J. Appl. Math. & Informatics Vol. **40***(2022), No. 1 - 2, pp. 15 - 24*
https://doi.org/10.14317/jami.2022.015

THE QUADRATIC HYPONORMALITY OF ONE-STEP EXTENSION OF THE BERGMAN-TYPE SHIFT

CHUNJI LI* AND WENTAO QI

ABSTRACT. Let $p > 1$ and $\alpha^{[p]}(x) : \sqrt{x}, \sqrt{\frac{p}{2p-1}}, \sqrt{\frac{2p-1}{3p-2}}, \cdots$, with $0 < x \leq \frac{p}{2p-1}$. In [10], the authors considered the subnormality, n-hyponormality and positive quadratic hyponormality of $W_{\alpha^{[p]}(x)}$. By continuing to study, in this paper, we give a sufficient condition of quadratic hyponormality of $W_{\alpha^{[p]}(x)}$. Finally, we give an example to characterize the gaps of $W_{\alpha^{[p]}(x)}$ distinctively.

AMS Mathematics Subject Classification : 47B37, 47B20.
Key words and phrases : Positive quadratically hyponormal, quadratically hyponormal, unilateral weighted shift.

1. Introduction

Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. We recall some basic definitions of some classes of operators. We say that T is *normal* if $T^*T = TT^*$; *hyponormal* if $T^*T \geq TT^*$, and *subnormal* if T has a normal extension. For $S, T \in B(\mathcal{H})$, let $[S,T] := ST - TS$. We say that an n-tuple $T = (T_1, \ldots, T_n)$ of bounded linear operators on $B(\mathcal{H})$ is *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of n copies of $\mathcal{H}$. For any $k \in \mathbb{N}$, we say $T \in B(\mathcal{H})$ is (strongly) *k-hyponormal* if $(I, T, \ldots, T^k)$ is hyponormal. It is well-known that T is subnormal if and only if T is *k-hyponormal* for all $k \in \mathbb{N}$. An operator T in $B(\mathcal{H})$ is said to be *weakly n-hyponormal* if $p(T)$ is hyponormal for any polynomial p with degree less than or equal to n. And an operator T is *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial p. In particular, the quadratical hyponormality (i.e. weak 2-hyponormality) of weight shift has been considered in detail in [1], [2], [4] and [7].

Received November 9, 2020. Revised February 1, 2021. Accepted April 14, 2021.
*Corresponding author.
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Recall that let $\alpha := \{\alpha_n\}_{n=0}^{\infty}$ be a bounded sequence in the set $\mathbb{R}_+$. The (unilateral) *weighted shift* W_α acting on $\ell^2(\mathbb{N}_0)$, with an orthonormal basis $\{e_i\}_{i=0}^{\infty}$, is defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. It follows straightforward that W_α is hyponormal if and only if the weight sequence $\{\alpha_n\}_{n=0}^{\infty}$ is non-decreasing.

If a weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ is given by $\alpha_n = \sqrt{\frac{n+1}{n+2}}$ $(n \geq 0)$, then the corresponding weighted shift is called the *Bergman shift*. Let $x > 0$ and $\alpha(x) : \alpha_0 = \sqrt{x}$, $\alpha_n = \sqrt{\frac{n+2}{n+3}}$ $(n \geq 1)$. The k-hyponormality, subnormality and quadratic hyponormality of $W_{\alpha(x)}$ were considered in detail in [3], [4], [5], [6], [7] and [9] etc. In [8], the authors considered the backward extension of *Bergman-type shift* $\alpha^{[p]}(x) : \sqrt{x}, \sqrt{\frac{1}{p}}, \sqrt{\frac{p}{2p-1}}, \sqrt{\frac{2p-1}{3p-2}}, \ldots$, with $p > 1$. Furthermore, let $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, p > 1$ and $\alpha^{[m,p]}(x) : \sqrt{x}, \left\{\sqrt{\frac{(m+n-1)p-(m+n-2)}{(m+n)p-(m+n-1)}}\right\}_{n=1}^{\infty}$, in [10], the authors considered the subnormality, k-hyponormality, and positive quadratic hyponormality of $W_{\alpha^{[m,p]}(x)}$, which extends all the results on Bergman weighted shift $W_{\alpha(x)}$ with $m \in \mathbb{N}$, and $\alpha(x) : \sqrt{x}, \sqrt{\frac{m}{m+1}}, \sqrt{\frac{m+1}{m+2}}, \sqrt{\frac{m+2}{m+3}}, \ldots$. By continuing to study, in this paper, we give a sufficient condition of quadratic hyponormality of $W_{\alpha^{[p]}(x)}$ with $\alpha^{[p]}(x) : \sqrt{x}, \sqrt{\frac{p}{2p-1}}, \sqrt{\frac{2p-1}{3p-2}}, \ldots$. Finally, we give an example to characterize the gaps of $W_{\alpha^{[p]}(x)}$ distinctively.

All of the calculations in this paper were taken by using the software *Scientific WorkPlace* [11].

2. Preliminaries and Notations

We know that a weighted shift W_α is quadratically hyponormal if $W_\alpha + sW_\alpha^2$ is hyponormal for arbitrary complex number s([7]), that is,

$$M(s) := [\left(W_\alpha + sW_\alpha^2\right)^*, W_\alpha + sW_\alpha^2] \geq 0$$

for arbitrary complex number s. We let $\{e_i\}_{i=0}^{\infty}$ be an orthonormal basis for $\ell^2(\mathbb{N}_0)$ and

$$M_n(s) := P_n[\left(W_\alpha + sW_\alpha^2\right)^*, W_\alpha + sW_\alpha^2]P_n,$$

where P_n is the orthogonal projection onto the subspace generated by $\{e_i\}_{i=0}^{n}$. Then $M_n(s)$ has the following form

$$M_n(s) = \begin{pmatrix} \rho_0 & \kappa_0 & 0 & \cdots & 0 & 0 \\ \overline{\kappa}_0 & \rho_1 & \kappa_1 & \cdots & 0 & 0 \\ 0 & \overline{\kappa}_1 & \rho_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \rho_{n-1} & \kappa_{n-1} \\ 0 & 0 & 0 & \cdots & \overline{\kappa}_{n-1} & \rho_n \end{pmatrix},$$

where

$$\begin{cases} \rho_n := \sigma_n + |s|^2\delta_n, \\ \kappa_n := s\sqrt{\phi_n}, \\ \sigma_n := \alpha_n^2 - \alpha_{n-1}^2, \\ \delta_n := \alpha_n^2\alpha_{n+1}^2 - \alpha_{n-1}^2\alpha_{n-2}^2, \\ \phi_n := \alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2)^2, \end{cases}$$

for any nonnegative integer n and $\alpha_n := 0$ for negative integer n.

Hence, W_α is quadratically hyponormal if and only if $M_n(s) \geq 0$ for arbitrary complex number s and $n \in \mathbb{N}_0$. Let $t := |s|^2$ and $d_n(t) := \det M_n(t)$ which is a polynomial in t of degree $n+1$, with Maclaurin expansion $d_n(t) := \sum_{k=0}^{n+1} \theta_{n,k} t^k$. It is easy to find that $d_n(t)$ satisfies

$$\begin{aligned} d_0(t) &= \rho_0, \\ d_1(t) &= \rho_0\rho_1 - |\kappa_0|^2, \\ d_{n+2}(t) &= \rho_{n+2}d_{n+1}(t) - |\kappa_{n+1}|^2 d_n(t), \quad (n \geq 0). \end{aligned}$$

Also we can get the followings

$$\begin{gathered} \theta_{n,0} = \sigma_0 \cdots \sigma_n,\ \theta_{n,n+1} = \delta_0 \cdots \delta_n,\ \theta_{1,1} = \sigma_1\delta_0 + \sigma_0\delta_1 - \phi_0, \\ \theta_{n+2,k} = \sigma_{n+2}\theta_{n+1,k} + \delta_{n+2}\theta_{n+1,k-1} - \phi_{n+1}\theta_{n,k-1}, \end{gathered} \tag{1}$$

for $n \geq 0$ and $k \geq 1$.

Lemma 1. $\theta_{n,1} = \sigma_0 \cdots \sigma_{n-1}\alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2) \geq 0$, *for all* $n \geq 1$.

3. Key Lemmas

In this section, we consider an one-step extension $W_{\alpha^{[p]}(x)}$ of the Bergman-type shift, where

$$\alpha^{[p]}(x) : \sqrt{x}, \sqrt{\frac{p}{2p-1}}, \sqrt{\frac{2p-1}{3p-2}}, \sqrt{\frac{3p-2}{4p-3}}, \cdots, \tag{2}$$

where $p > 1$ and $0 < x \leq \frac{p}{2p-1}$. We have $\theta_{n,k} \geq 0$ for all $0 \leq n \leq 4$ and $0 \leq k \leq 4$ with $0 \leq k \leq n+1$ except for $\theta_{4,3}$.

$$\begin{cases} \theta_{0,0} = x > 0, \\ \theta_{0,1} = \frac{p}{2p-1}x > 0, \end{cases}$$

$$\begin{cases} \theta_{1,0} = x\left(\frac{p}{2p-1} - x\right) \geq 0, \\ \theta_{1,1} = \frac{xp}{2p-1}\left(\frac{2p-1}{3p-2} - x\right) > 0, \\ \theta_{1,2} = \frac{p^2 x}{(3p-2)(2p-1)} > 0, \end{cases}$$

$$\begin{cases} \theta_{2,0} = \frac{(p-1)^2}{(3p-2)(2p-1)} x \left(\frac{p}{2p-1} - x \right) \geq 0, \\ \theta_{2,1} = \frac{2(p-1)^2 x}{(4p-3)(3p-2)} \left(\frac{p}{2p-1} - x \right) \geq 0, \\ \theta_{2,2} = xp\,(p-1)^2 \frac{(4p-1)-(4p-2)x}{(4p-3)(3p-2)(2p-1)^2} > 0, \\ \theta_{2,3} = xp^2 \frac{(2p-1)^2 - \left(4p^2-3p\right)x}{(4p-3)(3p-2)(2p-1)^2} > 0, \end{cases}$$

$$\begin{cases} \theta_{3,0} = \frac{x(p-1)^4}{(4p-3)(3p-2)^2(2p-1)} \left(\frac{p}{2p-1} - x \right) \geq 0, \\ \theta_{3,1} = \frac{2(p-1)^4 x}{(5p-4)(4p-3)(3p-2)(2p-1)} \left(\frac{p}{2p-1} - x \right) \geq 0, \\ \theta_{3,2} = \frac{(p-1)^4 x\left(\left(11p^2-4p\right)-\left(22p^2-24p+8\right)x\right)}{(5p-4)(4p-3)(3p-2)^2(2p-1)^2} > 0, \\ \theta_{3,3} = \frac{p(p-1)^2 x\left(\left(16p^3-31p^2+20p-4\right)-\left(21p^3-44p^2+32p-8\right)x\right)}{(5p-4)(4p-3)(3p-2)^2(2p-1)^2} > 0, \\ \theta_{3,4} = \frac{4p^2(p-1)^2 x\left((2p-1)^2-\left(4p^2-3p\right)x\right)}{(5p-4)(4p-3)(3p-2)^2(2p-1)^2} > 0, \end{cases}$$

$$\begin{cases} \theta_{4,0} = \frac{(p-1)^6 x}{(5p-4)(4p-3)^2(3p-2)^2(2p-1)} \left(\frac{p}{2p-1} - x \right) \geq 0, \\ \theta_{4,1} = \frac{2x(p-1)^6}{(6p-5)(5p-4)(4p-3)(3p-2)^2(2p-1)} \left(\frac{p}{2p-1} - x \right) \geq 0, \\ \theta_{4,2} = \frac{x(p-1)^6\left(\left(18p^2-11p\right)-\left(36p^2-46p+16\right)x\right)}{(6p-5)(5p-4)(4p-3)^2(3p-2)^2(2p-1)^2} \geq 0, \\ \theta_{4,3} = \frac{(p-1)^4 x\left(\left(44p^4-98p^3+71p^2-16p\right)-\left(94p^4-277p^3+312p^2-160p+32\right)x\right)}{(6p-5)(5p-4)(4p-3)^2(3p-2)^2(2p-1)^2}, \\ \theta_{4,4} = \frac{4(p-1)^4 px\left(16p^3-31p^2+20p-4\right)-(3p-2)\left(7p^2-10p+4\right)x}{(6p-5)(5p-4)(4p-3)^2(3p-2)^2(2p-1)^2} \geq 0, \\ \theta_{4,5} = \frac{16xp^2(p-1)^4\left((2p-1)^2-\left(4p^2-3p\right)x\right)}{(6p-5)(5p-4)(4p-3)^2(3p-2)^2(2p-1)^2} \geq 0. \end{cases}$$

Considering the $W_{\alpha^{[p]}(x)}$, we can obtain the following lemmas.

Lemma 2. *Let* $\alpha^{[p]}(x)$ *be as in* (2). *Then* $\theta_{n,2} \geq 0$ *for all* $n \geq 1$.
Proof. For $n \geq 2$, by (1) we have

$$\begin{aligned} & \delta_{n+2}\theta_{n+1,1} - \phi_{n+1}\theta_{n,1} \\ = \; & \delta_{n+2}\sigma_0 \cdots \sigma_n \alpha_{n+1}^2 (\alpha_{n+2}^2 - \alpha_n^2) - \phi_{n+1}\sigma_0 \cdots \sigma_{n-1}\alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2) \\ = \; & \sigma_0 \cdots \sigma_{n-1}(\delta_{n+2}\sigma_n\alpha_{n+1}^2(\alpha_{n+2}^2 - \alpha_n^2) - \phi_{n+1}\alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2)) \\ = \; & \frac{24\,(p-1)^8\,\sigma_0 \cdots \sigma_{n-1}}{(\Delta + 4p - 3)\,(\Delta + 2p - 1)^2\,(\Delta + p)^2\,(\Delta + 1)\,(\Delta + 3p - 2)^2} \geq 0, \end{aligned}$$

with $\Delta = n\,(p-1)$. It follows that if $\theta_{n+1,2} \geq 0$, then for $n \geq 2$,

$$\theta_{n+2,2} = u_{n+2}\theta_{n+1,2} + \delta_{n+2}\theta_{n+1,1} - \phi_{n+1}\theta_{n,1} \geq 0.$$

Since $\theta_{n,2} \geq 0$ for $n = 1, 2, 3$ with $0 < x \leq \frac{p}{2p-1}$ and $p > 1$, we can get $\theta_{n,2} \geq 0$ for all $n \geq 1$. □

Lemma 3. *Let* $\alpha^{[p]}(x)$ *be as in* (2). *Then* $\theta_{n,k} = \delta_n\theta_{n-1,k-1}$ *for all* $n \geq 4, k \geq 4$.

Proof. Clearly, $\sigma_{n+1}\delta_n = \phi_n$ ([10], Lemma 5.1), for all $n \geq 3$. So for all $n \geq 4$, it is simple that

$$\begin{aligned}
\theta_{n,k} &= \sigma_n\theta_{n-1,k} + \delta_n\theta_{n-1,k-1} - \phi_{n-1}\theta_{n-2,k-1} \\
&= \delta_n\theta_{n-1,k-1} - \phi_{n-1}\theta_{n-2,k-1} \\
&\quad +\sigma_n\left[\sigma_{n-1}\theta_{n-2,k} + \delta_{n-1}\theta_{n-2,k-1} - \phi_{n-2}\theta_{n-3,k-1}\right] \\
&= \delta_n\theta_{n-1,k-1} + \sigma_n\left[\sigma_{n-1}\theta_{n-2,k} - \phi_{n-2}\theta_{n-3,k-1}\right] \\
&= \delta_n\theta_{n-1,k-1} + \sigma_n\cdots\sigma_4 h_k, \quad \text{with} \\
h_k &:= \sigma_3\theta_{2,k} - \phi_2\theta_{1,k-1}, \quad k \geq 1.
\end{aligned}$$

Since $h_k = 0$ for all $k \geq 4$. Thus $\theta_{n,k} = \delta_n\theta_{n-1,k-1}$ for all $n \geq 4, k \geq 4$. □

Lemma 4. *Let $\alpha^{[p]}(x)$ be as in (2). If $\theta_{n,3} \geq 0$, then $\theta_{n+1,3} \geq 0$ for $n \geq 4$.*
Proof. Since ([10], Lemma 5.1) $\delta_{n+1}\sigma_n > \phi_n$, and for all $n \geq 4$,

$$\begin{aligned}
&\delta_{n+1}\theta_{n,2} - \phi_n\theta_{n-1,2} \\
=\ &\delta_{n+1}(\sigma_n\theta_{n-1,2} + \delta_n\theta_{n-1,1} - \phi_{n-1}\theta_{n-2,1}) - \phi_n\theta_{n-1,2} \\
=\ &(\delta_{n+1}\sigma_n - \phi_n)\theta_{n-1,2} + \delta_{n+1}(\delta_n\theta_{n-1,1} - \phi_{n-1}\theta_{n-2,1}) \geq 0,
\end{aligned}$$

and $\delta_n\theta_{n-1,1} - \phi_{n-1}\theta_{n-2,1} \geq 0$ by the proof of Lemma 2. Therefore if $\theta_{n,3} \geq 0$, then

$$\theta_{n+1,3} = \sigma_{n+1}\theta_{n,3} + \delta_{n+1}\theta_{n,2} - \phi_n\theta_{n-1,2} \geq 0$$

for all $n \geq 4$. □

Through Lemma 1, Lemma 2, Lemma 3 and Lemma 4, it follows that $\theta_{n,k} \geq 0$ for all $n, k \geq 0$ with $0 \leq k \leq n+1$ if and only if $\theta_{n,3} \geq 0$ for all $n \geq 4$, or equivalently $\theta_{4,3} \geq 0$. See Fig. 1 below.

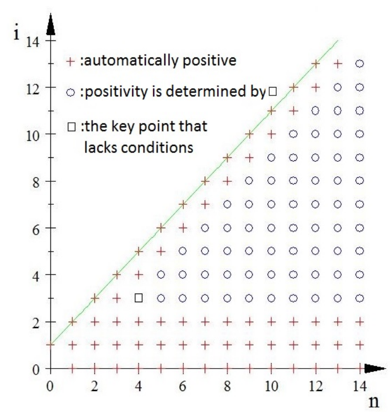

Figure 1: The positivity of $\theta_{n,i}$.

Proposition 5([10])**.** *Let $\alpha^{[p]}(x)$ be as in* (2).
(a) *If $1 < p \le \frac{25+\sqrt{241}}{12}$, then $W_{\alpha^{[p]}(x)}$ is positively quadratically hyponormal if and only if $0 < x \le \frac{p}{2p-1}$.*
(b) *If $p > \frac{25+\sqrt{241}}{12}$, then $W_{\alpha^{[p]}(x)}$ is positively quadratically hyponormal if and only if $0 < x \le \xi_1 := \frac{44p^4-98p^3+71p^2-16p}{94p^4-277p^3+312p^2-160p+32}$.*

Remark. When $1 < p \le \frac{25+\sqrt{241}}{12}$, $\theta_{4,3} \ge 0 \Leftrightarrow 0 < x \le \frac{p}{2p-1}$ and when $p > \frac{25+\sqrt{241}}{12}$, $\theta_{4,3} \ge 0 \Leftrightarrow 0 < x \le \xi_1$.

According to ([10]), it has the other interesting results.

Proposition 6. *Let $\alpha^{[p]}(x)$ be as in* (2).
(a) *$W_{\alpha^{[p]}(x)}$ is subnormal if and only if $0 < x \le \frac{1}{p}$.*
(b) *$W_{\alpha^{[p]}(x)}$ is n-hyponormal if and only if $0 < x \le \frac{1}{p}\frac{\prod_{l=1}^{n}[lp-(l-1)]^2}{\prod_{l=1}^{n}[lp-(l-1)]^2-(n!)^2(p-1)^{2n}}$.*

4. The Quadratic Hyponormality of $W_{\alpha^{[p]}(x)}$

Let $\alpha^{[p]}(x)$ be as in (2). Proposition 5 obtained equivalent condition of positive quadratical hyponormality of $W_{\alpha^{[p]}(x)}$. In this section we give a sufficient condition of the quadratical hyponormality of $W_{\alpha^{[p]}(x)}$. Let

$$\begin{aligned} \xi_0 &:= \frac{p}{2p-1}, \\ \xi_1 &:= \frac{44p^4-98p^3+71p^2-16p}{94p^4-277p^3+312p^2-160p+32}, \\ \xi_2 &:= \frac{72p^4-181p^3+154p^2-44p}{151p^4-478p^3+576p^2-312p+64}, \\ \xi_3 &:= \frac{856p^5-2791p^4+3418p^3-1857p^2+376p}{1809p^5-7126p^4+11335p^3-9104p^2+3696p-608}. \end{aligned} \tag{3}$$

Lemma 7. *Let $\alpha^{[p]}(x)$ be as in* (2).
(1) *If $1 < p \le \frac{15+\sqrt{85}}{7}$ (≈ 3.4599), then $\theta_{5,3} \ge 0$ if and only if $0 < x \le \xi_0$.*
(2) *If $p > \frac{15+\sqrt{85}}{7}$, then $\theta_{5,3} \ge 0$ if and only if $0 < x \le \xi_2$.*
Proof. In fact

$$\begin{aligned} \theta_{5,3} &= \sigma_5\theta_{4,3} + \delta_5\theta_{4,2} - \phi_4\theta_{3,2} \\ &= x\left(\xi_2 - x\right)\frac{(p-1)^6\left(151p^4-478p^3+576p^2-312p+64\right)}{(7p-6)(6p-5)(5p-4)^2(4p-3)^2(3p-2)^2(2p-1)^2}. \end{aligned}$$

And $\xi_2 < \xi_0$ if and only if $p > \frac{15+\sqrt{85}}{7}$. Thus we have our conclusions. □

Note that $d_n(t) \ge 0$ for $n = 0, 1, 2, 3$. Observe by Lemma 3 that if $n \ge 6$, then

$$\theta_{n,n-2}t^{n-2} + \theta_{n,n-1}t^{n-1} + \theta_{n,n}t^n = \delta_n \cdots \delta_6 t^{n-5}(\theta_{5,3}t^3 + \theta_{5,4}t^4 + \theta_{5,5}t^5).$$

Thus if $\theta_{5,3}t^3+\theta_{5,4}t^4+\theta_{5,5}t^5\geq 0$ for all $t\geq 0$, then $d_n(t)\geq 0$ for all $n\geq 6$ and $t\geq 0$ because other Maclaurin coefficients are nonnegative. So we will verify $\theta_{n,n-2}t^{n-2}+\theta_{n,n-1}t^{n-1}+\theta_{n,n}t^n\geq 0$ for $n=4,5$. That is ([3]),

$$\theta_{4,2}t^2+\theta_{4,3}t^3+\theta_{4,4}t^4\geq 0, \text{ and } \theta_{5,3}t^3+\theta_{5,4}t^4+\theta_{5,5}t^5\geq 0,$$

for all $t\geq 0$.

Theorem 8. *Let $\alpha^{[p]}(x)$ be as in* (2).
(a) *If $1<p\leq p_1$, then $W_{\alpha^{[p]}(x)}$ is quadratically hyponormal if and only if $0<x\leq\xi_0$.*
(b) *If $p>p_1$ and $0<x\leq\xi_3$, then $W_{\alpha^{[p]}(x)}$ is quadratically hyponormal, where*

$$p_1=\frac{494+2\sqrt{62743}\cos\omega}{291}\ (\approx 3.4188)\,,\ \textit{with}\ \omega=\frac{1}{3}\arccos\left(\frac{15684659}{3936684049}\sqrt{62743}\right). \tag{4}$$

Proof. From Proposition 5, we need to discuss the case of $p>\frac{25+\sqrt{241}}{12}\ (\approx 3.377)$. By Lemma 4 and Lemma 7, we know that $c(n,3)\geq 0$ for all $n\geq 5$, in one of the following two cases,

Case 1. $p>\frac{15+\sqrt{85}}{7}$ and $0<x\leq\xi_2$;

Case 2. $\frac{25+\sqrt{241}}{12}<p\leq\frac{15+\sqrt{85}}{7}$ and $0<x\leq\xi_0$.

Under **Case 1.** We have the following results.

Claim I. If $p>\frac{15+\sqrt{85}}{7}$ and $\xi_1<x\leq\xi_3$, then $\theta_{4,3}<0$ and $\theta_{4,2}t^2+\theta_{4,3}t^3+\theta_{4,4}t^4\geq 0$.

Proof of Claim I. Under the condition of the Claim, we can get

$$\sigma_5\theta_{4,3}+\delta_5\theta_{4,2}=\frac{(p-1)^6\,x\Phi_1}{(7p-6)\,(6p-5)^2\,(5p-4)^2\,(4p-3)^2\,(3p-2)^2\,(2p-1)^2}\geq 0,$$

where

$$\begin{aligned}\Phi_1 &= \left(740p^5-2438p^4+2985p^3-1602p^2+316p\right)\\ &\quad -\left(1522p^5-6055p^4+9662p^3-7744p^2+3128p-512\right)x.\end{aligned}$$

Since $\theta_{4,2}\geq 0$ and $\theta_{4,3}<0$, it follows that if $0<t\leq\frac{7p-6}{4(6p-5)}$, where $\frac{\sigma_5}{\delta_5}=\frac{7p-6}{4(6p-5)}$, then $\theta_{4,2}+\theta_{4,3}t\geq 0$. Since $\theta_{4,4}\geq 0$, we have $\theta_{4,2}t^2+\theta_{4,3}t^3+\theta_{4,4}t^4\geq 0$.

We also get that

$$\sigma_5\theta_{4,4}+\delta_5\theta_{4,3}=\frac{4\,(p-1)^6\,x\Phi_2}{(7p-6)\,(6p-5)^2\,(5p-4)^2\,(4p-3)^2\,(3p-2)^2\,(2p-1)^2}\geq 0.$$

where

$$\begin{aligned}\Phi_2 &= \left(376p^5-1121p^4+1242p^3-599p^2+104p\right)\\ &\quad -\left(711p^5-2566p^4+3745p^3-2768p^2+1040p-160\right)x.\end{aligned}$$

So if $t > \frac{7p-6}{4(6p-5)}$, then $t\theta_{4,4} + \theta_{4,3} \geq 0$. Since $\theta_{4,2} \geq 0$, we have that $\theta_{4,2}t^2 + \theta_{4,3}t^3 + \theta_{4,4}t^4 \geq 0$. ▲

Claim II. If $p > \frac{15+\sqrt{85}}{7}$ and $\xi_1 < x \leq \xi_3$, then $\theta_{5,3}t^3 + \theta_{5,4}t^4 + \theta_{5,5}t^5 \geq 0$.
Proof of Claim II. By the same argument as Claim I, it suffices to prove that if $\xi_1 < x \leq \xi_3$, then $\sigma_6\theta_{5,4} + \delta_6\theta_{5,3} \geq 0$ and $\sigma_6\theta_{5,5} + \delta_6\theta_{5,4} \geq 0$.

Indeed, a straightforward calculation shows that

$$\begin{aligned} & \sigma_6\theta_{5,4} + \delta_6\theta_{5,3} \\ = \quad & \frac{4x\,(p-1)^8\,\Phi_3}{(8p-7)\,(7p-6)^2\,(6p-5)^2\,(5p-4)^2\,(4p-3)^2\,(3p-2)^2\,(2p-1)^2} \geq 0, \end{aligned}$$

where

$$\begin{aligned} \Phi_3 \quad = \quad & \left(856p^5 - 2791p^4 + 3418p^3 - 1857p^2 + 376p\right) \\ & - \left(1809p^5 - 7126p^4 + 11\,335p^3 - 9104p^2 + 3696p - 608\right)x, \end{aligned}$$

and

$$\begin{aligned} & \sigma_6\theta_{5,5} + \delta_6\theta_{5,4} \\ = \quad & \frac{32x\,(p-1)^8\,\Phi_4}{(8p-7)\,(7p-6)^2\,(6p-5)^2\,(5p-4)^2\,(4p-3)^2\,(3p-2)^2\,(2p-1)^2} \geq 0. \end{aligned}$$

where

$$\begin{aligned} \Phi_4 \quad = \quad & \left(218p^5 - 655p^4 + 731p^3 - 355p^2 + 62p\right) \\ & - \left(413p^5 - 1501p^4 + 2205p^3 - 1640p^2 + 620p - 96\right)x. \end{aligned}$$

So $\theta_{5,3}t^3 + \theta_{5,4}t^4 + \theta_{5,5}t^5 \geq 0$. ▲

By Claim I and Claim II, we have proved that if $p > \frac{15+\sqrt{85}}{7}$ and $0 < x \leq \xi_3$, then $W_{\alpha^{[p]}(x)}$ is quadratically hyponormal.

Under **Case 2.** If $\frac{25+\sqrt{241}}{12} < p \leq \frac{15+\sqrt{85}}{7}$ and $\xi_1 < x \leq \xi_3\,(< \xi_2)$, then $\theta_{4,3} < 0$, $\theta_{5,3} \geq 0$. By Lemma 2, $\theta_{n,n-1} < 0$ for all $n \geq 4$. Note that if $\frac{25+\sqrt{241}}{12} < p \leq p_1$, then $\xi_3 \geq \xi_0$, and if $p_1 < p \leq \frac{15+\sqrt{85}}{7}$, then $\xi_3 < \xi_0$. By the same way as Claim I and Claim II, we can easily prove that if $\frac{25+\sqrt{241}}{12} < p \leq p_1$ and $\xi_1 < x \leq \xi_0$, or if $p_1 < p \leq \frac{15+\sqrt{85}}{7}$ and $\xi_1 < x \leq \xi_3$, then $\theta_{n,n-2}t^{n-2} + \theta_{n,n-1}t^{n-1} + \theta_{n,n}t^n \geq 0$ for $n = 4, 5$.

Therefore, if $1 < p \leq p_1$, then $W_{\alpha^{[p]}(x)}$ is quadratically hyponormal if and only if $0 < x \leq \xi_0$. If $p > p_1$ and $0 < x \leq \xi_3$, then $W_{\alpha^{[p]}(x)}$ is quadratically hyponormal. □

Remark. Let $\xi_0, \xi_1, \xi_2, \xi_3$ as in (3).
(1) When $\frac{25+\sqrt{241}}{12} < p < p_1$, we get $\xi_1 < \xi_0 < \xi_3 < \xi_2$.
(2) When $p_1 < p < \frac{15+\sqrt{85}}{7}$, we get $\xi_1 < \xi_3 < \xi_0 < \xi_2$.

(3) When $p > \frac{15+\sqrt{85}}{7}$, we get $\xi_1 < \xi_3 < \xi_2 < \xi_0$.

Example 9. If $p = 4$, then $\alpha^{[4]}(x) : \sqrt{x}, \sqrt{\frac{4}{7}}, \sqrt{\frac{7}{10}}, \sqrt{\frac{10}{13}}, \cdots$. By the results as above, we know that

- If $0 < x \leq \frac{22037}{38882} (\approx 0.56677)$, then $W_{\alpha^{[4]}(x)}$ is quadratically hyponormal. (By Theorem 8)
- $W_{\alpha^{[4]}(x)}$ is positively quadratically hyponormal if and only if $0 < x \leq \frac{379}{670} (\approx 0.56567)$. (By Proposition 5)
- If $\frac{379}{670} < x \leq \frac{22037}{38882}$, then $W_{\alpha^{[4]}(x)}$ is quadratically hyponormal but not positively quadratically hyponormal. In particular, $W_{\alpha^{[4]}(x_0)}$ is quadratically hyponormal but not positively quadratically hyponormal, here $x_0 = 0.566 = \frac{566}{1000} = \frac{283}{500}$.
- $W_{\alpha^{[4]}(x)}$ is 2-hyponormal if and only if $0 < x \leq \frac{49}{115} (\approx 0.426\,09)$.
- $W_{\alpha^{[4]}(x)}$ is 3-hyponormal if and only if $0 < x \leq \frac{4900}{13039} (\approx 0.37580)$.
- $W_{\alpha^{[4]}(x)}$ is 4-hyponormal if and only if $0 < x \leq \frac{207025}{591904} (\approx 0.34976)$.
- $W_{\alpha^{[4]}(x)}$ is n-hyponormal if and only if $0 < x \leq \frac{1}{4} \frac{1}{1-\left(\frac{3^n (n!)}{4\cdot 7\cdot \cdots \cdot (3n+1)}\right)^2}$.
- $W_{\alpha^{[4]}(x)}$ is subnormal if and only if $0 < x \leq \frac{1}{4}$.

5. Conclusion

After the subnormality, n-hyponormalty, and positively quadratic hyponormality [10], this paper considered the quadratic hyponormality of $W_{\alpha}^{[p]}(x)$. The cubic hyponormality, semi-weakly hyponormality and other topics, also in particular, new techniques for solving these problems can be considered for further research. We leave them to interested readers.

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Chunji Li received Ph.D. degree from Kyungpook National University, Korea. His research interests focus on the control theory, moment method, and unilateral weighted shifts.

Department of Mathematics, Northeastern University, Shenyang 110819, P.R. China.
e-mail: lichunji@mail.neu.edu.cn

Wentao Qi received Bachelor's degree from Northeastern University. He is graduate students in Zhejiang University.

Department of Mathematics, Zhejiang University, Hangzhou 310027, P.R. China.
e-mail: 21935053@zju.edu.cn