|(c|c) = 1$ and $(u|u)|(c|c) = 1$. Thus, $u \leq c$ and $u|u \leq c$ by Proposition 2.5 (7). So, $1 = u|(u|u) = (u|((u|u)|(u|u)))|((u|u)|(u|u)) = u \vee (u|u) \leq c \vee c = c$ from Proposition 2.5 (2), (S1)-(S2) and Corollary 2.7. This is a contradiction with $c < 1$. Hence, $u \in St(\{c\})$ or $u|u \in St(\{c\})$ which means that $St(\{c\})$ is an ultra filter of L. □

Theorem 4.7. *Let L be a Sheffer stroke BL-algebra and S be a nonempty subset of L. Then*

1. *$St(\{0\}) = \{1\}$ and*
2. *$St(\{0\}) \subseteq P$, for all filters P of L.*

Proof. It is obtained from Proposition 2.5 (5) and Theorem 4.4. □

Definition 4.8. *Let L be a Sheffer stroke BL-algebra, S and T be nonempty subsets of L. Then a stabilizer of S with respect to T is defined as follows:*

$$St(S,T) = \{c \in L : c \vee x \in T, \ \forall x \in S\}.$$

Example 4.9. *Consider the Sheffer stroke BL-algebra L in Example 2.4. For the subsets $S_1 = \{d,e\}$ and $T_1 = \{c,f\}$ of L, the stabilizer $St(S_1,T_1)$ of S_1 with respect to T_1 is empty set. Moreover, $St(S_2,T_2) = \{1\}$, for the subsets $S_2 = \{c,d,e\}$ and $T_2 = \{0,1\}$ of L.*

Theorem 4.10. *Let S, T, S_i and T_i $(i \in I)$ be nonempty subsets and P be a filter of L. Then*

1. *$St(S,T) = L$ implies $S \subseteq T$,*
2. *$P \subseteq T$ if and only if $St(P,T) = L$,*
3. *$St(P,P) = L$,*
4. *$St(S) \subseteq St(S,P)$,*
5. *if $S_i \subseteq T_i$ and $S_j \subseteq T_j$, then $St(T_i,S_j) \subseteq St(S_i,T_j)$,*
6. *$St(S,\{1\}) = St(S)$,*
7. *$St(S,\bigcap_{i\in I} T_i) = \bigcap_{i\in I} St(S,T_i)$,*
8. *$St(S,\bigcup_{i\in I} T_i) = \bigcup_{i\in I} St(S,T_i)$,*

9. $St(\emptyset, T) = \emptyset$,
10. $St(\{0\}, T) = T$,
11. $St(S, \{0\}) = \emptyset$ *and*
12. *If* $S = \{0\}$, *then* $St(S, \{0\}) = \{0\}$.

Proof. 1. Let $St(S,T) = L$. Since $c = c \vee c \in T$, for all $c \in S$, it is obtained that $c \in T$. Thus, $S \subseteq T$.

2. If $St(P,T) = L$, then $P \subseteq T$ from (1). Conversely, let P be a filter of L such that $P \subseteq T$, and $c \in L$. Then $x \leq c \vee x$, for all $x \in P$. Since P is a filter of L, it follows from $(SF-2)$ that $c \vee x \in P$, and so, $c \vee x \in T$, for all $x \in P$. Thus, $c \in St(S,T)$b which implies $St(P,T) = L$.
3. It follows from (2).
4. Let $c \in St(S)$. Then $c|(x|x) = x$, for all $x \in S$. Since $c \vee x = (c|(x|x))|(x|x) = x|(x|x) = 1 \in P$ from Corollary 2.7, Proposition 2.5 (2) and $(SF-3)$, respectively, it is obtained that $c \in St(S,P)$, i.e., $St(S) \subseteq St(S,P)$.
5. Let $S_i \subseteq T_i$, $S_j \subseteq T_j$ and $c \in St(T_i, S_j)$. Then $c \vee x \in S_j$, for all $x \in T_i$. Thus, $c \vee x \in T_j$, for all $x \in S_i$. Hence, $c \in St(S_i, T_j)$, and so, $St(T_i, S_j) \subseteq St(S_i, T_j)$.
6. Since $\{1\}$ is a filter of L, it follows from (4) that $St(S) \subseteq St(S,\{1\})$. Let $c \in St(S,\{1\})$. Then $c \vee x = 1$, for all $x \in S$. Since $1 = c \vee x = (c|(x|x))|(x|x)$ from Corollary 2.7, it is obtained from Proposition 2.5 (7)-(8) that $c|(x|x) = x$, for all $x \in S$. Thus, $c \in St(S)$ which implies $St(S,\{1\}) \subseteq St(S)$. Therefore, $St(S,\{1\}) = St(S)$.
7. Let $c \in St(S, \bigcap_{i\in I} T_i)$. Then $c \vee x \in \bigcap_{i\in I} T_i$, for all $x \in S$. Thus, $c \vee x \in T_i$, for all $i \in I$ and $x \in S$. Hence, $c \in St(S, T_i)$, for all $i \in I$, which means that $c \in \bigcap_{i\in I} St(S, T_i)$. So, $St(S, \bigcap_{i\in I} T_i) \subseteq \bigcap_{i\in I} St(S, T_i)$. Conversely, let $c \in \bigcap_{i\in I} St(S, T_i)$. Then $c \in St(S, T_i)$, for all $i \in I$, and so, $c \vee x \in T_i$, for all $i \in I$ and $x \in S$. Hence, $c \vee x \in \bigcap_{i\in I} T_i$, for all $x \in S$, i.e., $c \in St(S, \bigcap_{i\in I} T_i)$. Thus, $\bigcap_{i\in I} St(S, T_i) \subseteq St(S, \bigcap_{i\in I} T_i)$.
8. Let $c \in St(S, \bigcup_{i\in I} T_i)$. Then $c \vee x \in \bigcup_{i\in I} T_i$, for all $x \in S$, and so, $c \vee x \in T_{i_0}$, for some $i_0 \in I$ and all $x \in S$. Thus, $c \in St(S, T_{i_0})$, for some $i_0 \in I$. Hence, $c \in \bigcup_{i\in I} St(S, T_i)$ which implies $St(S, \bigcup_{i\in I} T_i) \subseteq \bigcup_{i\in I} St(S, T_i)$.

   Conversely, let $c \in \bigcup_{i\in I} St(S, T_i)$. Then $c \in St(S, T_{i_0})$, for some $i_0 \in I$. So, $c \vee x \in T_{i_0}$, for some $i_0 \in I$ and all $x \in S$. Hence, $c \vee x \in \bigcup_{i\in I} T_i$, for all $x \in S$, which means that $c \in St(S, \bigcup_{i\in I} T_i)$. Therefore,

$$\bigcup_{i\in I} St(S, T_i) \subseteq St(S, \bigcup_{i\in I} T_i).$$

9. $St(\emptyset, T) = \{c \in L : c \vee x \in T,\ \forall x \in \emptyset\} = \emptyset$.
10. $St(\{0\}, T) = \{c \in L : c = c \vee 0 \in T\} = T$.
11. $St(S, \{0\}) = \{c \in L : c \vee x = 0,\ \forall x \in S\} = \emptyset$.
12. If $S = \{0\}$, then $St(S, \{0\}) = \{c \in L : c = c \vee 0 = 0\} = \{0\}$.

□

Theorem 4.11. *Let T, S_1 and S_2 be nonempty subsets of L. Then $S_1 \subseteq S_2 \Rightarrow St(S_2, T) \subseteq St(S_1, T)$.*

Proof. Let $S_1 \subseteq S_2$ and $c \in St(S_2, T)$. Then $c \vee x \in T$, for all $x \in S_2$. Since $c \vee y \in T$, for all $y \in S_1 \subseteq S_2$, we get $c \in St(S_1, T)$ which means that $St(S_2, T) \subseteq St(S_1, T)$. □

Example 4.12. *Consider the Sheffer stroke BL-algebra L in Example 2.4. For the subsets $T = \{d, f\}$, $S_1 = \{a, b\}$ and $S_2 = \{c, e\}$ of L, $S_1 \nsubseteq S_2$ when $St(S_2, T) = \emptyset \subseteq \{d\} = St(S_1, T)$.*

Theorem 4.13. *Let L be a Sheffer stroke BL-algebra, P and Q be two filters of L. Then $St(P, Q)$ is a filter of L.*

Proof. Let P, Q be two filters of L and $c_1, c_2 \in St(P, Q)$. Then $c_1 \vee x \in Q$ and $c_2 \vee x \in Q$, for all $x \in P$. Since Q is a filter of L, it is obtained from $(SF-1)$ that $((c_1 \vee x)|(c_2 \vee x))|((c_1 \vee x)|(c_2 \vee x)) \in Q$. Since

$$\begin{aligned}
((c_1|c_2)|(c_1|c_2)) \vee x &= (((c_1|c_2)|(c_1|c_2))|(x|x))|(x|x) \\
&= (c_1|((c_2|(x|x))|(c_2|(x|x))))|(x|x) \\
&= (c_1|((((c_2|(x|x))|(x|x))|(x|x))| \\
&\quad (((c_2|(x|x))|(x|x))|(x|x))))|(x|x) \\
&= (c_1|(((c_2 \vee x)|(x|x))|((c_2 \vee x)|(x|x))))|(x|x) \\
&= (((c_1|(c_2 \vee x))|(c_1|(c_2 \vee x)))|(x|x))|(x|x) \\
&= ((c_1|(c_2 \vee x))|(c_1|(c_2 \vee x))) \vee x \\
&= ((c_1|(c_2 \vee x))|(c_1|(c_2 \vee x))) \vee ((((x|x)|(x|x))|((x \\
&\quad |x)|(c_2|(x|x))))|(((x|x)|(x|x))|((x|x)|(c_2|(x|x))))) \\
&= ((c_1|(c_2 \vee x))|(c_1|(c_2 \vee x))) vee ((x|(c_2 \vee x))|(x|(c_2 \vee x))) \\
&= ((c_1 \vee x)|(c_2 \vee x))|((c_1 \vee x)|(c_2 \vee x)) \in Q
\end{aligned}$$

from (S1)-(S3), Corollary 2.7, Lemma 2.8 and Proposition 2.5 (14), it follows that $((c_1|c_2)|(c_1|c_2)) \in St(P, Q)$. Also, let $c_1 \in St(P, Q)$ and $c_1 \leq c_2$. Then $c_1 \vee x \in Q$, for all $x \in P$. Since $c_1 \vee x \leq c_2 \vee x$ and Q is a filter of L, we get $c_2 \vee x \in Q$, for all $x \in P$. Thus, $c_2 \in St(P, Q)$. □

Theorem 4.14. *Let L be a Sheffer stroke BL-algebra, P be a filter and Q be an ultra filter of L. Then $St(P, Q)$ is an ultra filter of L.*

Proof. $St(P, Q)$ is a filter of L from Theorem 4.13. Let $c_1 \vee c_2 \in St(P, Q)$. Then $(c_1 \vee c_2) \vee x \in Q$, for all $x \in P$. Since $(c_1 \vee x) \vee (c_2 \vee x) = (c_1 \vee c_2) \vee (x \vee x) = (c_1 \vee c_2) \vee x \in Q$ and Q is an ultra filter of L, it is obtained from Lemma 2.14 that $c_1 \vee x \in Q$ or $c_2 \vee x \in Q$, for all $x \in P$. Thus, $c_1 \in St(P, Q)$ or $c_2 \in St(P, Q)$ which means that $St(P, Q)$ is an ultra filter of L. □

5. Conclusion

In the present paper, we have studied on new features and results in filters and stabilizers of Sheffer stroke BL-algebras. After presenting basic definitions and notions about Sheffer stroke BL-algebra, we investigate new properties of various filters of a Sheffer stroke BL-algebra and give relationships between them. Then we define a stabilizer $St(S)$ of a nonempty subset S of a Sheffer stroke BL-algebra L and show that the stabilizer is a filter of L. Also, it is proved that a stabilizer $St(\{c\})$ is an ultra filter of a Sheffer stroke BL-algebra L where there exist no an element $a \in L$ such that $c < a < 1$. It is demonstrated that the stabilizer of the subset $\{0\}$ of a Sheffer stroke BL-algebra is $\{1\}$, and so, it is contained by all filters of this algebraic structure. Besides, a stabilizer $St(S,T)$ of $S \neq \emptyset$ with respect to $T \neq \emptyset$ is described on a Sheffer stroke BL-algebra L and some properties are presented. Indeed, it is stated that the stabilizer $St(P,Q)$ is a filter of a Sheffer stroke BL-algebra if P and Q are two filters of this algebraic structure. Finally, it is shown that $St(P,Q)$ is an ultra filter of a Sheffer stroke BL-algebra when P is any filter and Q is an ultra filter of this algebra.

In the future works, we wish to study atoms and branches of Sheffer stroke BL-algebras.

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