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COMPUTATIONAL METHOD FOR SINGULARLY PERTURBED PARABOLIC REACTION-DIFFUSION EQUATIONS WITH ROBIN BOUNDARY CONDITIONS

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Abstract. In this study, the non-standard finite difference method for the numerical solution of singularly perturbed parabolic reaction-diffusion subject to Robin boundary conditions has presented. To discretize temporal and spatial variables, we use the implicit Euler and non-standard finite difference method on a uniform mesh, respectively. We proved that the proposed scheme shows uniform convergence in time with first-order and in space with second-order irrespective of the perturbation parameter. We compute three numerical examples to confirm the theoretical findings.

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1. Introduction

A singularly perturbed differential equation is a differential equation in which a small positive parameter $\varepsilon(0 < \varepsilon << 1)$ multiplied the highest order derivative term and the parameter ε is known as the perturbation parameter. The solution to such problems is characterized by layer regions which are narrow parts of the domain over which the solution undergoes abrupt changes. A boundary layer is defined as a thin layer of rapid change that occurs on a tiny interval around the boundary. If the characteristics of the reduced problem are parallel to the boundary when $\varepsilon \to 0$, then a parabolic boundary layer occur [6]. It is well-known fact that the solution of singular perturbation problems exhibits a multi-scale character (non-uniform behaviour), that is, there are thin transition layer(s) where the solution varies rapidly or jumps abruptly in some parts of the domain, which is known as boundary layer (inner) region while away from the layer(s) the solution behaves regularly and varies slowly, which is commonly known as outer region. In solving these types of problems using classical numerical methods

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on a uniform mesh, large oscillations may arise and pollute the solutions when the perturbation parameter becomes small in entire domain of interest due to the boundary layer behaviour. There is a vast literature about non-classical numerical methods. In the context of finite difference methods, we can group these methods into two. The first is fitted mesh finite difference methods and the second is fitted operator finite difference methods. Both types of methods have been used in the literature to solve singularly perturbed problems. The analysis of the fitted operator finite difference method is simpler due to the fact that it is based on a uniform mesh whereas fitted mesh finite difference methods are based on non-uniform meshes and its analysis is somewhat complicated than operator methods.

In this study, we consider the following class of singularly perturbed second-order linear parabolic partial differential equation of reaction-diffusion type

$$u_t + L_\varepsilon u = f(x,t), \qquad (x,t) \in \Omega = \Omega_x^N \times \Omega_t^M = (0,1) \times (0,T], \tag{1}$$

subject to the initial condition

$$u(x,0) = \phi_b(x), \qquad 0 \le x \le 1, \tag{2}$$

and boundary conditions of Robin type

$$\begin{cases} \beta_l u(0,t) \equiv \big(u - \sqrt{\varepsilon}\frac{\partial u}{\partial x}\big)(0,t) = \phi_l(t), & 0 \le t \le T, \\ \beta_r u(1,t) \equiv \big(u + \sqrt{\varepsilon}\frac{\partial u}{\partial x}\big)(1,t) = \phi_r(t), & 0 \le t \le T. \end{cases} \tag{3}$$

The spatial differential operator is defined as

$$L_\varepsilon = -\varepsilon u_{xx} + a(x,t),$$

where $\varepsilon(0 < \varepsilon << 1)$ is perturbation parameter. The coefficient $a(x,t)$, the source function $f(x,t)$ and the boundary functions $\phi_b(x)$, $\phi_l(t)$ and $\phi_r(t)$ are sufficiently smooth and bounded. The reaction term $a(x,t)$ is assumed to satisfy the following condition $a(x,t) \ge \alpha > 0, \ (x,t) \in \bar{\Omega}$. The solution u of (1)–(3) is expected to exhibit twin layers of width $O(\sqrt{\varepsilon})$ at $x = 0$ and $x = 1$.

Singularly perturbed problems of type (1) with initial-Dirichlet boundary conditions have been studied extensively in the literature using different numerical methods, to mention a few of recent studied (see [1]–[7] and the references therein). However, only few studies of such problems subject to Robin boundary conditions are there, for example, see [8]–[9]. Recently, authors in [10] studied singularly perturbed time delay parabolic reaction-diffusion equations subject to Robin boundary conditions. Authors in [11] studied singularly perturbed time delay convection-diffusion equation. Some robust fitted operator methods are developed in the literature for singularly perturbed problems, see [12]–[14]. Inspired by the simplicity of analysis on a uniform mesh and as far as the authors knowledge is concerned, the idea of non-standard finite difference methods have not been implemented for the problems of the type (1)–(3) so far. Our aim in this work is to design one such method. The time direction is discretized by the Euler method, both the spatial direction and Robin boundary conditions are discretized by the non-standard finite difference method. We provide the convergence analysis of the fully discrete problem and prove that the method is parameter-uniform. Three numerical experiments are conducted in order to validate our theoretical results.

The remaining parts of this study are outlined as follows. Section 2 is devoted to some properties of continuous problem and its bounds of derivatives. In section 3, we fully discretize the problem and discuss about the parameter-uniform convergence analysis of the fully discrete problem. In section 4, three numerical examples are solved using the present method. The paper ends with a brief conclusion in Section 5.

2. Properties of Continuous Problem

When studying the numerical aspects of singularly perturbed problems, the analytical counterpart plays a significant role. Here, we present the bound for the analytical solution of the continuous problem (1)–(3), which can be used for finding the bounds of the discrete solution and its derivatives. Setting the parameter value $\varepsilon = 0$, the reduced problem corresponding to (1)–(3) is

$$\begin{cases} \frac{\partial u_0}{\partial t}(x,t) + a(x,t)u_0(x,t) = f(x,t), & (x,t) \in \Omega, \\ u(x,0) = \phi_b(x), & 0 \le x \le 1. \end{cases} \tag{4}$$

Since the reduced problem (4) will not make use the two boundary conditions, the solution of problem in (1)–(3) will have both left and right boundary layers. The characteristics curve of the reduced problem in equation (4) is the vertical lines $x=$ constant, which implies that boundary layers arising in the solution are of parabolic type. The existence and uniqueness for a solution of (1)–(3) can be established under the assumption that the data are Hölder continuous and also satisfy an appropriate compatibility conditions at the corner points $(0,0)$ and $(1,0)$. We impose the compatibility conditions

$$\begin{aligned} &\phi_b(0) = \beta_l(0),\ \phi_b(1) = \beta_r(1), \\ &\frac{\partial \phi_l(0,0)}{\partial t} - \varepsilon \frac{\partial^2 \phi(0,0)}{\partial x^2} + a(0,0)\phi(0,0) = f(0,0), \\ &\frac{\partial \phi_r(1,0)}{\partial t} - \varepsilon \frac{\partial^2 \phi(1,0)}{\partial x^2} + a(1,0)\phi(1,0) = f(1,0). \end{aligned}$$

The boundary functions $\phi_l, \phi_r \in C^k([0,T]), \phi_b \in C^{1,k}([0,1]\times[0,T])$ are said to satisfy the k^{th} order compatibility condition at the initial function if

$$\begin{cases} \frac{\partial^k}{\partial t^k}\left(\phi_b - \sqrt{\varepsilon}\frac{\partial \phi_b}{\partial x}\right)(0,0) = \frac{d^k \phi_l(0)}{\partial t^k}, \\ \frac{\partial^k}{\partial t^k}\left(\phi_b + \sqrt{\varepsilon}\frac{\partial \phi_b}{\partial x}\right)(1,0) = \frac{d^k \phi_r(1)}{\partial t^k}. \end{cases} \tag{5}$$

Under the above compatibility conditions, it is clear that the solution of problem in (1)–(3) will have a unique solution which exhibits parabolic boundary layers at $x = 0$ and $x = 1$. The problem admits the following continuous maximum principle which ensures the stability of the solution for the problem in (1)–(3).

Lemma 2.1. *Assume that* $a \in C^{(0,0)}(\bar{\Omega})$ *and let* $\psi \in U^* = C^{(2,1)}(\Omega) \cap C^{(1,0)}(\Omega^*) \cap C^{(0,0)}(\bar{\Omega})$ *be a sufficiently smooth function defined on* Ω *such that* $L_\varepsilon \psi(x,t) \geq 0, (x,t) \in \Omega$, $\beta_l \psi(x,t) \geq 0, (x,t) \in \{0\} \times (0,T]$, $\beta_r \psi(x,t) \geq 0, (x,t) \in \{1\} \times (0,T]$ *and* $\psi(x,t) \geq 0, (x,t) \in [0,1] \times (0,T]$, *where* $\Omega^* = \Omega \cup \{0\} \times (0,T] \cup \{1\} \times (0,T]$. *Then,* $\psi(x,t) \geq 0$, *for all* $(x,t) \in \bar{\Omega}$.

Proof. Suppose the arbitrary function ψ takes its minimum value at the point $(x^*,t^*) \in \bar{\Omega}$ such that $\psi(x^*,t^*) = \min_{(x,t)\in\bar{\Omega}} \psi(x,t)$ and assume that $\psi(x^*,t^*) < 0$.
Case(i). For $(x^*,t^*) \in \{0\} \times (0,T]$, we have $\frac{\partial \psi}{\partial x}(x^*,t^*) \geq 0$. Hence, $\beta_l \psi(x^*,t^*) = \psi(x^*,t^*) - \sqrt{\varepsilon}\dfrac{\partial \psi}{\partial x}(x^*,t^*) < 0$, which is a contradiction.
Case(ii). For $(x^*,t^*) \in \{1\} \times (0,T]$, we have $\frac{\partial \psi}{\partial x}(x^*,t^*) \leq 0$. Hence, $\beta_r \psi(x^*,t^*) = \psi(x^*,t^*) - \sqrt{\varepsilon}\dfrac{\partial \psi}{\partial x}(x^*,t^*) < 0$ and for $t^* = 0, \psi(x^*,0) < 0$, which is a contradiction. This implies that (x^*,t^*) is not the boundary.
Case(iii). For $(x^*,t^*) \in \Omega$, as it attains minimum at (x^*,t^*), we have $\frac{\partial \psi}{\partial t}(x^*,t^*) \leq 0$ and $\frac{\partial^2 \psi}{\partial x^2}(x^*,t^*) \geq 0$. Hence,

$$L_\varepsilon \psi(x^*,t^*) = \frac{\partial \psi}{\partial t}(x^*,t^*) - \varepsilon \frac{\partial^2 \psi}{\partial x^2}(x^*,t^*) + a(x^*,t^*)\psi(x^*,t^*) < 0,$$

which is a contradiction to the assumption that $L_\varepsilon \psi(x,t) \geq 0$, $\forall (x,t) \in \Omega$. It follows that $\psi(x^*,t^*) \geq$ and thus $\psi(x,t) \geq 0$, $\forall (x,t) \in \bar{\Omega}$. □

The following Lemma proves the stability estimate to obtain unique solution.

Lemma 2.2. *Let* $u(x,t) \in C^{(2,1)}(\bar{\Omega})$ *be the solution to continuous problem in* (1)–(3) *satisfying the bound*

$$|u(x,t)| \leq \max\left\{|\phi_b(x)|, |\beta_l(0,t)|, |\beta_r(1,t)|\right\} + \alpha^{-1}\|f\|.$$

Proof. To proof this lemma, we define two smooth barrier functions $\Theta^\pm$ as

$$\Theta^\pm(x,t) = \max\left\{|\phi_b(x)|, |\beta_l(0,t)|, |\beta_r(1,t)|\right\} + \alpha^{-1}\|f\| \pm u(x,t).$$

Now, we evaluate the above defined barrier functions at the initial and boundary conditions, respectively as follows. At $t = 0$, we have

$$\begin{aligned}\Theta^\pm(x,0) &= \max\left\{|\phi_b(x)|, |\beta_l(0)|, |\beta_r(0)|\right\} + \alpha^{-1}\|f\| \pm u(x,0),\\ &= \max\left\{|\phi_b(x)|, |\beta_l(0)|, |\beta_r(0)|\right\} + \alpha^{-1}\|f\| \pm \phi_b(x) \geq 0.\end{aligned}$$

At $x = 0$, we have

$$\Theta^\pm(0,t) = \max\left\{|\phi_b(0)|, |\beta_l(0,t)|, |\beta_r(0,t)|\right\} + \alpha^{-1}\|f\| \pm u(0,t). \tag{6}$$

From equation (6), we deduce the following

$$u(0,t) = \pm\Theta^\pm(0,t) \mp \max\left\{|\phi_b(0)|, |\beta_l(0,t)|, |\beta_r(0,t)|\right\} + \alpha^{-1}\|f\|. \tag{7}$$

$$u_x(0,t) = \pm\Theta_x^{\pm}(0,t) \mp |\phi_b'(0)|. \tag{8}$$

Using equations (7) and (8) in the left boundary condition and rearranged gives

$$(\Theta^{\pm} - \sqrt{\varepsilon}\Theta_x^{\pm})(0,t) = \pm\phi_l(0,t) + \big[\max\big\{|\phi_b(0)|, |\beta_l(0,t)|, |\beta_r(0,t)|\big\} + \alpha^{-1}\|f\|\big] - \sqrt{\varepsilon}|\phi_b'(0)| \geq 0,$$

for $\phi_b'(0) = 0$; $\max\big\{|\phi_b(0)|, |\beta_l(0,t)|, |\beta_r(0,t)|\big\} + \alpha^{-1}\|f\| \pm \phi_l(0,t) \geq 0$.
At $x = 1$, we have

$$\Theta^{\pm}(1,t) = \max\big\{|\phi_b(1)|, |\beta_l(1,t)|, |\beta_r(1,t)|\big\} + \alpha^{-1}\|f\| \pm u(1,t). \tag{9}$$

From equation (9), we deduce the following

$$u(1,t) = \pm\Theta^{\pm}(1,t) \mp \max\big\{|\phi_b(1)|, |\beta_l(1,t)|, |\beta_r(1,t)|\big\} + \alpha^{-1}\|f\|. \tag{10}$$

$$u_x(1,t) = \pm\Theta_x^{\pm}(1,t) \mp |\phi_b'(1)|. \tag{11}$$

Using equations (10) and (11) in the right boundary condition gives

$$(\Theta^{\pm} + \sqrt{\varepsilon}\Theta_x^{\pm})(1,t) = \pm\phi_l(1,t) + \max\big\{|\phi_b(1)|, |\beta_l(1,t)|, |\beta_r(1,t)|\big\} + \alpha^{-1}\|f\| + \sqrt{\varepsilon}|\phi_b'(1)| \geq 0,$$

for $\phi_b'(1) = 0$; $\max\big\{|\phi_b(1)|, |\beta_l(1,t)|, |\beta_r(1,t)|\big\} + \alpha^{-1}\|f\| \pm \phi_l(1,t) \geq 0$. Now, on the domain Ω, we have

$$\begin{aligned}
L\Theta^{\pm}(x,t) &= \Theta_t^{\pm}(x,t) - \varepsilon\Theta_{xx}^{\pm}(x,t) + a(x,t)\Theta^{\pm}(x,t),\\
&= \max\{\beta_l'(0,t), \beta_r'(1,t)\} \pm u_t(x,t) - \varepsilon(|\phi_b''(x)| \pm u_{xx}(x,t))\\
&\quad + a(x,t)\Big(\Big[\max\big\{|\phi_b(x)|, |\beta_l(0,t)|, |\beta_r(1,t)|\big\} + \frac{1}{\alpha}\|f\|\Big] \pm u(x,t)\Big),\\
&= \max\{\beta_l'(0,t), \beta_r'(1,t)\} - \varepsilon|\phi_b''(x)|\\
&\quad + a(x,t)\max\big\{|\phi_b(x)|, |\beta_l(0,t)|, |\beta_r(1,t)|\big\} + \alpha^{-1}\|f\| \pm Lu(x,t),\\
&= \pm f(x,t) + \max\{\beta_l'(0,t), \beta_r'(1,t)\} - \varepsilon|\phi_b''(x)|\\
&\quad + a(x,t)\max\big\{|\phi_b(x)|, |\beta_l(0,t)|, |\beta_r(1,t)|\big\} + \alpha^{-1}\|f\|,\\
&= \pm f(x,t) + \alpha^{-1}a(x,t)\|f\| + a(x,t)\times\max\big\{|\phi_b(x)|, |\beta_l(0,t)|, |\beta_r(1,t)|\big\}\\
&\quad - \varepsilon|\phi_b''(x)| + \max\{\beta_l'(0,t), \beta_r'(1,t)\} \geq 0,
\end{aligned}$$

where $\phi_b''(x) = 0, \forall x \in [0,1]$ and $a(x,t) \geq \alpha$. From continuous maximum principle, it follows that $\Theta^{\pm}(x,t) \geq 0, \forall (x,t) \in \bar{\Omega}$. □

The next theorem states the classical bounds on the solution and its derivatives.

Theorem 2.3. *Let functions $a, f \in C^{(2+\alpha,1+\alpha/2)}(\bar{\Omega}), \phi_l, \phi_r \in C^{\frac{3+\alpha}{2}}[(0,T)], \phi_b \in C^{(4+\alpha,2+\alpha/2)}([0,1]\times[0,T]), \alpha \in (0,1)$. Assume that the compatibility conditions given in (5) for $k = 0,1,2$ are fulfilled. Then, the problem has a unique solution and the derivatives of the solution u satisfy the bound*

$$\left\|\frac{\partial^{i+j}u}{\partial x^i \partial t^j}\right\|_{\bar{\Omega}} \leq C\varepsilon^{-i/2}, \qquad i,j \geq 0,\ 0 \leq i+2j \leq 4,$$

where the constant C is independent of ε.

The non-classical bounds in singular and regular components and its derivatives are established in the following theorem.

Theorem 2.4. *Let functions $a, f \in C^{(4+\alpha,2+\alpha/2)}(\bar{\Omega}), \phi_l, \phi_r \in C^{\frac{5+\alpha}{2}}[(0,T)], \phi_b \in C^{(6+\alpha,3+\alpha/2)}([0,1]\times[0,T]), \alpha \in (0,1)$. Under the smoothness and compatibility conditions, we have the bounds for $i, j \geq 0,\ 0 \leq i+2j \leq 4$*

$$\left\| \frac{\partial^{i+j} v(x,t)}{\partial x^i \partial t^j} \right\| \leq C(1+\varepsilon^{1-i/2}),$$

$$\left| \frac{\partial^{i+j} w_l}{\partial x^i \partial t^j} \right| \leq C\varepsilon^{-i/2} \exp\left(\frac{-x}{\sqrt{\varepsilon}}\right), \qquad \left| \frac{\partial^{i+j} w_r}{\partial x^i \partial t^j} \right| \leq C\varepsilon^{-i/2} \exp\left(\frac{-(1-x)}{\sqrt{\varepsilon}}\right).$$

3. Fully Discretized Problem

In this section, we first develop exact scheme which will then be used to derive a non-standard finite difference scheme. Micken's presented rules for developing non-standard finite difference method for different problem types [15]. To develop a discrete scheme using Micken's rule, the denominator function for the second order derivative must be expressed in terms of more complicated functions of step sizes than those used in the standard finite difference procedures. This complicated function constitutes a general property of the schemes, which is useful while designing reliable schemes for such problems. In line to this, we need to derive a non-standard finite difference method which captures the layer behavior of the problem on a uniform mesh. Mickens in [15] and [16] gave a novel approach of non-standard finite difference method and the basic idea of this method is to replace the denominator function h^2 of the second order derivative with suitable function in the differential equation, which comes under the category of fitted operator finite difference method. On the spatial domain $[0,1]$, we introduce the equidistant meshes with uniform mesh length such that

$$\Omega_x^N = \{x_0 = 0, x_i = ih, i = 1(1)N, x_N = 1, h = \frac{1}{N}\},$$

where h is the step size and N is the number of mesh points in the spatial direction. Similarly, we divide the time interval $[0,T]$ into M equal intervals with uniform step size Δt defined by

$$\Omega_t^M = \{t_0 = 0, t_n = n\Delta t,\ \ n = 1(1)M-1,\ \ t_M = T,\ \ \Delta t = \frac{T}{M}\},$$

where M denotes the number of mesh points in time direction. We denote the approximation of $u(x_i,t_n)$ by U_i^n. According to [15], the concept of sub-equations is the major tool to derive the denominator function for a partial differential equation. Thus, from (1) we take the homogeneous form of constant coefficient sub-equation in spatial direction while assuming t as continuous

$$-\varepsilon \frac{d^2U}{dx^2}(x_i,t) + (aU)(x_i,t) = 0, \tag{12}$$

and design the non-standard finite difference scheme for this sub-equation. By discretizing (12) in space direction only, from the theory of non-standard finite difference methods, we have

$$-\varepsilon\frac{U_{i+1}(t)-2U_i(t)+U_{i-1}(t)}{\gamma_i^2}+a_i(t)U_i(t)=0, \tag{13}$$

Equation (13) has two linearly independent analytical solutions, namely, $\exp(-\rho x_i)$ and $\exp(\rho x_i)$, where $\rho_i(t)=\sqrt{\frac{a_i(t)}{\varepsilon}}$. Following Micken's in [15], we construct a difference equation for equation (13)

$$\begin{vmatrix} U_{i-1} & U_{1,i-1} & U_{2,i-1} \\ U_i & U_{1,i} & U_{2,i} \\ U_{i+1} & U_{1,i+1} & U_{2,i+1} \end{vmatrix} = \begin{vmatrix} U_{i-1} & \exp(-\rho x_{i-1}) & \exp(\rho x_{i-1}) \\ U_i & \exp(-\rho x_i) & \exp(\rho x_i) \\ U_{i+1} & \exp(-\rho x_{i+1}) & \exp(\rho x_{i+1}) \end{vmatrix} = 0. \tag{14}$$

Simplifying the above determinant by column expansion, we get

$$U_{i-1}-2\cosh(\rho h)U_i+U_{i+1}=0, \tag{15}$$

which is the exact scheme for equation (13) in the sense that equation (13) has the same general solution $U_i=A\exp(-\rho x_i)+B\exp(\rho x_i)$ as (15), see [17], [18] and [19]. Simplifying equation (15) by hyperbolic identity and using it into equation (13), we obtain

$$-\varepsilon\frac{4\sinh^2(\rho_i(t)\frac{h}{2})}{\gamma_i^2}U_i(t)+a_i(t)U_i(t)=0, \tag{16}$$

Simplification of equation (16) gives us the non-standard finite difference scheme

$$-\varepsilon\frac{U_{i+1}(t)-2U_i(t)+U_{i-1}(t)}{\frac{4\varepsilon}{a_i(t)}\sinh^2\left(\sqrt{\frac{a_i(t)}{\varepsilon}}\frac{h}{2}\right)}+a_i(t)U_i(t)=0, \tag{17}$$

From equation (17), we observe that the classical denominator h^2 was replaced by the complicated denominator function $\frac{4\varepsilon}{\rho}\sinh^2\left(\sqrt{\frac{\rho}{\varepsilon}}\frac{h}{2}\right)$ implying that the present method is non-standard finite difference method. For varying coefficient function, the denominator function becomes

$$\frac{4\varepsilon}{\rho_i}\sinh^2\left(\sqrt{\frac{\rho_i}{\varepsilon}}\frac{h}{2}\right).$$

Substituting equation (17) into problem (1), we get the fully-discrete scheme of the form

$$\frac{U_i^{n+1}-U_i^n}{\Delta t}-\varepsilon\left(\frac{U_{i-1}^{n+1}-2U_i^{n+1}+U_{i+1}^{n+1}}{\gamma_i^2(\varepsilon,h)}\right)+a_i^{n+1}U_i^{n+1}=f_i^{n+1}, \tag{18}$$

for $i = 1, 2, \cdots, N-1, n = 1, \cdots, M$. We use forward and backward non-standard finite differences to approximate the first derivative in the left and right boundary conditions, respectively as given below. Thus, we have the following semi-discrete initial and boundary conditions

$$\begin{aligned} U_i^0 &= \phi_b(x_i), && x_i \in \bar{\Omega}, \\ U_0^{n+1} - \sqrt{\varepsilon}\big(\frac{U_1^{n+1} - U_0^{n+1}}{\gamma_0}\big) &= \phi_l(t_{n+1}), && t \in [0, T], \\ U_N^{n+1} + \sqrt{\varepsilon}\big(\frac{U_N^{n+1} - U_{N-1}^{n+1}}{\gamma_N}\big) &= \phi_r(t_{n+1}), && t \in [0, T]. \end{aligned} \tag{19}$$

From the theory of non-standard finite difference methods, we have the following denominator functions at the two boundaries and interior points

$$\begin{aligned} \gamma_0 &= \sqrt{\varepsilon}\big(e^{\frac{h}{\sqrt{\varepsilon}}} - 1\big), \quad i = 0, \\ \gamma_i &= \frac{2}{\rho_i^{n+1}} \sinh\big(\rho_i^{n+1}\frac{h}{2}\big), \text{ where } \rho_i^{n+1} = \sqrt{\frac{a_i^{n+1}}{\varepsilon}}, i = 1, 2, \cdots, N-1 \\ \gamma_N &= \sqrt{\varepsilon}\big(1 - e^{\frac{-h}{\sqrt{\varepsilon}}}\big), \quad i = N. \end{aligned} \tag{20}$$

The discrete problem in equation (18) and the discrete conditions in equation (19) together with the denominator functions in equation (20) can be written in matrix form as

$$AU = F, \qquad i = 1, 2, ..., N-1, \quad n = 0, ..., M, \tag{21}$$

where U and F are column vectors of $N+1$ and the matrix A is a tri-diagonal matrix of $(N+1) \times (N+1)$. The entries of the coefficient matrix A is given by

$$\begin{cases} A_{0,0} = 1 + \frac{\sqrt{\varepsilon}}{\gamma_0}, \quad A_{0,1} = -\frac{\sqrt{\varepsilon}}{\gamma_0}, \\ A_{i,i-1} = -\frac{\varepsilon}{\gamma_i^2}, & i = 1, ..., N-1, \\ A_{i,i} = \frac{2\varepsilon}{\gamma_i^2} + \frac{1}{\Delta t} + a_i^{n+1}, & i = 1, ..., N-1, \\ A_{i,i+1} = -\frac{\varepsilon}{\gamma_i^2}, & i = 1, ..., N-1, \\ A_{N,N-1} = -\frac{\sqrt{\varepsilon}}{\gamma_N}, \quad A_{N,N} = 1 + \frac{\sqrt{\varepsilon}}{\gamma_N}. \end{cases} \tag{22}$$

The entries of column vectors F and U are given as follows

$$\begin{cases} F_0^{n+1} = f_0^{n+1} + \frac{U_0^n}{\Delta t} + \phi_l(t_{n+1}), \\ F_i^{n+1} = f_i^{n+1} + \frac{U_i^n}{\Delta t}, & i = 1(1)N-1, \\ F_N^{n+1} = f_N^{n+1} + \frac{U_N^n}{\Delta t} + \phi_r(t_{n+1}), \\ U = [U_0, U_1, \cdots, U_N]^T. \end{cases} \tag{23}$$

3.1. Stability Analysis for Discrete Problem. Next, we prove some useful attributes the discrete problem. The discrete operator defined in equation (18) satisfies the following discrete maximum principle.

Theorem 3.1. *Assume that $L^{N,M}$ be discrete operator given in equation* (18) *and Θ_i^n be any mesh function satisfying $L^{N,M}\Theta_i^n \leq 0$ for all $(i,n) \in \Omega^{N,M}$, initial condition $\Theta_i^0 \geq 0, 0 \leq i \leq N$ and boundary conditions $\beta_l(0,t_n) \equiv \Theta_0^n - \sqrt{\varepsilon}(\Theta_x)_0^n \geq 0, \beta_r(N,t_n) \equiv \Theta_N^n + \sqrt{\varepsilon}(\Theta_x)_N^n \geq 0$. If $L^{N,M}\Theta_n^n \leq 0$ in $\Omega^{N,M}$, then $\Theta_n^n \geq 0$ in $\bar{\Omega}^{N,M}$.*

Proof. Let s and p be indices such that $\Theta_s^p = \min_{\forall(i,n)} \Theta_i^n$ for $\Theta_i^n \in \bar{\Omega}^{N,M}$. Assume that $\Theta_s^p < 0$. Then, it is easy to see that $(s,p) \in \{1,\cdots,N-1\} \times \{1,\cdots,M\}$, because otherwise $\Theta_s^p \geq 0$. It follows that $\Theta_{s+1}^{p+1} - \Theta_s^{p+1} > 0, \Theta_{s-1}^{p+1} - \Theta_s^{p+1} > 0$ and $\Theta_s^{p+1} - \Theta_s^p > 0$. Thus, now

$$
\begin{aligned}
L^{N,M}\Theta_s^{p+1} &= \frac{1}{\Delta t}\Theta_s^{p+1} - \frac{\varepsilon}{\gamma_s^2}(\Theta_{s+1}^{p+1} - 2\Theta_s^{p+1} + \Theta_{s-1}^{p+1}) + a_s^{p+1}\Theta_s^{p+1}, \\
&= \frac{1}{\Delta t}\Theta_s^{p+1} - \frac{\varepsilon}{\gamma_s^2}[(\Theta_{s+1}^{p+1} - \Theta_s^{p+1}) - (\Theta_s^{p+1} - \Theta_{s-1}^{p+1})] + a_s^{p+1}\Theta_s^{p+1}, \\
&< 0,
\end{aligned}
$$

The discrete boundary conditions becomes

$$
\Theta_0^{p+1} - \frac{\sqrt{\varepsilon}}{\gamma_0}(\Theta_1^{p+1} - \Theta_0^{p+1}) < 0 \quad \text{and} \quad \Theta_N^{p+1} + \frac{\sqrt{\varepsilon}}{\gamma_N}(\Theta_N^{p+1} - \Theta_{N-1}^{p+1}) < 0,
$$

which is a contradiction. Hence, the assumption $\Theta_s^{p+1} < 0,\ \forall(s,p)$ is wrong. Thus, $\Theta_s^{p+1} > 0$ implies that $\Theta_n^m \geq 0$ in $\bar{\Omega}^{N,M}$. □

Now, we will prove the uniform stability analysis of the discrete problem.

Lemma 3.2. *Let U_i^{n+1} be the solution of discrete problem equation* (18) *satisfying the following bound*

$$
\|U_i^{n+1}\| \leq \alpha^{-1} \max_{\forall(i,n)} |L^{N,M}U_i^{n+1}| + \max_{\forall(i,n)}\{|(\phi_b)_i^0|, \max\{(\phi_l)_i^{n+1}, (\phi_r)_i^{n+1}\}\}.
$$

Proof. Let $Z = \alpha^{-1} \max_{\forall(i,n)} |L^{N,M}U_i^{n+1}| + \max_{\forall(i,n)}\{|(\phi_b)_i^0|, \max\{(\phi_l)_i^{n+1}, (\phi_r)_i^{n+1}\}\}$ and define the two barrier functions $(\Psi^{\pm})_i^{n+1}$ by $(\Psi^{\pm})_i^{n+1} = Z \pm U_i^{n+1}$. At the initial condition, we get $(\Psi^{\pm})_i^0 = Z \pm U_i^0 = Z \pm (\phi_b)_i^0 \geq 0$. At the boundary points, we have $(\Psi^{\pm})_0^{n+1} = Z \pm U_0^{n+1} = Z \pm (\phi_l)_i^{n+1} \geq 0$, and $(\Psi^{\pm})_N^{n+1} = Z \pm U_N^{n+1} = Z \pm (\phi_r)_i^{n+1} \geq 0$. On the discretized domain $1 \leq i \leq N-1$, we have

$$
\begin{aligned}
L^{N,M}(\Psi^{\pm})_i^{n+1} &\equiv \frac{1}{\Delta t}\Big((Z \pm U_i^{n+1}) - (Z \pm U_i^n)\Big) \\
&- \frac{\varepsilon}{\gamma_i^2}\Big((Z \pm U_{i-1}^{n+1}) - 2(Z \pm U_i^{n+1}) + (Z \pm U_{i+1}^{n+1})\Big) \\
&+ a_i^{n+1}(Z \pm U_i^{n+1}), \\
&= a_i^{n+1}Z \pm L^{N,M}U_i^{n+1}, \\
&= \pm f_i^{n+1} + a_i^{n+1}Z \geq 0,
\end{aligned}
$$

where $a_i^{n+1} \geq \alpha$ and from discrete maximum principle, we get $(\Psi^{\pm})_i^{n+1} \geq 0, \forall (x_i, t_n) \in \bar{\Omega}^{N,M}$. □

3.2. Convergence Analysis for Discrete Problem. We use the following lemma to prove the uniform convergence analysis of the discrete problem in (18).

Lemma 3.3. *For all positive integers j on a fixed mesh, we have*

$$\lim_{\varepsilon\to 0} \max_{1<i<N-1} \frac{\exp(\frac{-Cx_i}{\sqrt{\varepsilon}})}{\varepsilon^{\frac{j}{2}}} = 0 \quad \textit{and} \quad \lim_{\varepsilon\to 0} \max_{1<i<N-1} \frac{\exp(\frac{-C(1-x_i)}{\sqrt{\varepsilon}})}{\varepsilon^{\frac{j}{2}}} = 0,$$

where $x_i = ih, h = \frac{1}{N}, \ \ \forall i = 1, ..., N-1.$

Proof. We transform the domain [0,1] into the discrete domain $0 = x_0 < \cdots < x_N = 1$. The interior grid points satisfy the inequalities

$$\max_{1<i<N-1} \frac{\exp(\frac{-Cx_i}{\sqrt{\varepsilon}})}{\varepsilon^{\frac{j}{2}}} \leq \frac{\exp(\frac{-Cx_1}{\sqrt{\varepsilon}})}{\varepsilon^{\frac{j}{2}}} = \frac{\exp(\frac{-Ch}{\sqrt{\varepsilon}})}{\varepsilon^{\frac{j}{2}}},$$

and

$$\max_{1<i<N-1} \frac{\exp(\frac{-C(1-x_n)}{\sqrt{\varepsilon}})}{\varepsilon^{\frac{j}{2}}} \leq \frac{\exp(\frac{-C(1-x_{N-1})}{\sqrt{\varepsilon}})}{\varepsilon^{\frac{j}{2}}} = \frac{\exp(\frac{-Ch}{\sqrt{\varepsilon}})}{\varepsilon^{\frac{j}{2}}}.$$

Since $x_1 = h, 1 - x_{N-1} = 1-(N-1)h = 1 - Nh + h = h$, applying L'Hospital's rule repeatedly results in

$$\lim_{\varepsilon\to 0} \frac{\exp(\frac{-Ch}{\sqrt{\varepsilon}})}{\varepsilon^{\frac{j}{2}}} = \lim_{p=\frac{1}{\sqrt{\varepsilon}}\to \infty} \frac{p^j}{\exp(Chp)} = \lim_{p\to\infty} \frac{j!}{(Ch)^j \exp(Chp)} = 0.$$

Hence, the proof is completed. □

Now, we analyze convergence analysis of the proposed method. The truncation error of the proposed method is given by

$$\begin{aligned} L^{N,M}(U_i^{n+1} - u_i^{n+1}) &= f_i^{n+1} - L^{N,M}u_i^{n+1}, \\ &= Lu_i^{n+1} - L^{N,M}u_i^{n+1}, \\ &= (u_t)_i^{n+1} - \varepsilon(u_{xx})_i^{n+1} + a_i^{n+1}u_i^{n+1}, \\ &- \left[\frac{u_i^{n+1} - u_i^n}{\Delta t} - \varepsilon\left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\gamma_i^2}\right) + a_i^{n+1}u_i^{n+1}\right], \\ &= (u_t)_i^{n+1} - \varepsilon(u_{xx})_i^{n+1} - \left[\frac{u_i^{n+1} - u_i^n}{\Delta t}\right] \\ &+ \varepsilon\left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\gamma_i^2}\right). \end{aligned}$$

Taylor's expansions of the terms u_i^{n+1}, u_{i+1}^{n+1} and u_{i-1}^{n+1} are give as following

$$u_{i+1}^{n+1} = u_i^{n+1} + h(u_x)_i^{n+1} + \frac{h^2}{2}(u_{xx})_i^{n+1} + \frac{h^3}{6}(u_{xxx})_i^{n+1} + \frac{h^4}{24}(u_{xxxx})_i^{n+1} + \cdots$$
$$u_{i-1}^{n+1} = u_i^{n+1} - h(u_x)_i^{n+1} + \frac{h^2}{2}(u_{xx})_i^{n+1} - \frac{h^3}{6}(u_{xxx})_i^{n+1} + \frac{h^4}{24}(u_{xxxx})_i^{n+1} + \cdots$$
$$u_i^{n+1} = u_i^n + \Delta t(u_t)_i^n + \frac{\Delta t^2}{2}(u_{tt})_i^n + \cdots$$

Adding the first two equations in the above Taylor's expansion, we have

$$u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} = h^2(u_{xx})_i^{n+1} + \frac{h^4}{12}(u_{xxxx})_i^{n+1} + \cdots$$

From the last equation of Taylor's expansion, we obtain

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = (u_t)_i^n + \frac{\Delta t}{2}(u_{tt})_i^n + \cdots$$

Using the truncated values of the above results, we have

$$L^{N,M}(U_i^{n+1} - u_i^{n+1}) = (u_t)_i^{n+1} - \varepsilon(u_{xx})_i^{n+1} - \left[(u_t)_i^n + \frac{\Delta t}{2}(u_{tt})_i^n\right] + \frac{\varepsilon}{\gamma_i^2}\left(h^2(u_{xx})_i^{n+1} + \frac{h^4}{12}(u_{xxxx})_i^{n+1}\right).$$

A truncated Taylor's expansion of the denominator function $\frac{1}{\gamma_i^2}$ reads [20]

$$\frac{1}{\gamma_i^2} = \frac{\rho_i^2}{4}\left(\frac{4}{\rho_i^2 h^2} - \frac{1}{3} + \frac{\rho_i^2 h^2}{60}\right).$$

Substituting this into the above expression and rearranging gives

$$L^{N,M}(U_i^{n+1} - u_i^{n+1}) = \left(\frac{\varepsilon}{12}(u_{xxxx})_i^{n+1} - \varepsilon\frac{\rho_i^2}{12}(u_{xx})_i^{n+1}\right)h^2 - \left(\varepsilon\frac{\rho_i^2}{144}(u_{xxxx})_i^{n+1} - \varepsilon\frac{\rho_i^4}{240}(u_{xx})_i^{n+1}\right)h^4 + \left(\varepsilon\frac{\rho_i^4}{2880}(u_{xxxx})_i^{n+1}\right)h^6 - \frac{(u_{tt})_i^n}{2}\Delta t.$$

Applying Theorem (2.4) for the bounds on the derivatives and Lemma (3.3) on the above expression gives

$$L^{N,M}(U_i^{n+1} - u_i^{n+1}) = \left(\frac{\varepsilon}{12} - \varepsilon\frac{\rho_i^2}{12}\right)h^2 - \left(\varepsilon\frac{\rho_i^2}{144} - \varepsilon\frac{\rho_i^4}{240}\right)h^4 + \left(\varepsilon\frac{\rho_i^4}{2880}\right)h^6 - \frac{(u_{tt})_i^n}{2}\Delta t.$$

Taking maximum on both sides of the above expression and using the value of ρ, we have

$$\left|L^{N,M}(U_i^{n+1} - u_i^{n+1})\right| \le C_1 h^2 + C_2 h^4 + C_3 h^6 + 0.5\left|(u_{tt})_i^n\right|\Delta t.$$

Applying the relation $h^2 > h^4 > h^6$, the discrete problem satisfy the bound

$$\|L^{N,M}(U_i^{n+1} - u_i^{n+1})\| \le C(h^2 + \Delta t).$$

Using Lemma (3.2), we get the result

$$\|(U_i^{n+1} - u_i^{n+1})\| \le C(h^2 + \Delta t).$$

where C is a constant independent of ε, N and Δt.
The error bound at the left boundary x_0 is estimated as follows

$$U_0^{n+1} - u_0^{n+1} = u_0^{n+1} - \sqrt{\varepsilon}(u_x)_0^{n+1} - \phi_l^{n+1} - \big[u_0^{n+1} - \frac{\sqrt{\varepsilon}}{\gamma_0}(u_1^{n+1} - u_0^{n+1}) - \phi_l^{n+1}\big].$$

Taylor series expansion of the term u_1^{n+1} and the denominator function γ_0 yields

$$U_0^{n+1} - u_0^{n+1} = -\sqrt{\varepsilon}(U_x)_0^{n+1} + \big(\frac{\sqrt{\varepsilon}}{h} - \frac{1}{2}\big)\big(h(U_x)_0^{n+1} + \frac{h^2}{2}(U_{xx})_0^{n+1} + \cdots\big).$$

Using the relation $h > h^2$ and applying the bound in Lemma (3.2), we get

$$\|U_0^{n+1} - u_0^{n+1}\| \le Ch. \tag{24}$$

In similar way, we can find the error bound at the right boundary x_N as

$$\|U_N^{n+1} - u_N^{n+1}\| \le Ch. \tag{25}$$

From equations (24)-(25), we conclude that the error bound at the two boundaries satisfies first-order uniform convergence. The error bound at the interior mesh points can be established in the following theorem.

Theorem 3.4. *Let $u_i^{n+1} \in C^{4,2}(\bar{\Omega})$ be the solution to problem in (1)–(3) and U_i^{n+1} be the solution to discrete problem in (18) and (19). Then, the error bound at the interior mesh points satisfies*

$$\sup_{0<\varepsilon\le 1} \max_{0\le i\le N; 0\le n\le M} \|U_i^{n+1} - u_i^{n+1}\| \le C(h^2 + \Delta t),$$

where C is a constant independent of ε and the mesh lengths h and Δt.

4. Numerical Results

In this section, we carry out numerical experiment in order to corroborate the applicability of the proposed method with the theoretical results claimed in the previous sections. Since the exact solution for the first two examples are not known, we use the double mesh principle to calculate maximum point-wise errors. For each ε, we can find the maximum point-wise errors for different values of mesh points and ε using the following formula

$$E_\varepsilon^{N,\Delta t} = \max_{0\le i\le N; t\in[0,T]} \big|U^{N,\Delta t}(x_i, t_n) - U^{2N,\Delta t/2}(x_i, t_n)\big|,$$

where $U^{N,\Delta t}(x_i, t_n)$ denotes the numerical solution obtained at $(N, \Delta t)$ mesh points where $U^{2N,\Delta t/2}(x_i, t_n)$ denotes the numerical solution at $(2N, \Delta t/2)$ mesh points. Whereas the exact solution for the third example is known, we use the following formula to calculate the maximum point-wise errors.

$$E_\varepsilon^{N,\Delta t} = \max_{0\le i\le N; t\in[0,T]} \big|u(x_i, t_n) - U^{N,\Delta t}(x_i, t_n)\big|,$$

where $U^{N,\Delta t}(x_i,t_n)$ denotes the numerical solution obtained at $(N,\Delta t)$ mesh points where $u(x_i,t_n)$ denotes the exact solution at $(N,\Delta t)$ mesh points. The numerical ε-uniform rate of convergence and ε-uniform maximum point-wise errors were calculated using the following formulas, respectively

$$R^{N,\Delta t} = \log_2\left(\frac{E^{N,\Delta t}}{E^{2N,\Delta t/2}}\right) \quad \text{and} \quad E^{N,\Delta t} = \max_{\varepsilon} E_{\varepsilon}^{N,\Delta t}.$$

Example 4.1. Consider singularly perturbed reaction-diffusion problem [9]

$$\begin{cases} \frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + \frac{1+x^2}{2}u = t^3, & (x,t)\in(0,1)\times(0,1],\\ u(x,0)=0, & x\in[0,1],\\ \left(u-\sqrt{\varepsilon}\frac{\partial u}{\partial x}\right)(0,t) = -\frac{128}{35}\pi^{-1/2}t^{7/2}, & t\in[0,1],\\ \left(u+\sqrt{\varepsilon}\frac{\partial u}{\partial x}\right)(1,t) = -\frac{128}{35}\pi^{-1/2}t^{7/2}, & t\in[0,1]. \end{cases}$$

where the exact solution is not available.

Example 4.2. Consider singularly perturbed reaction-diffusion problem

$$\begin{cases} \frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + (1+x+t)u = 4^3x^3t^2(1-x)^3, & (x,t)\in(0,1)\times(0,1],\\ u(x,0)=(4x(1-x))^3, & x\in[0,1],\\ \left(u-\sqrt{\varepsilon}\frac{\partial u}{\partial x}\right)(0,t) = t^3, & t\in[0,1],\\ \left(u+\sqrt{\varepsilon}\frac{\partial u}{\partial x}\right)(1,t) = t^3, & t\in[0,1]. \end{cases}$$

where the analytical solution for this example is not available.

Example 4.3. Consider singularly perturbed reaction-diffusion problem

$$\begin{cases} \frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + (1+xe^{-t})u = f(x,t), & (x,t)\in(0,1)\times(0,1],\\ u(x,0)=0, & x\in[0,1],\\ \left(u-\sqrt{\varepsilon}\frac{\partial u}{\partial x}\right)(0,t) = \phi_l(t), & t\in[0,1],\\ \left(u+\sqrt{\varepsilon}\frac{\partial u}{\partial x}\right)(1,t) = \phi_r(t), & t\in[0,1]. \end{cases}$$

where the functions $f(x,t),\phi_l(t)$ and $\phi_r(t)$ are chosen from the exact solution

$$u(x,t) = (1-e^{-t})\left(\frac{e^{-x/\sqrt{\varepsilon}} + e^{-(1-x)/\sqrt{\varepsilon}}}{1+e^{-1/\sqrt{\varepsilon}}} - \cos^2(\pi x)\right).$$

Table 1 gives the maximum point-wise errors using the present method and the method in literature. Table 2 gives numerical results for Example 4.1 for equal number of mesh points. From Tables 1–2, one can observe that the present method gives an ε-uniform numerical results.

Table 1. Maximum errors and rate of convergence for Example 4.1.

$\forall\varepsilon$	$N=32$ $M=8$	$N=64$ $M=16$	$N=128$ $M=32$	$N=256$ $M=64$	$N=512$ $M=128$
Our Result					
$E^{N,M}$	2.5491e-2	1.2508e-2	6.1884e-3	3.0781e-3	1.5349e-3
$R^{N,M}$	1.0226	1.0138	1.0075	1.0039	
Result in [9]					
$E^{N,M}$	3.2130e-2	1.6470e-2	8.3343e-3	4.1915e-3	2.1018e-3
$R^{N,M}$	0.9641	0.9827	0.9916	0.9958	

Table 2. Maximum errors and rate of convergence for Example 4.1.

$\varepsilon\downarrow$	$N=32$ $M=32$	64 64	128 128	256 256	512 512	1024 1024
10^{-09}	6.1884e-3	3.0781e-3	1.5349e-3	7.6638e-4	3.8292e-4	1.9140e-4
10^{-10}	6.1884e-3	3.0781e-3	1.5349e-3	7.6638e-4	3.8292e-4	1.9140e-4
10^{-11}	6.1884e-3	3.0781e-3	1.5349e-3	7.6638e-4	3.8292e-4	1.9140e-4
10^{-12}	6.1884e-3	3.0781e-3	1.5349e-3	7.6638e-4	3.8292e-4	1.9140e-4
10^{-13}	6.1884e-3	3.0781e-3	1.5349e-3	7.6638e-4	3.8292e-4	1.9140e-4
10^{-14}	6.1884e-3	3.0781e-3	1.5349e-3	7.6638e-4	3.8292e-4	1.9140e-4
10^{-15}	6.1884e-3	3.0781e-3	1.5349e-3	7.6638e-4	3.8292e-4	1.9140e-4
10^{-16}	6.1884e-3	3.0781e-3	1.5349e-3	7.6638e-4	3.8292e-4	1.9140e-4
$E^{N,M}$	6.1884e-3	3.0781e-3	1.5349e-3	7.6638e-4	3.8292e-4	1.9140e-4
$R^{N,M}$	1.0075	1.0039	1.0020	1.0010	1.0005	

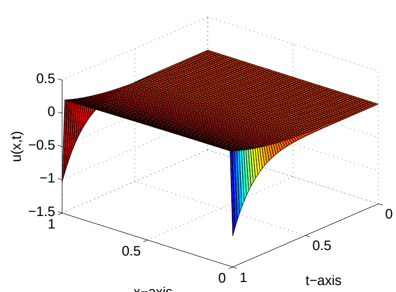

Figure 1. Numerical solution at $N=2^6, \Delta t = 1/N$, $\varepsilon = 10^{-12}$ for Example 4.1.

Surface plot in Figure 1 and line graph in Figure 2 for Example 4.1 shows numerical simulations. To confirm the theoretical order of convergence in spatial direction graphically, the maximum point-wise errors for Example 4.1 is plotted using log-log scale as can be seen in Figure 3 showing the ε-uniform convergence.

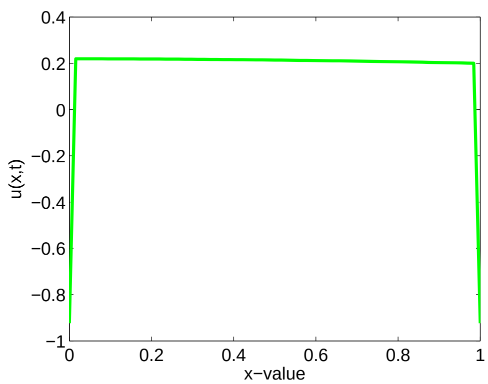

FIGURE 2. Numerical solution in terms of line graph at $N = 2^6, \Delta t = 1/N$, $\varepsilon = 10^{-12}$ for Example 4.1.

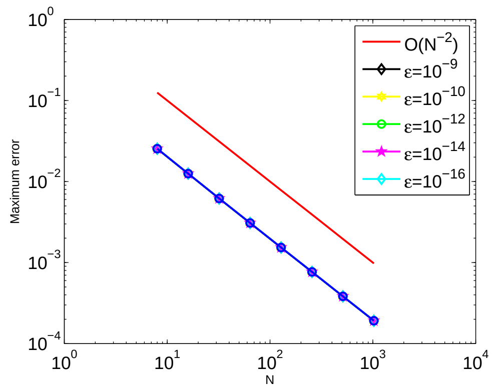

FIGURE 3. Plot of maximum point-wise errors for Example 4.1 via log-log scale for the result in Table 2.

Table 3 gives numerical results for Example 4.2 for equal number of mesh points.

Table 3. Maximum errors and rate of convergence for Example 4.2.

$\varepsilon \downarrow$	$N=32$	64	128	256	512	1024
	$M=32$	64	128	256	512	1024
10^{-09}	7.5968e-3	3.8794e-3	1.9599e-3	9.8521e-4	4.9389e-4	2.4727e-4
10^{-10}	7.5968e-3	3.8794e-3	1.9599e-3	9.8521e-4	4.9389e-4	2.4727e-4
10^{-11}	7.5968e-3	3.8794e-3	1.9599e-3	9.8521e-4	4.9389e-4	2.4727e-4
10^{-12}	7.5968e-3	3.8794e-3	1.9599e-3	9.8521e-4	4.9389e-4	2.4727e-4
10^{-13}	7.5968e-3	3.8794e-3	1.9599e-3	9.8521e-4	4.9389e-4	2.4727e-4
10^{-14}	7.5968e-3	3.8794e-3	1.9599e-3	9.8521e-4	4.9389e-4	2.4727e-4
10^{-15}	7.5968e-3	3.8794e-3	1.9599e-3	9.8521e-4	4.9389e-4	2.4727e-4
10^{-16}	7.5968e-3	3.8794e-3	1.9599e-3	9.8521e-4	4.9389e-4	2.4727e-4
$E^{N,M}$	7.5968e-3	3.8794e-3	1.9599e-3	9.8521e-4	4.9389e-4	2.4727e-4
$R^{N,M}$	9.6956e-1	9.8505e-1	9.9228e-01	9.9624e-1	9.9810e-1	

Similarly, surface plot in Figure 4 and line graph in Figure 5 for Example 4.2 depicts the solution profile.

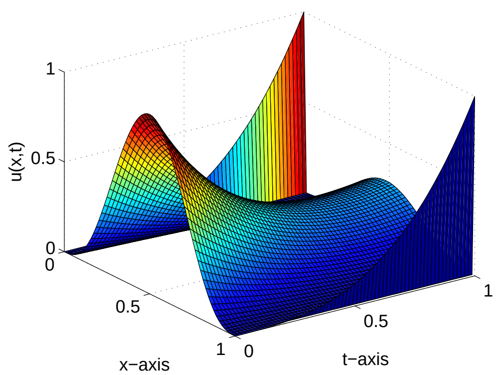

Figure 4. Numerical solution at $N = 2^6, \Delta t = 1/N$, $\varepsilon = 10^{-12}$ for Example 4.2.

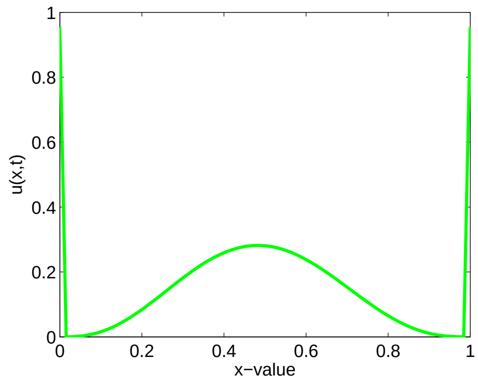

FIGURE 5. Numerical solution in terms of line graph at $N = 2^6, \Delta t = 1/N$, $\varepsilon = 10^{-12}$ for Example 4.2.

To confirm the theoretical order of convergence in spatial direction graphically, the maximum point-wise errors for Example 4.2 is plotted using log-log scale as can be seen in Figure 6 showing the ε-uniform convergence.

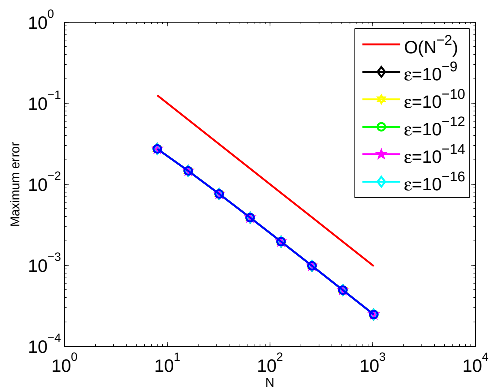

FIGURE 6. Plot of maximum point-wise errors for Example 4.2 via log-log scale for the result in Table 3.

Table 4 gives numerical results for Example 4.3 for equal number of mesh points.

Table 4. Maximum errors and rate of convergence for Example 4.3.

$\varepsilon \downarrow$	$N = 32$	64	128	256	512	1024
	$M = 32$	64	128	256	512	1024
10^{-09}	1.2310e-2	6.4654e-3	3.2345e-3	1.6142e-3	7.8942e-4	3.7678e-4
10^{-10}	1.2310e-2	6.4654e-3	3.2345e-3	1.6142e-3	7.8942e-4	3.7678e-4
10^{-11}	1.2310e-2	6.4654e-3	3.2345e-3	1.6142e-3	7.8942e-4	3.7678e-4
10^{-12}	1.2310e-2	6.4654e-3	3.2345e-3	1.6142e-3	7.8942e-4	3.7678e-4
10^{-13}	1.2310e-2	6.4654e-3	3.2345e-3	1.6142e-3	7.8942e-4	3.7678e-4
10^{-14}	1.2310e-2	6.4654e-3	3.2345e-3	1.6142e-3	7.8942e-4	3.7678e-4
10^{-15}	1.2310e-2	6.4654e-3	3.2345e-3	1.6142e-3	7.8942e-4	3.7678e-4
10^{-15}	1.2310e-2	6.4654e-3	3.2345e-3	1.6142e-3	7.8942e-4	3.7678e-4
$E^{N,M}$	1.2310e-2	6.4654e-3	3.2345e-3	1.6142e-3	7.8942e-4	3.7678e-4
$R^{N,M}$	9.2902e-1	9.9920e-1	1.0027e+0	1.0320e+0	1.0671e+0	

Again, Figure 7 show the numerical solution for Example 4.3 through surface plot and Figure 8 is line graph for Example 4.3.

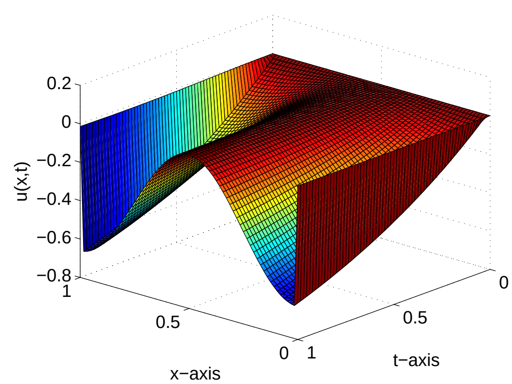

Figure 7. Numerical solution at $N = 2^6, \Delta t = 1/N, \varepsilon = 10^{-12}$ for Example 4.3.

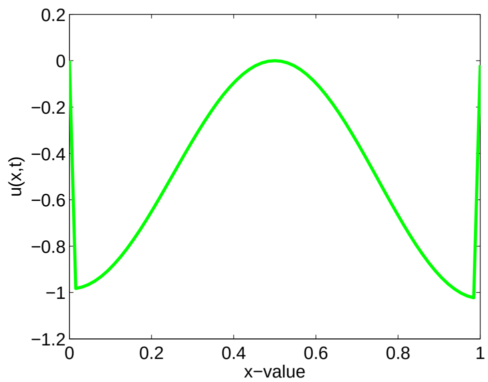

FIGURE 8. Numerical solution in terms of line graph at $N = 2^6, \Delta t = 1/N$, $\varepsilon = 10^{-12}$ for Example 4.3.

The theoretical order of convergence in spatial direction is confirmed graphically through log-log plot of the maximum point-wise errors for Example 4.3 in Figure 9.

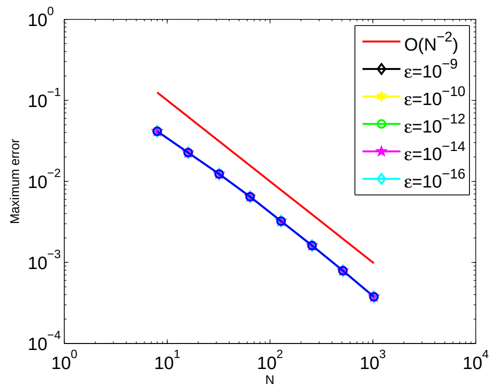

FIGURE 9. Plot of maximum point-wise errors for Example 4.3 via log-log scale for the result in Table 4.

From all the tables of values, we deduce that when the mesh points increases the maximum point-wise errors decreases. All the numerical simulations for the examples considered depict that the problem (1)–(3) has a parabolic boundary layer at $x = 0$ and $x = 1$.

5. Conclusion

In this study, the non-standard finite difference method for the numerical solution of singularly perturbed parabolic reaction-diffusion subject to Robin boundary conditions has presented. The problem is discretized in space direction via non-standard finite difference method and in time direction via implicit Euler scheme. Convergence analysis shows that the proposed method is second-order at the interior points and first-order at the two boundaries in space direction and first-order in time direction. Thus, the overall spatial direction convergence is first-order. The numerical solutions indicate that the proposed method is $\varepsilon-$uniformly convergent of first-order which agrees with the theoretical estimates. To see the solution profile, graphs of the numerical solution have been plotted for the three examples considered.

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