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ANALYSIS OF SOLUTIONS FOR THE BOUNDARY VALUE PROBLEMS OF NONLINEAR FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS INVOLVING GRONWALL'S INEQUALITY IN BANACH SPACES

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ABSTRACT. We study the existence and uniqueness of solutions for a class of boundary value problems of nonlinear fractional order differential equations involving the Caputo fractional derivative by employing the Banach's contraction principle and the Schauder's fixed point theorem. In addition, an example is given to demonstrate the application of our main results.

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1. Introduction

The purpose of this article is to extend the earlier works [1],[2],[6],[7] on fractional boundary value problems (BVP for short), for fractional differential equations in R to the abstract Banach space $\mathscr{Y}$ of the type

$$\begin{cases} {}^c\mathscr{D}_a^\alpha \wp(t) = \mathscr{E}(t, \wp(t), \mathscr{L}\wp(t)), \text{ for } t \in J = [a, b], \; n-1 < \alpha \leq n, \\ \wp^{(k)}(a) = \wp_k, \; k = 0, 1, 2, \ldots, n-2; \; \wp^{(n-1)}(b) = \wp_b, \end{cases} \tag{1.1}$$

where ${}^c\mathscr{D}_a^\alpha$ is the Caputo fractional derivative, $\mathscr{E} : J \times \mathscr{Y} \to \mathscr{Y}$ is a continuous function and

$\wp_0, \wp_1, \ldots, \wp_{n-2}, \wp_b$ are real constants and $\mathscr{L}$ is a nonlinear integral operator given by $\mathscr{L}\wp(t) = \int_0^t \gamma(t,s)\wp(s)ds$ with $\gamma_0 = \max\int_0^t \gamma(t,s)ds : (t,s) \in J \times J$ where $k \in C(J \times J, R^+)$.

In fact, the abrupt changes such as shocks, harvesting, or natural disasters, and many changes may happens in the dynamics of evolving processes. These short-term perturbations are often treated as in the form of boundary value

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problems of partial differential equations (We refer to [13], [14] and [19] for theoretical and numerical analysis related to this research issue.) and nonlinear fractional order differential equations involving the Caputo fractional derivative. Recently, in the published works, Caputo fractional order derivative.

The paper is structured as follows: we have presented some information in the section 2 about Caputo fractional order derivative, mild solutions of equations (1.1) along with some basic definitions, results and lemmas. We discuss the main results for mild solutions for the equations (1.1) in the section 3. Finally, an example is discussed to illustrate the main result.

2. Preliminaries

Some notations and lemmas are important in order to state our results. Denote by $\mathscr{C}(J,R)$ the Banach space of all continuous functions from J into R with the norm

$$\|x\|_\infty := \sup_{t\in J}\{|\wp(t)|\}, J = [a,b].$$

Definition 2.1([1],[7]) For a function $h \in AC^{nJ}$ given on the interval $[a,b]$, the α-th Caputo fractional-order derivative of h is defined by

$$({}^cD_a^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds, \tag{2.1}$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.
A solution of the problem (1.1) is defined as follows.

Definition 2.2([1],[7]) The fractional order $\alpha > 0$ integral of the function $h(t) \in L^1([a,b],R_+)$ is defined by

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)ds, \tag{2.2}$$

where Γ is the classical gamma function.

Definition 2.3 A function $x \in AC^n(J)$ and such that $Fx \in L^2(J)$, where $Fx(s) = \mathscr{E}(t,\wp(t),\mathscr{L}\wp(t))$ that satisfies (1.1) is called a solution of (1.1).

Lemma 2.1 ([24]) Let $\alpha > 0$. Then the differential equation

$${}^cD_a^\alpha h(t) = 0$$

has solutions

$$h(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + \cdots + c_{n-1}(t-a)^{n-1},$$
$$c_i \in R,\ i = 0,1,2,\ldots,n-1, n = [\alpha]+1.$$

Lemma 2.2 Let $\alpha > 0$. Then

$$I_a^\alpha\ {}^cD_a^\alpha h(t) = h(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + \cdots + c_{n-1}(t-a)^{n-1}.$$

In particular, when $a = 0$,

$$I^\alpha\ {}^cD^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

for some $c_i \in R\ i = 0, 1, 2, \ldots, n-1, n = [\alpha] + 1$.

Lemma 2.3 ([18]) The relations

$$ {}^cD_a^\alpha I_a^\alpha h(t) = h(t), I_a^\alpha I_a^\beta h(t) = I_a^{\alpha+\beta} h(t) \tag{2.3}$$

are valid in following case $Re\ \alpha > 0,\ Re\ \beta > 0,\ h(t) \in L^1(a, b)$.

3. Existence and Uniqueness Results

Theorem 3.1 Suppose that

(H1) $\exists$ real valued functions $\eth_1, \eth_2 \in \mathscr{C}(J, R)$ such that

$$\begin{gathered} |\mathscr{E}(t, \sigma(t), \mathscr{L}\sigma(t)) - \mathscr{E}(t, \beta(t), \mathscr{L}\beta(t))| \\ \leq \eth_1(t)(|\sigma(t) - \beta(t)| + \eth_2(t)(|\mathscr{L}\sigma(t) - \beta(t)|) \\ \forall\ t \in J = [a, b];\ \sigma(t), \beta(t) \in R. \end{gathered}$$

If

$$\begin{aligned} \theta = &\left(I^\alpha(\eth_1(t) + \eth_2(t)\gamma_0) \right) \\ &\times \left(\frac{(b-a)^\alpha}{(n-2)!\Gamma(\alpha - n + 2)} + \frac{(b-a)^{n-1}}{(n-1)!} I^{\alpha-n+1} \right) \|x - y\|_\infty. < 1, \end{aligned} \tag{3.1}$$

then the BVP (1.1) has unique solution on J.

Proof. Transform the problem (1.1) into a fixed point problem. Consider the operator

$$\Im : \mathscr{C}^{n-1}(J, R) \to \mathscr{C}^{n-1}(J, R)$$

defined

$$\begin{aligned} \Im\wp(t) = &\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathscr{E}(s, \wp(s), \mathscr{L}\wp(s)) ds \\ &+ \left(\frac{x_b}{(n-1)!} + \frac{\mathscr{E}(a, \wp(a))(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)} \right)(t-a)^{n-1} \\ &- \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \mathscr{E}(s, \wp(s), \mathscr{L}\wp(s)) ds \end{aligned} \tag{3.2}$$

$$+\sum_{k=0}^{n-2}\frac{\wp_k}{k!}(t-a)^k.$$

The Banach contraction principle is used to prove that $\Im$ has a fixed point.

Let $\wp(t),\iota(t)\in\mathscr{C}^{n-1}(J,R)$. Then $\forall t\in J$,

$$\begin{aligned}
&|\Im\wp(t)-\Im\iota(t)|\\
&\le \frac{1}{\Gamma(\alpha)}\int_a^t|\mathscr{E}(s,\wp(s),\mathscr{L}\wp(s))-\mathscr{E}(s,\iota(s),\mathscr{L}\iota(s))|ds\\
&+\frac{(t-a)^{n-1}(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}|\mathscr{E}(a,\wp(a),\mathscr{L}\wp(a))-\mathscr{E}(a,\iota(a),\mathscr{L}\iota(a))|\\
&+\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b(b-s)^{\alpha-n}|\mathscr{E}(s,\wp(s),\mathscr{L}\wp(s))-\mathscr{E}(s,\iota(s),\mathscr{L}\iota(s))|ds\\
&\le\frac{\|x-y\|_\infty}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}(\eth_1(t)+\eth_2(t)\gamma_0)ds+\frac{(b-a)^{\alpha-n+1}(\eth_1(t)+\eth_2(t)\gamma_0)\|x-y\|_\infty}{(n-2)!\Gamma(\alpha-n+2)}\\
&+\frac{(b-a)^{n-1}\|x-y\|_\infty}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b(b-s)^{\alpha-n}(\eth_1(t)+\eth_2(t)\gamma_0)ds\\
&=I^\alpha\Bigg(\eth_1(t)+\eth_2(t)\gamma_0)+\frac{(b-a)^\alpha(\eth_1(t)+\eth_2(t)\gamma_0)}{(n-2)!\Gamma(\alpha-n+2)}+\frac{(b-a)^{n-1}}{(n-1)!}I^{\alpha-n+1}(\eth_1(t)\\
&+\eth_2(t)\gamma_0)\Bigg)\|\wp-\iota\|_\infty.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|\Im\wp-\Im\iota\|_\infty\\
&\le\left(I^\alpha(\eth_1(t)+\eth_2(t)\gamma_0)\right)\left(\frac{(b-a)^\alpha}{(n-2)!\Gamma(\alpha-n+2)}+\frac{(b-a)^{n-1}}{(n-1)!}I^{\alpha-n+1}\right)\|\wp-\iota\|_\infty.
\end{aligned}$$

Consequently, by (3.1) $\Im$ is a contraction operator. As a consequence of the Banach Fixed point theorem, $\Im$ has a fixed point which is the unique solution of the problem (1.1). The proof is completed. □

Theorem 3.2 Assume that

(H2) $\exists$ a constant $\mathscr{V}>0$ such that

$$|\mathscr{E}(t,\sigma(t),\mathscr{L}\sigma(t))-\mathscr{E}(t,\beta(t),\mathscr{L}\beta(t)))|\le\mathscr{V}\Big(|\sigma(t)-\beta(t)|+|\mathscr{L}\sigma(t)-\mathscr{L}\beta(t)|\Big)$$
$$\forall t\in J=[a,b];\ \sigma(t),\beta(t)\in R.$$

If

$$\theta = \mathscr{V}(b-a)^{a}\left(\frac{1}{\Gamma(\alpha+1)} + \frac{n}{(n-1)!\Gamma(\alpha-n+2)}\right) < 1, \tag{3.3}$$

then the BVP (1.1) has a unique solution on J.

The third result is based on Schauder's Fixed point theorem.

Theorem 3.3 Assume that

(H3) The function $\mathscr{E} : J \times \mathscr{Y} \times \mathscr{Y} \to \mathscr{Y}$ is continuous.

(H4) There exists a constant $\mathscr{M} > 0$, such that

$$|\mathscr{E}(t, \sigma(t), \mathscr{L}\wp(t))| \leq \mathscr{M} \text{ for each } t \in J = [a, b] \text{ and } \forall \sigma(t) \in R. \tag{3.4}$$

Then the BVP (1.1) has at least one solution on J.

Proof. Schauder's Fixed point theorem is used to prove that $\Im$ defined by (3.2) has a fixed point. The proof will be given in several steps.
Step 1: $\Im$ is continuous.
Let $\{\wp_m\}$ be a sequence such that $\wp_m \to \wp$ in $\mathscr{C}(J, R)$. Then for each $t \in J$

$$\begin{aligned}
&|\Im\wp_m(t) - \Im(t)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} |\mathscr{E}(s, \wp_m(s), \mathscr{L}\wp_m(s)) - \mathscr{E}(t, \wp(t), \mathscr{L}\wp(t))| ds \\
&\quad + \frac{(t-a)^{n-1}(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)} |\mathscr{E}(a, \wp_m(a), \mathscr{L}\wp_m(a)) - \mathscr{E}(a, \wp(a), \mathscr{L}\wp(a))| \\
&\quad + \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} |\mathscr{E}(s, \wp_m(s), \mathscr{L}\wp_m(s)) - \mathscr{E}(t, \wp(t), \mathscr{L}\wp(t))| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \sup_{s\in J} |\mathscr{E}(s, \wp_m(s), \mathscr{L}\wp_m(s)) - \mathscr{E}(t, \wp(t), \mathscr{L}\wp(t))| ds \\
&\quad + \frac{(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)} \sup_{s\in J} |\mathscr{E}(s, \wp_m(s)\mathscr{L}\wp_m(s)) - \mathscr{E}(t, \wp(t), \mathscr{L}\wp(t))| \\
&\quad + \frac{(b-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \sup_{s\in J} |\mathscr{E}(s, \wp_m(s), \mathscr{L}\wp_m(s)) - \mathscr{E}(t, \wp(t), \mathscr{L}\wp(t))| ds
\end{aligned}$$

then

$$\begin{aligned}
&|\Im\wp_m(t) - \Im(t)| \\
&\leq \|\mathscr{E}(\cdot, \wp_m(\cdot)) - \mathscr{E}(\cdot, \wp(\cdot))\|_\infty \left(\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} ds \right.
\end{aligned}$$

$$+\frac{(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}+\frac{(b-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b (b-s)^{\alpha-n}ds\Bigg)$$
$$\leq \|\mathscr{E}(\cdot,\wp_m(\cdot),\mathscr{L}\wp_m(.))-\mathscr{E}(\cdot,\wp(\cdot),\mathscr{L}\wp(\cdot))\|_{\infty}(b-a)^{\alpha}$$
$$\times\left(\frac{1}{\Gamma(\alpha+1)}+\frac{n}{(n-1)!\Gamma(\alpha-n+2)}\right).$$

Since $\mathscr{E}$ is a continuous function, it can be shown that

$$\|\Im\wp_m-\Im\wp\|_{\infty}$$
$$\leq (b-a)^{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}+\frac{n}{(n-1)!\Gamma(\alpha-n+2)}\right)\|\mathscr{E}(\cdot,\wp_m(\cdot))-\mathscr{E}(\cdot,\wp(\cdot))\|_{\infty}$$

and hence

$$\|\Im\wp_m-\Im\wp\|_{\infty}\to 0,\ m\to\infty.$$

Step 2: $\Im$ maps the bounded sets into the bounded sets in $\mathscr{C}(J,R)$.

For any $\eta^*>0$, it can be shown that there exists a positive ℓ such that $\forall\wp\in B_{\eta^*}=\{\wp\in\mathscr{C}(J,R):\|\wp\|_{\infty}\leq\eta^*\},\ \|\Im\wp\|_{\infty}\leq\ell$.

In fact, $\forall t\in J$, by (3.2) and (H4)

$$|\Im(t)|\leq\sum_{k=0}^{n-2}\frac{|\wp_k|}{k!}(b-a)^k$$
$$+\left(\frac{|\wp_b|}{(n-1)!}+\frac{|\mathscr{E}(a,\wp(a)\mathscr{L}\wp(a))|(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\right)(b-a)^{n-1}$$
$$+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|\mathscr{E}(t,\wp(t),\mathscr{L}\wp(t))|ds$$
$$+\frac{(b-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b (b-s)^{\alpha-n}|\mathscr{E}(t,\wp(t),\mathscr{L}\wp(t))|ds$$
$$\leq\sum_{k=0}^{n-2}\frac{|\wp_k|}{k!}(b-a)^k+\frac{|\wp_b|(b-a)^{n-1}}{(n-1)!}$$
$$+\mathscr{M}(b-a)^{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}+\frac{n}{(n-1)!\Gamma(\alpha-n+2)}\right).$$

Thus

$$\|\Im\wp\|_{\infty}\leq\ell$$

where

$$\ell=\sum_{k=0}^{n-2}\frac{|\wp_k|}{k!}(b-a)^k+\frac{|\wp_b|(b-a)^{n-1}}{(n-1)!}$$

$$+\mathscr{M}(b-a)^{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}+\frac{n}{(n-1)!\Gamma(\alpha-n+2)}\right).$$

Step 3: $\Im$ maps the bounded sets into the equicontinuous sets of $\mathscr{C}(J,R)$.

Let $t_1,t_2\in J,\ t_1<t_2,\ B_{\eta^*}$ be abounded set of $\mathscr{C}(J,R)$ as above, and $\wp\in B_{\eta^*}$.

$$\begin{aligned}
&|\Im\wp(t_2)-\Im\wp(t_1)|\\
&\leq\Bigg|\frac{1}{\Gamma(\alpha)}\Big(\int_a^{t_2}(t_2-s)^{\alpha-1}\mathscr{E}(t,\wp(t),\mathscr{L}\wp(t))ds-\int_a^{t_1}(t_1-s)^{\alpha-1}\mathscr{E}(t,\wp(t),\mathscr{L}\wp(t))ds\Big)\\
&+\left(\frac{\wp_b}{(n-1)!}+\frac{\mathscr{E}(a,\wp(a),\mathscr{L}\wp(a))(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\right)\Big((t_2-a)^{n-1}-(t_1-a)^{n-1}\Big)\\
&-\frac{(t_2-a)^{n-1}-(t_1-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b(b-s)^{\alpha-n}\mathscr{E}(t,\wp(t),\mathscr{L}\wp(t))ds\\
&+\sum_{k=0}^{n-2}\frac{\wp_k}{k!}\Big((t_2-a)^k-(t_1-a)^k\Big)\Bigg|.
\end{aligned}$$

Then

$$\begin{aligned}
&|\Im\wp(t_2)-\Im\wp(t_1)|\\
&\leq\frac{\mathscr{M}}{\Gamma(\alpha)}\int_a^{t_1}\Big((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big)ds+\frac{\mathscr{M}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\\
&+\left(\frac{|x_b|}{(n-1)!}+\frac{\mathscr{M}(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\right)\Big((t_2-a)^{n-1}-(t_1-a)^{n-1}\Big)\\
&+\frac{\mathscr{M}\big((t_2-a)^{n-1}-(t_1-a)^{n-1}\big)}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b(b-s)^{\alpha-n}ds\\
&+\sum_{k=0}^{n-2}\frac{|\wp_k|}{k!}\Big((t_2-a)^k-(t_1-a)^k\Big)\\
&\leq\frac{\mathscr{M}}{\Gamma(\alpha+1)}\Big((t_2-a)^{\alpha}-(t_1-a)^{\alpha}\Big)+\sum_{k=0}^{n-2}\frac{|\wp_k|}{k!}\Big((t_2-a)^k-(t_1-a)^k\Big)\\
&+\left(\frac{|\wp_b|}{(n-1)!}+\frac{n\mathscr{M}(b-a)^{\alpha-n+1}}{(n-1)!\Gamma(\alpha-n+2)}\right)\Big((t_2-a)^{n-1}-(t_1-a)^{n-1}\Big)
\end{aligned}$$

As $t_1\to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of steps 1 to 3 together with the *Arzelá* $-$ *Ascoli* theorem, $\Im:\mathscr{C}(J,R)\to\mathscr{C}(J,R)$ is completely continuous.

Step 4: A priori bounds.

Let $\varepsilon = \{\wp \in \mathscr{C}(J, R) : \wp = \eth\Im\wp$ for some $0 < \eth < 1\}$. It shall be shown that the set is bounded.

Let $\wp \in \varepsilon$, then $\wp = \eth\Im\wp$ for some $0 < \eth < 1$. Thus $\forall t \in J$,

$$\begin{aligned}
&\wp = \eth\Im\wp \\
&\quad = \frac{\eth}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathscr{E}(t, \wp(t), \mathscr{L}\wp(t)) ds \\
&\quad + \eth\left(\frac{\wp_b}{(n-1)!} + \frac{\mathscr{E}(a, \wp(a), \mathscr{L}\wp(a))(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\right)(t-a)^{n-1} \\
&\quad - \frac{\eth(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \mathscr{E}(t, \wp(t), \mathscr{L}\wp(t)) ds + \eth\sum_{k=0}^{n-2} \frac{\wp_k}{k!}(t-a)^k.
\end{aligned}$$

By the condition (H4) and Step 2,

$$\begin{aligned}
|\wp(t)| \le & \sum_{k=0}^{n-2} \frac{|\wp_k|}{k!}(b-a)^k + \frac{|\wp_b|(b-a)^{n-1}}{(n-1)!} \\
& + \mathscr{M}(b-a)^\alpha \left(\frac{1}{\Gamma(\alpha+1)} + \frac{n}{(n-1)!\Gamma(\alpha-n+2)}\right).
\end{aligned}$$

Thus for every $\forall t \in J$,

$$\begin{aligned}
\|\wp\|_\infty \le & \sum_{k=0}^{n-2} \frac{|\wp_k|}{k!}(b-a)^k + \frac{|\wp_b|(b-a)^{n-1}}{(n-1)!} \\
& + \mathscr{M}(b-a)^\alpha \left(\frac{1}{\Gamma(\alpha+1)} + \frac{n}{(n-1)!\Gamma(\alpha-n+2)}\right) := R.
\end{aligned}$$

This shows that the set ε is bounded. As a consequence of Schauder's fixed point theorem, $\Im$ has a fixed point which is a solution of the problem (1.1). $\square$

Theorem 3.4 Assume that (H3) and the following conditions hold.

(H5) There exist a functional $\psi_f \in L^1(J, R^+)$ and a continuous and nondecreasing $\varphi : [0, \infty) \to (0, \infty)$, such that

$|\mathscr{E}(t, \wp(t), \mathscr{L}\wp(t))| \le \psi_f(t)\varphi(|\wp(t)|)$ for each $t \in J = [a, b]$ and $\forall \wp(t) \in R$.

(H6) There exists a number $\mathscr{S} > 0$, such that

$$\begin{aligned}
\theta = \mathscr{S}^{-1}\Bigg(&\varphi(\mathscr{S})\|I^\alpha\psi_f\|_{L^1} + \frac{\eth(b-a)^{n-1}\varphi(\mathscr{S})}{(n-1)!} I^{\alpha-n+1}\psi_f(b) + \sum_{k=0}^{n-2} \frac{|\wp_k|}{k!}(b-a)^k \\
&+ \frac{|\wp_b|}{(n-1)!}(b-a)^{n-1} + \frac{\psi_f(a)\varphi(|\wp(a)|)}{(n-2)!\Gamma(\alpha-n+2)}(b-a)^\alpha\Bigg) < 1. \qquad (3.5)
\end{aligned}$$

Then the BVP (1.1) has at least one solution on J.

Proof. Considering the operator $\Im$ defined by (3.2), $\forall\eth \in [0,1],\ t \in J = [a,b]$, letting $\wp(t)$ meet $\wp(t) = \eth(\Im\wp)(t)$, then from (H5) and (H6),

$$
\begin{aligned}
|\wp(t)| = |\eth(\Im\wp)(t)| \leq |\Im(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,x(s)Sx(s))ds \\
&+ \left|\left(\frac{x_b}{(n-1)!} + \frac{\mathscr{E}(a,\wp(a),\mathscr{L}\wp(a))(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\right)(t-a)^{n-1}\right| \\
&+ \frac{|(t-a)^{n-1}|}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b (b-s)^{\alpha-n}|\mathscr{E}(t,\wp(t),\mathscr{L}\wp(t))|ds + \sum_{k=0}^{n-2}\frac{|\wp_k|}{k!}(t-a)^k \\
&\leq \frac{\varphi(\|\wp\|_\infty)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\psi_f(s)ds \\
&+ \frac{|\wp_b|}{(n-1)!}(b-a)^{n-1} + \frac{\psi_f(a)\varphi(|\wp(a)|)(b-a)^\alpha}{(n-2)!\Gamma(\alpha-n+2)} \\
&+ \frac{\varphi(\|\wp\|_\infty)(b-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b (b-s)^{\alpha-n}\psi_f(s)ds + \sum_{k=0}^{n-2}\frac{|x_k|}{k!}(b-a)^k \\
&\leq \varphi(\|\wp\|_\infty)\|I^\alpha\psi_f\|_{L^1} + \frac{|x_b|}{(n-1)!}(b-a)^{n-1} + \frac{\psi_f(a)\varphi(|x(a)|)(b-a)^\alpha}{(n-2)!\Gamma(\alpha-n+2)} \\
&+ \frac{\varphi(\|\wp\|_\infty)(b-a)^{n-1}}{(n-1)!} I^{\alpha-n+1}\psi_f(b) + \sum_{k=0}^{n-2}\frac{|\wp_k|}{k!}(b-a)^k.
\end{aligned}
$$

By (H6), there exists $\mathscr{S}$ such that $\|\wp\|_\infty \neq \mathscr{S}$. Let $\mathscr{U} = \{\wp \in \mathscr{C}(J,R) : \|\wp\|_\infty < K\}$. The operator $\Im : \bar{\mathscr{U}} \to \mathscr{C}(J,R)$ is completely continuous. Through proper selection of $\mathscr{U}$, there exists no $\wp(t) \in \partial\mathscr{U}$ such that $\wp(t) = \eth(\Im\wp)(t)$ for some $\eth \in (0,1)$.

Therefore, $\Im$ is *Leray − Schauder type* operator, so that it has a fixed point $\wp(t)$ in $\bar{U}$, which is a solution of the BVP (1.1).

4. Illustrated an example

Boundary value problem:

$$
\begin{cases} {}^c\mathscr{D}_a^\alpha\wp(t) = \frac{|\wp(t)|t}{1+\wp(t)},\ t \in J = [0,1],\ n-1 < \alpha \leq n, \\ \wp^{(k)}(0) = 0,\ k = 0,1,2,\ldots,n-2;\ \wp^{(n-1)}(1) = 1, \end{cases} \qquad \textit{where} \quad 1+\wp(t) \neq 0. \tag{4.1}
$$

Take

$$
\mathscr{E}(t,\sigma(t),\mathscr{L}\wp(t)) = \frac{\sigma(t)t}{1+\sigma(t)},\ (t,\sigma(t),S\sigma(t)) \in J \times [0,\infty).
$$

Let $\wp(t), y(t) \in [0,\infty),\ t \in J$. Then

$$\begin{aligned}|\mathscr{E}(t,\wp(t),\mathscr{L}\wp(t)) - \mathscr{E}(t,\iota(t),\mathscr{L}\iota(t))| &= t\left|\frac{\wp(t)}{1+\wp(t)} - \frac{y(t)}{1+y(t)}\right| \\ &= \frac{t|\wp(t)-\iota(t)|}{(1+\wp(t))(1+\iota(t))} \\ &\le t_1|\wp(t)-\iota(t)| + t_2|\mathscr{L}\wp - |\mathscr{L}\iota|. \quad (4.2)\end{aligned}$$

Hence the condition (H1) holds with $\eth(t) = t \in \mathscr{C}(J,R)$. It can be checked that condition (3.2) is satisfied with $b = 1$. In fact,

$$\begin{aligned}\theta &= I^{\alpha}\eth(t) + \frac{\eth(0)}{(n-2)!\Gamma(\alpha-n+2)} + \frac{1}{(n-1)!}I^{\alpha-n+1}\eth(1) \\ &= \frac{1}{\Gamma(\alpha+2)}t^{\alpha+1} + \frac{1}{(n-1)!\Gamma(\alpha-n+3)} < 1,\ (t \le 1,\ \eth(0) = 0) \quad (4.3)\end{aligned}$$

only if

$$\frac{1}{\Gamma(\alpha+2)} + \frac{1}{(n-1)!\Gamma(\alpha-n+3)} < 1. \quad (4.4)$$

For example, $\alpha = \frac{5}{2}$, then $n = [\alpha]+1 = 3,\ \Gamma(\alpha+2) = \Gamma(\frac{9}{2}) = \frac{105\sqrt{\pi}}{16}$, $\Gamma(\alpha-n+3) = \Gamma(\alpha) = \Gamma(\frac{5}{2}) = \frac{3\sqrt{\pi}}{4},\ (n-1)! = 2! = 2.$Then

$$\begin{aligned}\theta &\le \frac{1}{\Gamma(\alpha+2)} + \frac{1}{(n-1)!\Gamma(\alpha-n+3)} \\ &= \frac{1}{\Gamma(\alpha+2)} + \frac{1}{2\Gamma(\alpha)} = \Big(\frac{16}{105} + \frac{2}{3}\Big)\frac{1}{\sqrt{\pi}} \\ &= 0.4621 < 1. \quad (4.5)\end{aligned}$$

Then by theorem 3.1 the boundary value problem (4.1) has a unique solution on $J = [0,1]$ for the values of $\alpha \in (2,3]$.

5. Conclusion

In this paper, we establish the existence and uniqueness of solutions for a class of boundary value problems of nonlinear fractional order differential equations involving the Caputo fractional derivative by employing the Banach's contraction principle and the Schauder's fixed point theorem. We will try to investigate the controllability of similar problem in our future research work.

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