J. Appl. Math. & Informatics Vol. **40***(2022), No. 1 - 2, pp. 371 - 391*
https://doi.org/10.14317/jami.2022.371

SOME PROPERTIES OF POLY-COSINE TANGENT AND POLY-SINE TANGENT POLYNOMIALS†

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ABSTRACT. In this paper we give some prperties of the poly-cosine tangent polynomials and poly-sine tangent polynomials.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.
Key words and phrases : Cosine Bernoulli polynomials, sine Bernoulli polynomials, tangent numbers and polynomials, poly-cosine tangent polynomials, poly-sine tangent polynomials.

1. Introduction

Many mathematicians have been working on Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, and tangent numbers and polynomials (see [1, 2, 3, 4, 5, 6, 10, 11, 12]). It is well known that the Bernoulli polynomials are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{1}$$

When $x=0, B_n=B_n(0)$ are called the Bernoulli numbers. The tangent polynomials are given by the generating function to be

$$\left(\frac{2}{e^{2t}+1}\right)e^{xt}=\sum_{n=0}^{\infty}T_n(x)\frac{t^n}{n!}. \tag{2}$$

When $x=0, T_n=T_n(0)$ are called the tangent numbers (see [5, 6]).

Received April 10, 2021. Revised July 5. 2021. Accepted January 24, 2022.
†This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

The Bernoulli polynomials $\mathbf{B}_n^{(r)}(x)$ of order r are defined by the following generating function

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} \mathbf{B}_n^{(r)}(x)\frac{t^n}{n!}, \quad (|t| < 2\pi). \tag{3}$$

The Frobenius–Euler polynomials of order r, denoted by $\mathbf{H}_n^{(r)}(u,x)$, are defined as

$$\left(\frac{1-u}{e^t-u}\right)^r e^{xt} = \sum_{n=0}^{\infty} \mathbf{H}_n^{(r)}(u,x)\frac{t^n}{n!}. \tag{4}$$

The values at $x = 0$ are called Frobenius-Euler numbers of order r; when $r = 1$, the polynomials or numbers are called ordinary Frobenius-Euler polynomials or numbers. The cosine-tangent polynomials $T_n^{(C)}(x,y)$ and sine-tangent polynomials $T_n^{(S)}(x,y)$ are defined by means of the generating functions

$$\sum_{n=0}^{\infty} T_n^{(C)}(x,y)\frac{t^n}{n!} = \frac{2}{e^{2t}+1}e^{xt}\cos yt, \tag{5}$$

and

$$\sum_{n=0}^{\infty} T_n^{(k,S)}(x,y)\frac{t^n}{n!} = \frac{2}{e^{2t}+1}e^{xt}\sin yt, \tag{6}$$

respectively.

In this paper, we introduce some special polynomials which are related to tangent polynomials. In addition, we give some identities for these polynomials. Finally, we investigate the distribution of zeros of these polynomials.

2. Poly-cosine tangent polynomials and poly-sine tangent polynomials

In this section, we define the poly-cosine tangent and poly-sine tangen polynomials. For any integer k, let $Li_k(t)$ be the power series given by

$$Li_k(t) = \sum_{m=1}^{\infty} \frac{t^m}{m^k}. \tag{7}$$

When $k = 1$, $Li_1(t) = -\log(1-t)$. In [8], we introduced poly-tangent numbers and polynomials. After that we investigated some their properties. We also obtained some relationships both between these polynomials and tangent polynomials and between these polynomials and cauchy numbers. Now, we define modified poly-tangent numbers and polynomials.

Definition 2.1. For any integer k, the modified poly-tagent polynomials $T_n^{(k)}(z)$ are defined by means of the generating function

$$\sum_{n=0}^{\infty} T_n^{(k)}(z)\frac{t^n}{n!} = \frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{zt}. \tag{8}$$

The numbers $T_n^{(k)}(0) := T_n^{(k)}$ are called the modified poly-tagent numbers. If $k = 1$, then

$$T_n^{(1)}(x) = T_n(x), T_n^{(1)} = T_n.$$

Now, we consider the poly-tagent polynomials that are given by the generating function to be

$$\sum_{n=0}^{\infty} T_n^{(k)}(x+iy)\frac{t^n}{n!} = \frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{(x+iy)t}. \tag{9}$$

On the other hand, we note that

$$e^{(x+iy)t} = e^{xt}e^{iyt} = e^{xt}(\cos yt + i\sin yt). \tag{10}$$

From (9) and (10), we obtain

$$\sum_{n=0}^{\infty} T_n^{(k)}(x+iy)\frac{t^n}{n!} = \frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{xt}(\cos yt + i\sin yt), \tag{11}$$

and

$$\sum_{n=0}^{\infty} T_n^{(k)}(x-iy)\frac{t^n}{n!} = \frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{xt}(\cos yt - i\sin yt). \tag{12}$$

Hence, by (11) and (12), we obtain

$$\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{xt}\cos yt = \sum_{n=0}^{\infty}\left(\frac{T_n^{(k)}(x+iy)+T_n^{(k)}(x-iy)}{2}\right)\frac{t^n}{n!}, \tag{13}$$

and

$$\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{xt}\sin yt = \sum_{n=0}^{\infty}\left(\frac{T_n^{(k)}(x+iy)+T_n^{(k)}(x-iy)}{2i}\right)\frac{t^n}{n!}. \tag{14}$$

It follows that we define the following poly-cosine tangent and poly-sine-tangent polynomials.

Definition 2.2. The poly-cosine tangent polynomials $T_n^{(k,C)}(x,y)$ and poly-sine tangent polynomials $T_n^{(k,S)}(x,y)$ are defined by means of the generating functions

$$\sum_{n=0}^{\infty} T_n^{(k,C)}(x,y)\frac{t^n}{n!} = \frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{xt}\cos yt, \tag{15}$$

and

$$\sum_{n=0}^{\infty} T_n^{(k,S)}(x,y)\frac{t^n}{n!} = \frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{xt}\sin yt, \tag{16}$$

respectively.

Note that $T_n^{(k,C)}(x,0) = T_n^{(k)}(x), T_n^{(k,S)}(x,0) = 0, (n \geq 0)$.
By (13)-(16), we have

$$T_n^{(k,C)}(x,y) = \frac{T_n^{(k)}(x+iy) + T_n^{(k)}(x-iy)}{2},$$
$$T_n^{(k,S)}(x,y) = \frac{T_n^{(k)}(x+iy) - T_n^{(k)}(x-iy)}{2i}.$$

Clearly, we obtain the following explicit representations of $T_n^{(k)}(x+iy)$

$$T_n^{(k)}(x+iy) = \sum_{l=0}^{n} \binom{n}{l} T_l^{(k)}(x+iy)^{n-l},$$
$$T_n^{(k}(x+iy) = \sum_{l=0}^{n} \binom{n}{l} T_l^{(k)}(x)(iy)^{n-l}.$$

Let

$$e^{xt}\cos yt = \sum_{l=0}^{\infty} C_l(x,y)\frac{t^l}{l!}, \qquad e^{xt}\sin yt = \sum_{l=0}^{\infty} S_l(x,y)\frac{t^l}{l!}. \tag{17}$$

Then, by Taylor expansions of $e^{xt}\cos yt$ and $e^{xt}\sin yt$, we get

$$e^{xt}\cos yt = \sum_{l=0}^{\infty}\left(\sum_{m=0}^{[\frac{l}{2}]}\binom{l}{2m}(-1)^m x^{l-2m}y^{2m}\right)\frac{t^l}{l!} \tag{18}$$

and

$$e^{xt}\sin yt = \sum_{l=0}^{\infty}\left(\sum_{m=0}^{[\frac{l-1}{2}]}\binom{l}{2m+1}(-1)^m x^{l-2m-1}y^{2m+1}\right)\frac{t^l}{l!}, \tag{19}$$

where [ ] denotes taking the integer part (see [Axiom]). By (17), (18) and (19), we get

$$C_l(x,y) = \sum_{m=0}^{[\frac{l}{2}]}\binom{l}{2m}(-1)^m x^{l-2m}y^{2m},$$

and

$$S_l(x,y) = \sum_{m=0}^{[\frac{l-1}{2}]}\binom{l}{2m+1}(-1)^m x^{l-2m-1}y^{2m+1}, (l \geq 0).$$

Now, we observe that

$$\begin{aligned}\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{xt}\cos yt &= \left(\sum_{l=0}^{\infty} T_l^{(k)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} C_m(x,y)\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}T_l^{(k)}C_{n-l}(x,y)\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, we obtain the following theorem

Theorem 2.3. *For $n \geq 0$, we have*

$$T_n^{(k,C)}(x,y) = \sum_{l=0}^{n} \binom{n}{l} T_l^{(k)} C_{n-l}(x,y)$$

and

$$T_n^{(k,S)}(x,y) = \sum_{l=0}^{n} \binom{n}{l} T_l^{(k)} S_{n-l}(x,y).$$

Let

$$\begin{aligned} Li_k(1-e^{-t})e^{xt}\cos yt &= \sum_{l=0}^{\infty} C_l^{(k)}(x,y)\frac{t^l}{l!}, \\ Li_k(1-e^{-t})e^{xt}\sin yt &= \sum_{l=0}^{\infty} S_l^{(k)}(x,y)\frac{t^l}{l!}. \end{aligned} \tag{20}$$

Then we get

$$\begin{aligned} \sum_{n=0}^{\infty} C_n^{(k)}(x,y)\frac{t^n}{n!} &= \sum_{l=0}^{\infty} \frac{(1-e^{-t})^{l+1}}{(l+1)^k} e^{xt}\cos yt, \\ &= \sum_{l=0}^{\infty} \frac{1}{(l+1)^k} \sum_{i=0}^{l+1} \binom{l+1}{i} (-1)^i e^{(x-i)t}\cos(yt) \\ &= \sum_{l=0}^{\infty} \frac{1}{(l+1)^k} \sum_{i=0}^{l+1} \binom{l+1}{i} (-1)^i \sum_{n=0}^{\infty} C_n(x-i,y)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{\infty} \frac{1}{(l+1)^k} \sum_{i=0}^{l+1} \binom{l+1}{i} (-1)^i C_n(x-i,y) \right) \frac{t^n}{n!}. \end{aligned} \tag{21}$$

By (20) and (21), we get

$$C_n^{(k)}(x,y) = \sum_{l=0}^{\infty} \frac{1}{(l+1)^k} \sum_{i=0}^{l+1} \binom{l+1}{i} (-1)^i C_n(x-i,y),$$

and

$$S_n^{(k)}(x,y) = \sum_{l=0}^{\infty} \frac{1}{(l+1)^k} \sum_{i=0}^{l+1} \binom{l+1}{i} (-1)^i S_n(x-i,y).$$

A few of them are

$$\begin{aligned} &C_0^{(k)}(x,y) = 1, \quad C_1^{(k)}(x,y) = 1, \\ &C_2^{(k)}(x,y) = -1 + 2^{1-k} + 2x, \\ &C_3^{(k)}(x,y) = 1 - 3\cdot 2^{1-k} + 2\cdot 3^{1-k} - 3x + 3\cdot 2^{1-k}x + 3x^2 - 3y^2, \end{aligned}$$

and

$$S_0^{(k)}(x,y) = 0, \quad S_1^{(k)}(x,y) = 0,$$
$$S_2^{(k)}(x,y) = 2y,$$
$$S_3^{(k)}(x,y) = -3y + 3\cdot 2^{1-k}y + 6xy.$$
$$S_4^{(k)}(x,y) = 4y - 3\cdot 2^{3-k}y + 8\cdot 3^{1-k}y - 12xy + 3\cdot 2^{3-k}xy + 12x^2y - 4y^3.$$

Now, we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n^{(k,C)}(x,y)\frac{t^{n+1}}{n!} &= \frac{2}{e^{2t}+1} Li_k(1-e^{-t})e^{xt}\cos yt \\
&= \left(\sum_{n=0}^{\infty} C_n^{(k)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} T_n\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}C_l^{(k)}(x,y)T_{n-l}\right)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, we obtain the following theorem

Theorem 2.4. *For $n > 0$, we have*

$$nT_{n-1}^{(k,C)}(x,y) = \sum_{l=0}^{n}\binom{n}{l}C_l^{(k)}(x,y)T_{n-l}$$

and

$$nT_{n-1}^{(k,S)}(x,y) = \sum_{l=0}^{n}\binom{n}{l}S_l^{(k)}(x,y)T_{n-l}.$$

From (15), we have

$$\begin{aligned}
&2Li_k(1-e^{-t})e^{xt}\cos yt \\
&= \left(\sum_{n=0}^{\infty} T_n^{(k,C)}(x,y)\frac{t^{n+1}}{n!}\right)\left(e^{2t}+1\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}lT_{l-1}^{(k,C)}(x,y)2^{n-l} + nT_{n-1}^{(k,C)}(x,y)\right)\frac{t^n}{n!}.
\end{aligned} \tag{22}$$

By (15) and (22), we get

$$C_n^{(k)}(x,y) = \frac{1}{2}\left(\sum_{l=0}^{n}\binom{n}{l}lT_{l-1}^{(k,C)}(x,y)2^{n-l} + nT_{n-1}^{(k,C)}(x,y)\right). \tag{23}$$

Therefore, we obtain the following theorem

Theorem 2.5. *For $n > 0$, we have*

$$C_n^{(k)}(x,y) = \frac{1}{2}\left(\sum_{l=0}^{n}\binom{n}{l}lT_{l-1}^{(k,C)}(x,y)2^{n-l} + nT_{n-1}^{(k,C)}(x,y)\right),$$

and

$$S_n^{(k)}(x,y)=\frac{1}{2}\left(\sum_{l=0}^{n}\binom{n}{l}lT_{l-1}^{(k,S)}(x,y)2^{n-l}+nT_{n-1}^{(k,S)}(x,y)\right).$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}T_n^{(k,C)}(x+2,y)\frac{t^n}{n!}&=\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{(x+2)t}\cos yt\\&=\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{xt}(e^{2t}-1+1)\cos yt\\&=\frac{2}{t}Li_k(1-e^{-t})e^{xt}\cos yt-\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{xt}\cos yt\end{aligned}$$

Hence we have

$$\sum_{n=0}^{\infty}\left(T_n^{(k,C)}(x+2,y)+T_n^{(k,C)}(x,y)\right)\frac{t^{n+1}}{n!}=\sum_{n=0}^{\infty}\left(2C_n^{(k)}(x,y)\right)\frac{t^n}{n!}.$$

By comparing the coefficients on the both sides, we get

$$T_{n-1}^{(k,C)}(x+2,y)+T_{n-1}^{(k,C)}(x,y)=\frac{2}{n}C_n^{(k)}(x,y),\ (n\geq 1).$$

Therefore, we obtain the following theorem:

Theorem 2.6. *For $n\geq 1$, we have*

$$T_{n-1}^{(k,C)}(x+2,y)+T_{n-1}^{(k,C)}(x,y)=\frac{2}{n}C_n^{(k)}(x,y),$$

and

$$T_{n-1}^{(k,S)}(x+2,y)+T_{n-1}^{(k,S)}(x,y)=\frac{2}{n}S_n^{(k)}(x,y).$$

By (15), we have

$$\begin{aligned}\sum_{n=0}^{\infty}T_n^{(k,C)}(x+r,y)\frac{t^n}{n!}&=\left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{xt}\cos yt\right)e^{rt}\\&=\left(\sum_{l=0}^{\infty}T_l^{(k,C)}(x,y)\frac{t^l}{l!}\right)\left(\sum_{k=0}^{\infty}r^k\frac{t^k}{k!}\right)\qquad(24)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}T_k^{(k,C)}(x,y)r^{n-k}\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, by comparing the coefficients on the both sides, we obtain the following theorem:

Theorem 2.7. *For $n\geq 0, r\in\mathbb{N}$, we have*

$$T_n^{(k,C)}(x+r,y)=\sum_{k=0}^{n}\binom{n}{k}T_k^{(k,C)}(x,y)r^{n-k},$$

and

$$T_n^{(k,S)}(x+r,y)=\sum_{k=0}^{n}\binom{n}{k}T_k^{(k,S)}(x,y)r^{n-k}.$$

By (15), we get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\partial}{\partial x}T_n^{(k,C)}(x,y)\frac{t^n}{n!}&=\frac{\partial}{\partial x}\left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}e^{xt}\cos yt\right)\\&=\frac{2Li_k(1-e^{-t})}{e^{2t}+1}e^{xt}\cos yt\\&=\sum_{n=1}^{\infty}\left(nT_{n-1}^{(k,C)}(x,y)\right)\frac{t^n}{n!}.\end{aligned}\tag{25}$$

Comparing the coefficients on the both sides of (25), we have

$$\frac{\partial}{\partial x}T_n^{(k,C)}(x,y)=nT_{n-1}^{(k,C)}(x,y).$$

Similarly, for $n\geq 1$, we have

$$\begin{aligned}\frac{\partial}{\partial x}T_n^{(k,S)}(x,y)&=nT_{n-1}^{(k,S)}(x,y),\\\frac{\partial}{\partial y}T_n^{(k,C)}(x,y)&=-nT_{n-1}^{(k,S)}(x,y),\\\frac{\partial}{\partial y}T_n^{(k,S)}(x,y)&=nT_{n-1}^{(k,C)}(x,y).\end{aligned}$$

We remember that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations (see [12])

$$x^n=\sum_{k=0}^{n}S_2(n,k)(x)_k \text{ and } (x)_n=\sum_{k=0}^{n}S_1(n,k)x^k,\tag{26}$$

respectively. Here, $(x)_n=x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. The numbers $S_2(n,m)$ also admit a representation in terms of a generating function

$$\frac{(e^t-1)^m}{m!}=\sum_{n=m}^{\infty}S_2(n,m)\frac{t^n}{n!}.\tag{27}$$

By (15), (27) and by using Cauchy product, we get

$$\begin{aligned}\sum_{n=0}^{\infty} T_n^{(k,C)}(x,y)\frac{t^n}{n!} &= \left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}\right)(1-(1-e^{-t}))^{-x}\cos yt \\ &= \left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}\right)\cos yt\sum_{l=0}^{\infty}\binom{x+l-1}{l}(1-e^{-t})^l \\ &= \sum_{l=0}^{\infty} <x>_l \frac{(e^t-1)^l}{l!}\left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}\right)e^{-lt}\cos yt \qquad (28) \\ &= \sum_{l=0}^{\infty} <x>_l \sum_{n=0}^{\infty} S_2(n,l)\frac{t^n}{n!}\sum_{n=0}^{\infty} T_n^{(k,C)}(-l,y)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)T_{n-i}^{(k,C)}(-l,y) <x>_l\right)\frac{t^n}{n!},\end{aligned}$$

where $<x>_l = x(x+1)\cdots(x+l-1)(l\geq 1)$ with $<x>_0 = 1$.

By comparing the coefficients on both sides of (28), we have the following theorem:

Theorem 2.8. *For $n>0$, we have*

$$T_n^{(k,C)}(x,y) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)T_{n-i}^{(k,C)}(-l,y) <x>_l,$$
$$T_n^{(k,S)}(x,y) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)T_{n-i}^{(k,S)}(-l,y) <x>_l .$$

Now, we define the new type polynomials that are given by the generating functions to be

$$\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}\cos yt = \sum_{n=0}^{\infty} T_n^{(k,C)}(y)\frac{t^n}{n!}, \qquad (29)$$

and

$$\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}\sin yt = \sum_{n=0}^{\infty} T_n^{(k,S)}(y)\frac{t^n}{n!}, \qquad (30)$$

respectively.

Note that $T_n^{(k,C)}(0,y) = T_n^{(k,C)}(y), T_n^{(k,S)}(0,y) = T_n^{(k,S)}(y), T_n^{(k,C)}(0) = T_n^{(k)}$, $T_n^{(k,S)}(0) = 0$. The new type polynomials can be determined explicitly. A few of them are

$$T_0^{(k,C)}(y) = 1, \quad T_1^{(k,C)}(y) = -\frac{3}{2} + 2^{-k},$$
$$T_2^{(k,C)}(y) = \frac{4}{3} - 2^{2-k} + 2 \cdot 3^{-k} - y^2,$$
$$T_3^{(k,C)}(y) = \frac{3}{4} + 3 \cdot 2^{1-2k} + 7 \cdot 2^{-1-k} + 3 \cdot 2^{1-k} - 2 \cdot 3^{1-k} - 3^{2-k} + \frac{9y^2}{2} - 3 \cdot 2^{-k} y^2,$$

and

$$T_0^{(k,S)}(y) = 0, \quad T_1^{(k,S)}(y) = y,$$
$$T_2^{(k,S)}(y) = -3y + 2^{1-k} y,$$
$$T_3^{(k,S)}(y) = 4y - 3 \cdot 2^{2-k} y + 2 \cdot 3^{1-k} y - y^3.$$

From (8), (17), (24) and (25), we derive the following equations:

$$\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)} \cos yt = \sum_{k=0}^{\infty} \left(\sum_{m=0}^{[\frac{k}{2}]} \binom{k}{2m} (-1)^m T_{k-2m}^{(k)} y^{2m} \right) \frac{t^k}{k!}, \tag{31}$$

and

$$\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)} \sin yt = \sum_{k=0}^{\infty} \left(\sum_{m=0}^{[\frac{k-1}{2}]} \binom{k}{2m+1} (-1)^m T_{k-2m-1}^{(k)} y^{2m+1} \right) \frac{t^k}{k!}. \tag{32}$$

By (29), (30), (31), (32), we get

$$T_n^{(C)}(y) = \sum_{m=0}^{[\frac{n}{2}]} \binom{n}{2m} (-1)^m y^{2m} T_{n-2m}^{(k)},$$

and

$$T_n^{(S)}(y) = \sum_{m=0}^{[\frac{n-1}{2}]} \binom{n}{2m+1} (-1)^m y^{2m+1} T_{n-2m-1}^{(k)}.$$

From (13) and (29), we derive the following theorem:

Theorem 2.9. *For $n \geq 0$, we have*

$$T_n^{(k,C)}(x,y) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} T_l^{(k,C)}(y),$$

and

$$T_n^{(k,S)}(x,y) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} T_l^{(k,S)}(y).$$

By (13), (27), (31), and by using Cauchy product, we have

$$\begin{aligned}\sum_{n=0}^{\infty} T_n^{(C)}(x,y)\frac{t^n}{n!} &= \left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}\right)((e^t-1)+1)^x \cos yt \\ &= \frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)} \cos yt \sum_{l=0}^{\infty} \binom{x}{l}(e^t-1)^l \\ &= \sum_{l=0}^{\infty} (x)_l \frac{(e^t-1)^l}{l!} \left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)} \cos yt\right) \\ &= \sum_{l=0}^{\infty} (x)_l \sum_{n=0}^{\infty} S_2(n,l)\frac{t^n}{n!} \sum_{n=0}^{\infty} T_n^{(k,C)}(y)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{\infty}\sum_{i=l}^{n} \binom{n}{i}(x)_l S_2(i,l) T_{n-i}^{(k,C)}(y)\right)\frac{t^n}{n!}.\end{aligned} \tag{33}$$

By comparing the coefficients on both sides of (33), we have the following theorem:

Theorem 2.10. *For $n \geq 0$, we have*

$$T_n^{(k,C)}(x,y) = \sum_{l=0}^{\infty}\sum_{i=l}^{n} \binom{n}{i}(x)_l S_2(i,l) T_{n-i}^{(k,C)}(y),$$
$$T_n^{(k,S)}(x,y) = \sum_{l=0}^{\infty}\sum_{i=l}^{n} \binom{n}{i}(x)_l S_2(i,l) T_{n-i}^{(k,S)}(y).$$

By (3), (27), (29) and by using Cauchy product, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} T_n^{(k,C)}(x,y)\frac{t^n}{n!} \\ &= \left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)}\right) e^{xt}\cos(yt) \\ &= \frac{(e^t-1)^r}{r!}\frac{r!}{t^r}\left(\frac{t}{e^t-1}\right)^r e^{xt} \sum_{n=0}^{\infty} T_n^{(k,C)}(y)\frac{t^n}{n!} \\ &= \frac{(e^t-1)^r}{r!}\left(\sum_{n=0}^{\infty} \mathbf{B}_n^{(r)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} T_n^{(k,C)}(y)\frac{t^n}{n!}\right)\frac{r!}{t^r} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r,r)\sum_{i=0}^{n-l}\binom{n-l}{i}\mathbf{B}_i^{(r)}(x) T_{n-l-i}^{(k,C)}(y)\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem:

Theorem 2.11. *For $n \geq 0$ and $r \in \mathbb{N}$, we have*

$$T_n^{(k,C)}(x,y) = \sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r,r) \sum_{i=0}^{n-l} \binom{n-l}{i} T_{n-l-i}^{(k,C)}(y) \mathbf{B}_i^{(r)}(x),$$

$$T_n^{(k,S)}(x,y) = \sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r,r) \sum_{i=0}^{n-l} \binom{n-l}{i} T_{n-l-i}^{(k,S)}(y) \mathbf{B}_i^{(r)}(x).$$

By (4), (13), (29) and by using the Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_n^{(k,C)}(x,y) \frac{t^n}{n!} = \left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)} \right) e^{xt} \cos(yt) \\
&= \frac{(e^t-u)^r}{(1-u)^r} \left(\frac{1-u}{e^t-u} \right)^r e^{xt} \left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)} \right) \cos yt \\
&= \sum_{n=0}^{\infty} \mathbf{H}_n^{(r)}(u,x) \frac{t^n}{n!} \sum_{i=0}^{r} \binom{r}{i} e^{it} (-u)^{r-i} \frac{1}{(1-u)^r} \left(\frac{2Li_k(1-e^{-t})}{t(e^{2t}+1)} \right) \cos yt \\
&= \frac{1}{(1-u)^r} \sum_{i=0}^{r} \binom{r}{i} (-u)^{r-i} \sum_{n=0}^{\infty} \mathbf{H}_n^{(r)}(u,x) \frac{t^n}{n!} \sum_{n=0}^{\infty} T_n^{(k,C)}(i,y) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{(1-u)^r} \sum_{i=0}^{r} \binom{r}{i} (-u)^{r-i} \sum_{l=0}^{n} \binom{n}{l} \mathbf{H}_l^{(r)}(u,x) T_{n-l}^{(k,C)}(i,y) \right) \frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem:

Theorem 2.12. *For $n \geq 0$ and $r \in \mathbb{N}$, we have*

$$T_n^{(k,C)}(x,y) = \frac{1}{(1-u)^r} \sum_{i=0}^{r} \sum_{l=0}^{n} \binom{r}{i} \binom{n}{l} (-u)^{r-i} T_{n-l}^{(k,C)}(i,y) \mathbf{H}_l^{(r)}(u,x),$$

$$T_n^{(k,S)}(x,y) = \frac{1}{(1-u)^r} \sum_{i=0}^{r} \sum_{l=0}^{n} \binom{r}{i} \binom{n}{l} (-u)^{r-i} T_{n-l}^{(k,S)}(i,y) \mathbf{H}_l^{(r)}(u,x).$$

By Theorem 2.11, Theorem 2.12, and Theorem 2.13 we have the following corollary.

Corollary 2.13. *For $n \geq 0$ and $r \in \mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{l=0}^{\infty} \sum_{i=l}^{n} \binom{n}{i} (x)_l S_2(i,l) T_{n-i}^{(k,C)}(y) \\
&= \frac{1}{(1-u)^r} \sum_{i=0}^{r} \sum_{l=0}^{n} \binom{r}{i} \binom{n}{l} (-u)^{r-i} \mathbf{H}_l^{(r)}(u,x) T_{n-l}^{(k,C)}(i,y) \\
&= \sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r,r) \sum_{i=0}^{n-l} \binom{n-l}{i} \mathbf{B}_i^{(r)}(x) T_{n-l-i}^{(k,C)}(y).
\end{aligned}$$

3. Zeros of the poly-cosine tangent and poly-sine polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the poly-cosine tangent polynomials $T_n^{(k,C)}(x,y)$ and poly-sine tangent polynomials $T_n^{(k,S)}(x,y)$. The poly-cosine tangent polynomials $T_n^{(k,C)}(x,y)$ and poly-sine tangent polynomials $T_n^{(k,S)}(x,y)$ can be determined explicitly. A few of them are

$$
\begin{aligned}
T_0^{(k,S)}(x,y) &= 0,\\
T_1^{(k,S)}(x,y) &= y,\\
T_2^{(k,S)}(x,y) &= -3y + 2^{1-k}y + 2xy\\
T_3^{(k,S)}(x,y) &= 4y - 3\cdot 2^{2-k}y + 2\cdot 3^{1-k}y - 9xy + 3\cdot 2^{1-k}xy + 3x^2y - y^3,\\
T_4^{(k,S)}(x,y) &= 3y + 3\cdot 2^{3-2k}y + 7\cdot 2^{1-k}y + 3\cdot 2^{3-k}y - 8\cdot 3^{1-k}y - 4\cdot 3^{2-k}y,\\
&\quad + 16xy - 3\cdot 2^{4-k}xy + 8\cdot 3^{1-k}xy - 18x^2y + 3\cdot 2^{2-k}x^2y\\
&\quad + 4x^3y + 6y^3 - 2^{2-k}y^3 - 4xy^3,
\end{aligned}
$$

and

$$
\begin{aligned}
T_0^{(k,C)}(x,y) &= 1,\\
T_1^{(k,C)}(x,y) &= -\frac{3}{2} + 2^{-k} + x,\\
T_2^{(k,C)}(x,y) &= \frac{4}{3} - 2^{2-k} + 2\cdot 3^{-k} - 3x + 2^{1-k}x + x^2 - y^2\\
T_3^{(k,C)}(x,y) &= \frac{3}{4} + 3\cdot 2^{1-2k} + 7\cdot 2^{-1-k} + 3\cdot 2^{1-k} - 2\cdot 3^{1-k} - 3^{2-k} + 4x\\
&\quad - 3\cdot 2^{2-k}x + 2\cdot 3^{1-k}x - \frac{9x^2}{2} + 3\cdot 2^{-k}x^2 + x^3 + \frac{9y^2}{2}\\
&\quad - 3\cdot 2^{-k}y^2 - 3xy^2,\\
T_4^{(k,C)}(x,y) &= -\frac{14}{5} - 3\cdot 2^{3-2k} - 3\cdot 2^{4-2k} - 5\cdot 2^{2-k} + 2^{3-k} + 10\cdot 3^{1-k}\\
&\quad + 4\cdot 3^{2-k} + 24\cdot 5^{-k} + 3x + 3\cdot 2^{3-2k}x + 7\cdot 2^{1-k}x + 3\cdot 2^{3-k}x\\
&\quad - 8\cdot 3^{1-k}x - 4\cdot 3^{2-k}x + 8x^2 - 3\cdot 2^{3-k}x^2 + 4\cdot 3^{1-k}x^2 - 6x^3\\
&\quad + 2^{2-k}x^3 + x^4 - 8y^2 + 3\cdot 2^{3-k}y^2 - 4\cdot 3^{1-k}y^2 + 18xy^2\\
&\quad - 3\cdot 2^{2-k}xy^2 - 6x^2y^2 + y^4.
\end{aligned}
$$

We investigate the beautiful zeros of the poly-sine tangent polynomials $T_n^{(k,S)}(x,y)$ by using a computer. We plot the zeros of the poly-sine tangent polynomials $T_n^{(k,S)}(x,y)$ for $n = 50$ (Figure 1). In Figure 1(top-left), we choose $n = 50, k = -2$ and $x = 2$. In Figure 1(top-right), we choose $n = 50, k = -1$ and $x = 2$. In Figure 1(bottom-left), we choose $n = 50, k = 1$ and $x = 4$. In Figure 1(bottom-right), we choose $n = 50, k = 2$ and $x = 6$.

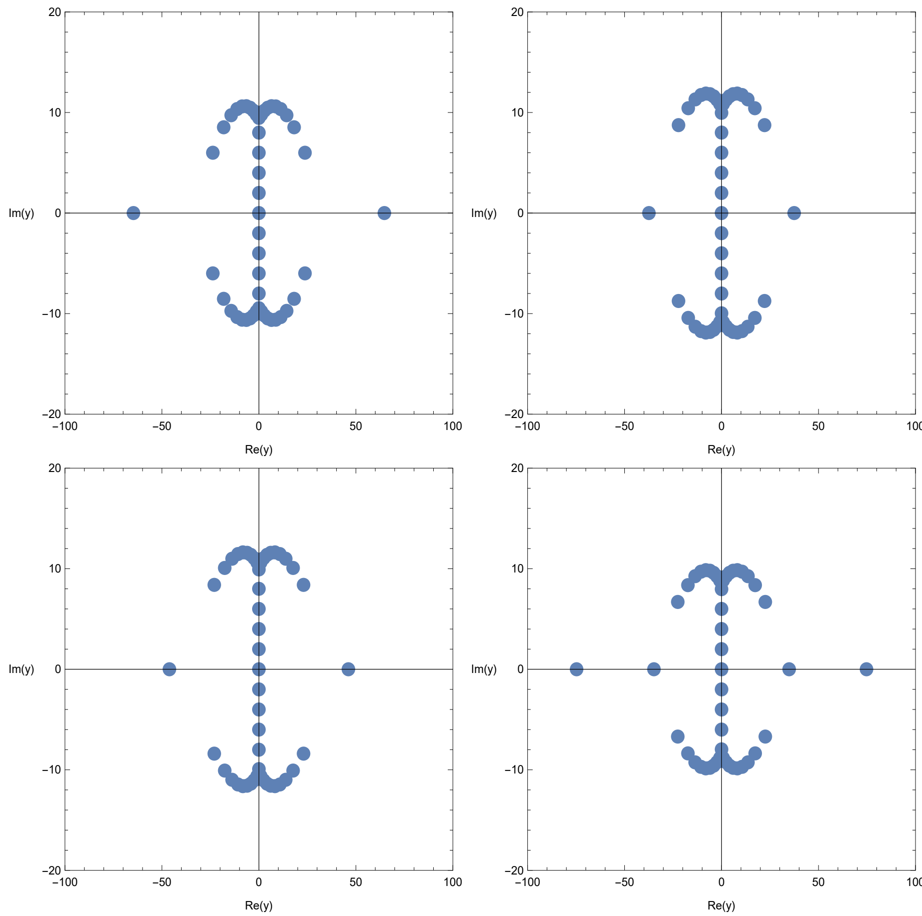

FIGURE 1. Zeros of $T_n^{(k,S)}(x,y)$

Stacks of zeros of $T_n^{(S)}(x,y)$ for $1 \le n \le 50$ from a 3-D structure are presented(Figure 2). In Figure 2(top-left), we choose $k=-2$ and $x=2$. In Figure

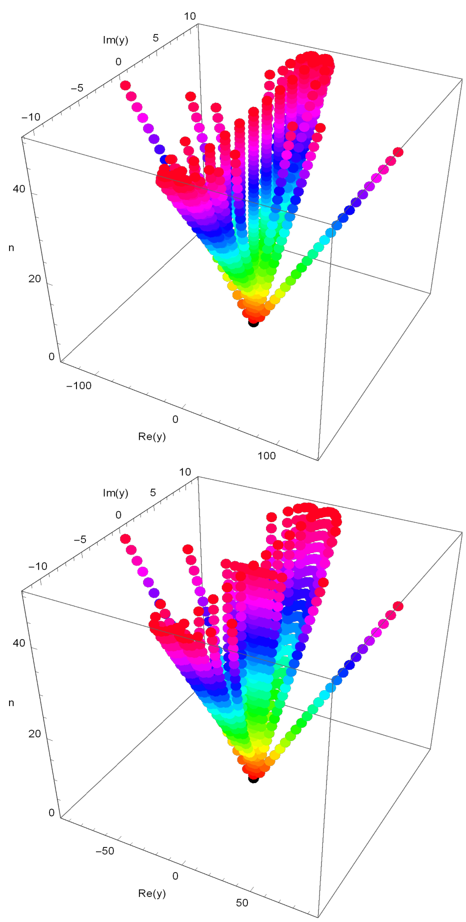

FIGURE 2. Stacks of zeros of $T_n^{(k,S)}(x,y)$ for $1 \le n \le 50$

2(top-right), we choose $k=-1$ and $x=2$. In Figure 2(bottom-left), we choose $k=1$ and $x=4$. In Figure 2(bottom-right), we choose $k=2$ and $x=6$.

The plot of real zeros of $T_n^{(S)}(x,y)$ for $1 \leq n \leq 50$ structure are presented(Figure 3). In Figure 3(top-left), we choose $k=-2$ and $x=2$. In Figure 3(top-right),

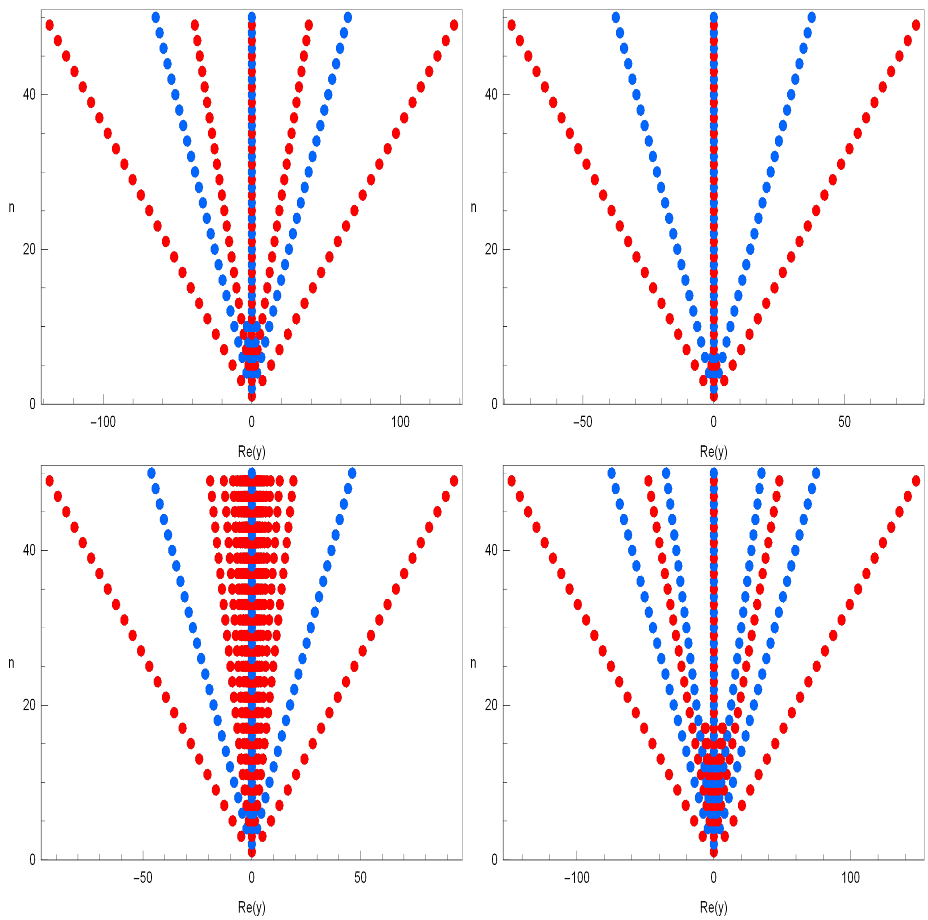

Figure 3. Stacks of zeros of $T_n^{(k,S)}(x,y)$ for $1 \leq n \leq 50$

we choose $k=-1$ and $x=2$. In Figure 3(bottom-left), we choose $k=1$ and $x=4$. In Figure 3(bottom-right), we choose $k=2$ and $x=6$.

We investigate the beautiful zeros of the poly-cosine tangent polynomials $T_n^{(k,C)}(x,y)$ by using a computer. We plot the zeros of the poly-cosine tangent polynomials $T_n^{(k,C)}(x,y)$ for $n = 50$ (Figure 4). In Figure 4(top-left), we choose

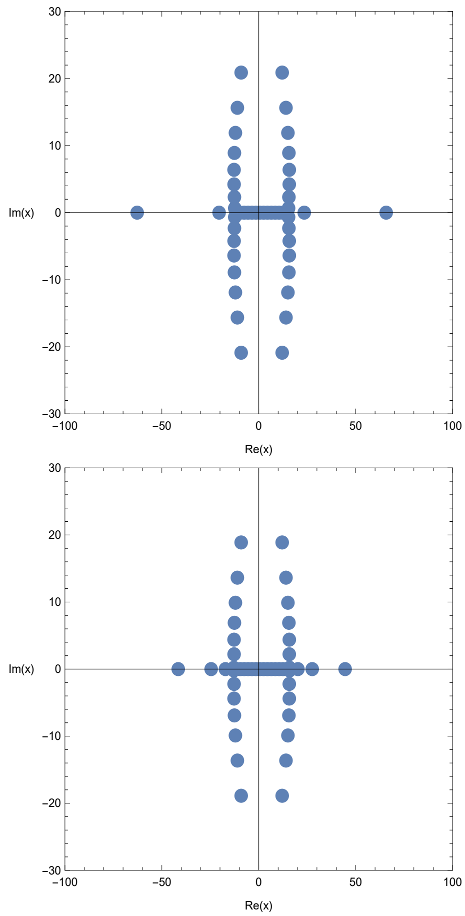

FIGURE 4. Zeros of $T_n^{(k,S)}(x,y)$

$n = 50, k = -2$ and $y = 2$. In Figure 4(top-right), we choose $n = 50, k = -1$ and $y = 2$. In Figure 4(bottom-left), we choose $n = 50, k = 1$ and $y = 4$. In Figure 4(bottom-right), we choose $n = 50, k = 2$ and $y = 6$.

Stacks of zeros of $T_n^{(k,C)}(x,y)$ for $1 \leq n \leq 50$ from a 3-D structure are presented(Figure 5). In Figure 5(top-left), we choose $k=-2$ and $y=2$. In

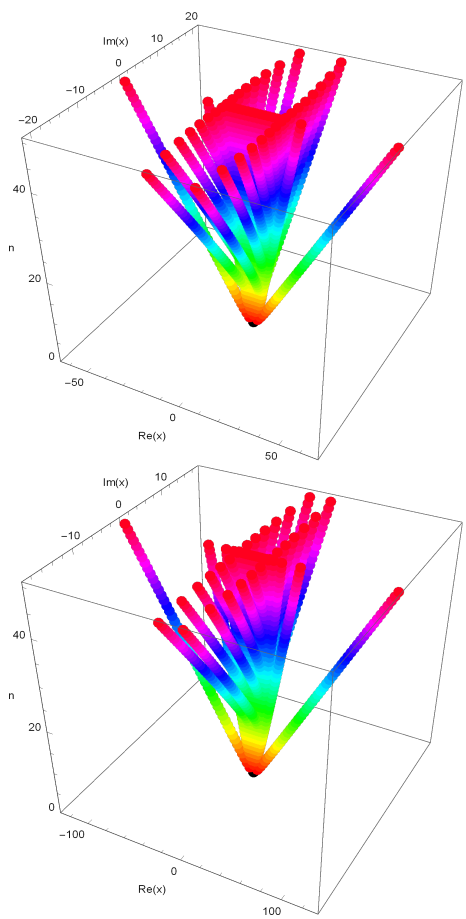

FIGURE 5. Stacks of zeros of $T_n^{(k,C)}(x,y)$ for $1 \leq n \leq 50$

Figure 5(top-right), we choose $k=-1$ and $y=2$. In Figure 5(bottom-left), we choose $k=1$ and $y=4$. In Figure 5(bottom-right), we choose $k=2$ and $y=6$.

The plot of real zeros of $T_n^{(k,C)}(x,y)$ for $1 \leq n \leq 50$ structure are presented(Figure 6). In Figure 6(top-left), we choose $k=-2$ and $y=2$. In

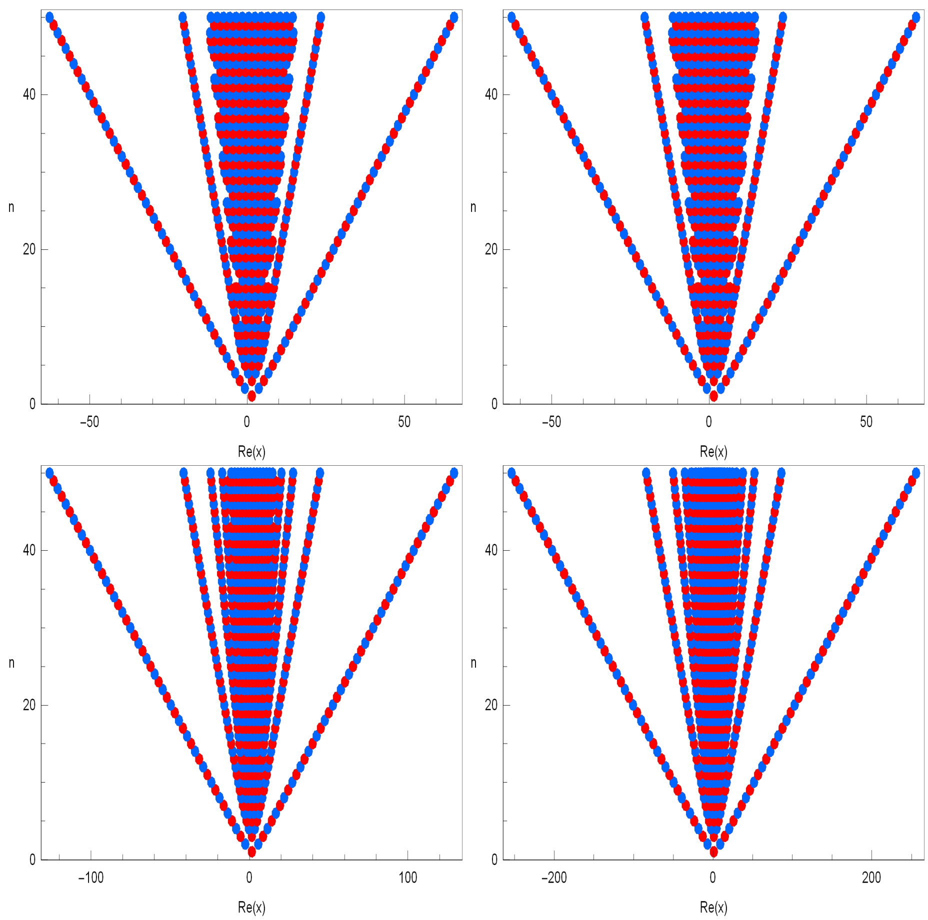

FIGURE 6. Stacks of zeros of $T_n^{(k,C)}(x,y)$ for $1 \leq n \leq 50$

Figure 6(top-right), we choose $k=-1$ and $y=2$. In Figure 6(bottom-left), we choose $k=1$ and $y=4$. In Figure 6(bottom-right), we choose $k=2$ and $y=6$.

Next, we calculated an approximate solution satisfying poly-sine tangent polynomials $T_n^{(k,S)}(x,y)=0$ for $y\in\mathbb{R}$. The results are given in Table 1.

Table 1. Approximate solutions of $T_n^{(2,S)}(4,y)=0$

degree n	y
1	0
2	0
3	4.4347, 0, 4.4347
4	−2.1320, 0, 2.1320
5	−8.0162, −1.1386, 0, 1.1386, 8.0162
6	−3.8352, −0.66523, 0, 0.66523, 3.8352
7	−11.547, −1.6773, −1.2360, 0, 1.2360, 1.6773, 11.547

We also calculated an approximate solution satisfying poly-cosine tangent polynomials $T_n^{(k,C)}(x,y)=0$ for $x\in\mathbb{R}$.

Table 2. Approximate solutions of $T_n^{(2,C)}(x,6)=0$

degree n	x
1	1.5000
2	−4.5759, 7.5759
3	−9.0238, 1.5000, 12.024
4	−13.146, −1.1430, 4.1430, 16.146
5	−17.144, −3.1427, 1.5000, 6.1427, 20.1442
6	−21.082, −4.8708, −0.29881, 3.2988, 7.8708, 24.082

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