J. Appl. Math. & Informatics Vol. **40**(2022), No. 1 - 2, pp. 185 - 190
https://doi.org/10.14317/jami.2022.185

FUZZY COMPACTNESS, FUZZY REGULARITY VIA FUZZY MAXIMAL OPEN AND FUZZY MINIMAL CLOSED SETS

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Abstract. The aim of this article is to define fuzzy maximal open cover and discuss its few properties. we also defined and study fuzzy m-compact space and discussed its properties. Also we obtain few more results on fuzzy minimal c-regular and fuzzy minimal c-normal spaces. We have proved that a fuzzy Haussdorff m-compact space is fuzzy minimal c-normal.

AMS Mathematics Subject Classification : 54A40, 03E72.
Key words and phrases : Fuzzy minimal open, fuzzy maximal open cover, fuzzy minimal c-regular(resp.c-normal).

1. Introduction

Zadeh[11] established fuzzy set in 1965. Chang[3] introduced fuzzy topology in 1968. Since fuzzy minimal (resp.maximal) open sets[4], Swaminathan developed fuzzy mean open sets in [6]. An addendum to fuzzy sets,few more properties of fuzzy minimal(resp.maximal)set investigated by Swaminathan and Sivaraja [10] and they have showed "if a fuzzy topological space having both fuzzy minimal open and fuzzy maximal open set, then it may be fuzzy disconnected".Swaminathan and Sivaraja[9] introduced and investigated fuzzy cut-point space and related results.

The following terminilogies, "fuzzy minimal open set, fuzzy maximal open set, fuzzy mean open set, fuzzy clopen set, fuzzy cut-point space, fuzzy connected topological space, fuzzy disconnected topological space and fuzzy topological space"are respectively abbreviated as "FMIO, FMAO, FMEO, FCLO, FCS, FCTS,FDTS and FTS."

In section 2 of this article we define fuzzy maximal open cover.Fuzzy m-compact space and few properties discussed in Section 3. In section 4, the idea of fuzzy minimal c-regular (resp.c-normal spaces) are introduced from which We show that a fuzzy Haussdorff m-compact space is fuzzy minimal c-normal.

Received September 23, 2021. Revised November 23, 2021. Accepted December 3, 2021.
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2. Preliminaries

Definition 2.1. [8] *A proper FCLO set δ of X is called a FMICLO set if ϑ is a FCLO set such that $\vartheta < \delta$, then $\vartheta = \delta$ or $\vartheta = 0_X$.*

Definition 2.2. [8] *A proper FCLO set δ of X is called a FMACLO set if ϑ is a FCLO set such that $\delta < \vartheta$, then $\delta = \vartheta$ or $\vartheta = 1_X$.*

Definition 2.3. [6] *In a fts X, α is called a FMEO(resp.γ FMEC) if $\exists\ \lambda, \mu(\neq \alpha)$ two distinct proper fuzzy open sets (resp. two distinct proper fuzzy closed sets $\beta, \delta(\neq \gamma)$) such that $\lambda < \alpha < \mu$(resp. $\beta < \gamma < \delta$)*

Lemma 2.4. [9] *Each nonzero fuzzy open set γ of a T_1-fcts X is infinite and is not a FMIO in X.*

Theorem 2.5. [9] *A proper fuzzy open set γ of a T_1-fcts X is a FMEO set in X iff $\gamma \neq 1_X - \{x_\alpha\}$ for any $x_\alpha \in X$.*

3. Fuzzy maximal open cover and Fuzzy m-compact space

Firstly we introduce FMAO covers. Further, the idea of fuzzy m-compact space is studied by means of FMAO covers.

A fuzzy cover $\mathfrak{C}$ of X is an fuzzy refinement of the fuzzy cover $\mathfrak{D}$ of X if $\forall$ $\alpha \in \mathfrak{C}$, $\exists \beta \in \mathfrak{C}$ such that $\alpha < \beta$.

Definition 3.1. Let $\mathfrak{C}$ and $\mathfrak{D}$ be two fuzzy covers of a FTS X.$\mathfrak{C}$ is an fuzzy s-refinement of $\mathfrak{D}$ if $\forall\ \alpha \in \mathfrak{C}\ \exists\ \beta \in \mathfrak{D}$ such that $\alpha < \beta$. A fuzzy s-refinement $\mathfrak{C}$ of $\mathfrak{D}$ is said to be a fuzzy open s-refinement of $\mathfrak{D}$ if all members of $\mathfrak{C}$ and $\mathfrak{D}$ are fuzzy open.

It is observed that if $\mathfrak{D} = \{1_X\}$ and $\alpha \neq 1_X$ for each $\alpha \in \mathfrak{C}$, then $\mathfrak{C}$ is an fuzzy s-refinement of $\mathfrak{D}$. If $\mathfrak{C}$ is fuzzy s-refinement of $\mathfrak{D}$ then $\mathfrak{C}$ is an fuzzy refinement of $\mathfrak{D}$. Further we see that no element of an s-fuzzy refinement of any fuzzy cover of X is FMAO.

Definition 3.2. A fuzzy open cover $\mathfrak{C}$ of a fts X is called a FMAO cover of X if $\mathfrak{C}$ is not an fuzzy s-refinement of any other fuzzy open cover of X.

Lemma 3.3. *A fuzzy open cover containing a FMAO set is fuzzy maximal.*

Proof. Obvious □

Theorem 3.4 (Existence of FMAO covers)**.** *There exists a FMAO cover in an infinite T_1-fts.*

Proof. Let X be an infinite T_1-fts. Then $\forall\ p_x^\alpha \in X$, $1_X - \{p_x^\alpha\}$ is FMAO set in X. Let $p_x^\beta \in X$. Consider a finite fuzzy subset $G = \{p_x^{\alpha_i} | p_x^{\alpha_i} \neq \delta, i \in Z; 1 \leq i \leq n\}$.Also α in X is fuzzy closed as X is T_1 fts. Henceforth $\{1_X - \{p_x^\beta\}, 1_X - G\}$ is fuzzy open cover of X having FMAO set $1_X - \{p_x^\beta\}$. Hence by Lemma 3.1,$\{1_X - \{p_x^\beta\}, 1_X - G\}$ is FMAO cover of X. □

Theorem 3.5. *Any fuzzy open cover $\mathfrak{M}$ of an infinite T_1-fts is a FMAO cover of X iff $\mathfrak{M}$ contains a FMAO set.*

Proof. Let $\mathfrak{M} = \{U_k | k \in V\}$ be a FMAO cover of X such that no U_k,$k \in V$ is FMAO. By Theorem 2.5, U_k is not also FMIO $\forall\ k \in V$ which implies that U_k,$k \in V$ is FMEO. So $\forall\ k \in V$, $\exists V_k$ a proper fuzzy open set such that $U_k \lneq V_k$. Let $\mathfrak{N} = \{V_k | U_k \lneq V_k, U_k \in \mathfrak{M}\}$. Clearly $\mathfrak{N}$ is fuzzy cover of X. Therefore $\mathfrak{M}$ is an fuzzy s-refinement of $\mathfrak{N}$ a contradiction to tha fact that $\mathfrak{M}$ is a FMAO cover of X. Hence $\mathfrak{M}$ has a FMAO sets as one among its members. The converse part follows by Lemma 3.3. $\square$

Definition 3.6. If every FMAO cover of a fts X has a finite fuzzy open s-refinement then X is said to be fuzzy m-compact.

Theorem 3.7. *Every infinite T_1 fcts is fuzzy m-compact.*

Proof. Let $\mathfrak{M}$ be FMAO cover of an infinite T_1 fcts X. By Theorem 3.5, $\mathfrak{M}$ contains a FMAO set U. By Theorem 2.5, take $U = 1_X - \{p_x^\alpha\}$ for some $p_x^\alpha \in X$. There is an $V \in \mathfrak{M}$ such that $p_x^\alpha \in V$. By Lemma 3.7, for fuzzy points $p_x^\alpha, p_x^\beta \in V$ with $p_x^\alpha \neq p_x^\beta$ there are fuzzy open sets $V_1 = 1_X - \{p_x^\alpha, p_x^\beta\}$,$V_2 = V - \{p_x^\alpha\}$,$V_3 = V - \{p_x^\beta\}$ of X. Then $\{V_1, V_2, V_3\}$ is an fuzzy s-refinement of $\mathfrak{M}$. $\square$

Example 3.8. Let $X = I$. Then $\mathfrak{F} = \{0_X, \beta_1, \beta_2, \beta_3, \beta_4, 1_X\}$ be a FTS where

$$\beta_1 = \begin{cases} 1 & if \quad x \neq \frac{1}{2} \\ 0 & if \quad x = \frac{1}{2} \end{cases} \quad \beta_2 = \begin{cases} 0 & if \quad x \neq \frac{1}{2} \\ 1 & if \quad x = \frac{1}{2} \end{cases}$$

$$\beta_3 = \begin{cases} \frac{1}{2} & if \quad x \neq \frac{1}{2} \\ 1 & if \quad x = \frac{1}{2} \end{cases} \quad \beta_4 = \begin{cases} \frac{1}{2} & if \quad x \neq \frac{1}{2} \\ 0 & if \quad x = \frac{1}{2} \end{cases}$$

Clearly $(X, \mathfrak{F})$ is fuzzy compact but not fuzzy m-compact.

Remark 3.1. By Theorem 3.4,the real number space with the usual fuzzy topology is fuzzy m-compact but generally it is not fuzzy compact. Since by Theorem 3.7 together with Example 3.8, we conclude that both fuzzy compactness and fuzzy m-compactness are independent.

Definition 3.9. A function $f : X \to Y$ for any two FTSs X and Y is said to be fuzzy m-continuous,if inverse image of each proper fuzzy open set in Y is FMAO in X.

Theorem 3.10. *Let X be a fuzzy m-compact topological space and $f : X \to Y$ be a bijective fuzzy m-continuous function. Then Y is fuzzy m-compact.*

Proof. By assuming the contrary there is no finite fuzzy s-refinement for any FMAO cover of Y, $\mathfrak{S}^{(Y)} = \{U | U \in \mathfrak{S}^{(Y)}\}$.As X is fuzzy m-compact ${\mathfrak{S}_1}^{(X)} = \{f^{-1}(U_k) | U_k \in \mathfrak{S}^{(Y)}, k \in Z; 1 \leq k \leq n\}$ is finite fuzzy s-refinement of FMAO cover $\mathfrak{S}^{(X)} = \{f^{-1}(U) | U \in \mathfrak{S}^{(Y)}\}$ .As Y is not fuzzy m-compact for each $k \in Z; 1 \leq k \leq n$ $U_k > U$ for $U \in \mathfrak{S}^{(Y)}$ which implies $f^{-1}(U_k) > f^{-1}(U)$ a contradiction to fuzzy s-refinement of X.This completes the proof. $\square$

Definition 3.11. A fuzzy point p_x^α of a fts X is fuzzy m-complete accumulation point of any fuzzy subset M of X if $|U \wedge M| = |M|$ for each FMAO set U containing p_x^α.

Theorem 3.12. *Each infinite fuzzy subset of a fuzzy m-compact space has an fuzzy m-complete accumulation point.*

Proof. Let ϑ be an infinite fuzzy subset of a fuzzy m-compact fts X. Assume for each $p_x^\gamma \in X$,there is a FMAO set $S_{p_x^\alpha}$ containing p_x^γ and satisfying $|S_{p_x^\alpha} \wedge \vartheta| < |\vartheta|$. Since $\{S_{p_x^\alpha} | p_x^\gamma \in X\}$ is an fuzzy open cover of X consists of FMAO sets, by Lemma 3.3, $\{S_{p_x^\alpha} | p_x^\alpha \in X\}$ is a FMAO cover of X. Therefore a finite fuzzy s-refinement $\{S_{p_x^\alpha} | p_x^{\gamma_i} \in X, i \in Z; 1 \leq i \leq n\}$ of $\{S_{p_x^\alpha} | p_x^\gamma \in X\}$.$|\vartheta| = |\bigvee_{i=1}^{n} (S_{p_x^\alpha} \wedge \vartheta)| < |\vartheta|$, a contradiction. □

4. FUZZY MINIMAL c-REGULAR AND FUZZY c-NORMAL SPACES

Definition 4.1. A fts X is called a fuzzy minimal c-regular if for each $p_x^\alpha \in X$ and each FMIC set γ with $p_x^\alpha \notin \gamma$, there exists disjoint fuzzy open sets λ,μ such that $p_x^\alpha \in \lambda$ and $\lambda < \mu$.

Theorem 4.2. *Let X be a fts. Then:*
(i) X is fuzzy minimal c-regular.
(ii) Given a fuzzy point $p_x^\alpha \in X$ and a FMAO set ω containing p_x^α ,$\exists\ \vartheta$,a fuzzy open set such that $p_x^\alpha \in \vartheta < Cl(\vartheta) < \lambda$.
(iii) Given a fuzzy point $p_x^\alpha \in X$ and a FMIC set γ with $p_x^\alpha \notin \omega$, $\exists\ \omega$a fuzzy open set containing p_x^α such that $Cl(\omega) \wedge \gamma = 0_X$.

Proof. $(i) \Rightarrow (ii), (ii) \Rightarrow (iii), (iii) \Rightarrow (i)$: Proof follows. □

Definition 4.3. A fts X is called a fuzzy minimal c-normal if for each pair of distinct FMIC sets η, γ there exists disjoint fuzzy open sets λ,μ such that $\eta < \lambda$ and $\gamma < \mu$.

Theorem 4.4. *Let X be a fts. Then:*
(i) X is fuzzy minimal c-normal.
(ii) For each FMIC set α and each FMAO set ω with $\alpha < \omega$,$\exists\ \vartheta$,a fuzzy open set such that $\alpha < \mu < Cl(\vartheta) < \omega$.
(iii) For each pair of distinct FMIC sets α,β ,$\exists\ \omega, \vartheta$ disjoint fuzzy open sets such that $\alpha < \omega$, $Cl(\omega) \wedge \beta = 0_X$ and $\beta < \vartheta$, $Cl(\vartheta) \wedge \alpha = 0_X$.
(iv) For each pair of distinct FMIC sets α,β,$\exists\ \omega, \vartheta$,disjoint fuzzy open sets such that $\alpha < \omega$, $\beta < \vartheta$ and $Cl(\omega) \wedge Cl(\vartheta) = 0_X$.

Proof. $(i) \Rightarrow (ii)$: Obvious.

$(ii) \Rightarrow (iii)$:Suppose that $\alpha < 1_X - \beta$ for any FMAO set $1_X - \beta$.By (ii) $\exists\ \omega$ an fuzzy open set such that $\alpha < \omega < Cl(\omega) < 1_X - \beta$.Clearly $Cl(\omega) \wedge \beta = 0_X$ as $Cl(\omega) < 1_X - \beta$. By assuming $\vartheta = 1_X - Cl(\omega)$,we get $\beta < \vartheta < 1_X - \omega < 1_X -$

α.Since,$1_X - \vartheta$ is fuzzy closed,$\beta < Cl(\vartheta) < 1_X - \omega < 1_X - \alpha$.Clearly,$Cl(\vartheta) \wedge \alpha = 0_X$ as $Cl(\vartheta) < 1_X - \alpha$.It is evident that ω,ϑ are distinct.

$(iii) \Rightarrow (iv)$: By (iii),For any distinct fuzzy open sets ω,ϑ such that $\alpha < \omega$,$Cl(\omega) \wedge \beta = 0_X$ and $\beta < \vartheta$,$Cl(\vartheta) \wedge \alpha = 0_X$. As $Cl(\omega) \wedge \beta = 0_X$, $Cl(\vartheta) \wedge \alpha = 0_X$ imply that $Cl(\omega) \wedge Cl(\vartheta) = 0_X$.

$(iv) \Rightarrow (i)$: Proof is easy and hence omitted. □

Theorem 4.5. *Every fuzzy Hausdorff m-compact space is fuzzy minimal fuzzy c-regular.*

Proof. Let fts X be a fuzzy Hausdorff m-compact. Suppose $\gamma \in X$ is FMIC set and $p_x^\alpha \in X$ such that $p_x^\alpha \notin \lambda$. Since X is fuzzy Hausdorff, for each $p_x^\beta \in \lambda$, we have λ_β, μ_β disjoint fuzzy open sets such that $p_x^\alpha \in \lambda_\beta$,$p_x^\beta \in \mu_\beta$. Let $\mathfrak{G} = \{\mu_\beta | p_x^\beta \in \lambda\} \vee \{1_X - \lambda\}$. Then $\mathfrak{G}$ is FMAO cover of X by Lemma 3.1. By fuzzy m-compactness of X, then we have a finite fuzzy s-refinement $\mathfrak{H}$ of $\mathfrak{G}$.Let $\lambda = \{\Lambda \in \mathfrak{H} | \Lambda \wedge \lambda \neq 0_X\}$.So λ is an fuzzy open set which contains λ.Let $\Lambda_1, \Lambda_2 ... \Lambda_n$ be the only fuzzy members of $\mathfrak{H}$ such that $\Lambda_k \wedge \lambda \neq 0_X$,$k \in Z, 1 \leq k \leq n$. For each $k \in Z, 1 \leq k \leq n$,$\exists$ $p_x^{\beta_k} \in \lambda$ such that $\Lambda_k \lneq \mu_{\beta_k}$,$k \in Z, 1 \leq k \leq n$. We put $\mu = \bigwedge_{k=1}^{n} \lambda_{\beta_k}$. Then $p_x^\alpha \in \mu$.It is easy to show that $\lambda \wedge \mu = 0_X$. □

Corollary 4.6. *A fuzzy Hausdorff m-compact space is fuzzy minimal c-normal.*

Proof. Let α,β be distinct FMIC sets in fuzzy Hausdorff fuzzy m-compact space X. By theorem 4.3, X is fuzzy minimal c-regular. Hence for each $p_x^\delta \in \alpha$, $\exists$ U, V fuzzy open sets such that $p_x^\delta \in U$,$\beta < V$ and $U \wedge V = 0_X$. The collection $\mathfrak{G} = \{\eta_\alpha | p_x^\delta \in \alpha\} \vee \{1_X - \alpha\}$ is a FMAO cover of X by Lemma 3.1.Now proceeding like the proof of Theorem 4.3, we get two fuzzy open sets η and μ such that $\alpha < \lambda$, $\beta < \mu$ and $U \wedge V = 0_X$. □

Lemma 4.7. *If Y is a fuzzy closed (resp.fuzzy open) subset of a fts X, then FMIC (resp.FMIO) sets in the subspace Y of X are FMIC (resp.FMIO) sets in X.*

Proof. Let α be a FMIC set in Y,a fuzzy closed subset of a fts X.Evidently α is also fuzzy closed in X as $\alpha = \eta \wedge Y$ for any fuzzy closed set η in X . If possible, suppose we have a fuzzy closed set γ in X such that $\gamma < \alpha$. Clearly $\gamma \wedge Y$ is fuzzy closed in Y such that $\gamma \wedge Y < \gamma < \alpha$; either $\gamma \wedge Y = \alpha$ or $\gamma \wedge Y = 0_X$ as α is FMIC in Y.$\gamma \wedge Y = \alpha$ implies that $\gamma \wedge Y = \alpha = \gamma$. Now it is enough to prove that $\gamma = 0_X$ for $\gamma \wedge Y = 0_X$. We see that $\gamma < \alpha < Y$ as α is a fuzzy subset of Y. So we have $\gamma \wedge Y = \gamma \neq 0_X$ if $\gamma \neq 0_X$. Hence ,$\gamma = 0_X$.
Similarly, we can prove for the fuzzy open sets. □

Definition 4.8. A fuzzy subspace Y of a fts X is said to be FMIC (resp.FMIO) invariant if FMIC (resp.FMIO) sets of Y are also FMIC (resp.FMIO) sets of X.

Theorem 4.9. *FMIC invariant fuzzy subspaces of fuzzy minimal c-normal spaces are fuzzy minimal c-normal.*

Proof. Let α,β be two distinct FMIC sets in Y,a FMIC invariant fuzzy subspaces of a fuzzy minimal c-normal space X.Hence α,β are FMIC sets in X. As X is fuzzy minimal c-normal space, $\exists$ η,μ distinct fuzzy open sets in X such that $\alpha < \eta$,$\beta < \mu$ and $(Y \wedge \eta) \wedge (Y \wedge \mu) = 0_X$.That is $Y \wedge \eta$; $Y \wedge \mu$ are distinct fuzzy open sets in Y such that $\alpha < (Y \wedge \eta)$ and $\beta < (Y \wedge \mu)$. □

Corollary 4.10. *Each fuzzy closed subspace of a fuzzy minimal fuzzy c-normal space is fuzzy minimal fuzzy c-normal.*

Proof. Using Lemma 4.5, we have to proceed like that of theorem 4.6. □

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