J. Appl. Math. & Informatics Vol. **40***(2022), No. 1 - 2, pp. 133 - 139*
https://doi.org/10.14317/jami.2022.133

PACKING LATIN SQUARES BY *BCL* ALGEBRAS

YONGHONG LIU

Abstract. We offered a new method for constructing Latin squares. We introduce the concept of a standard form via example for Latin squares of order n and we also call it symmetric *BCL* algebras matrix, and thereby become *BCL* algebra representations of the picture of Latin squares. Our research shows that some new properties of the Latin squares with *BCL* algebras are in $\mathbb{Z}_n$.

AMS Mathematics Subject Classification : 05B15, 03G25.
Key words and phrases : Latin squares, *BCL* algebras, matrix, BIBD, NP-complete.

1. Introduction

The Latin squares of order n, i.e., an n-by-n array where 1 through n occurs precisely once in each row and each column. Bose *et al*. [1] proved that Euler's conjecture (the reason stems from the nonrepresentational formulation of the problem of the 36 officers) was false for all $n > 6$. But it often common concerns in Latin squares design that the construct question requires extensive study, e.g., a natural generalization of orthogonality of Latin squares is given in an article by Liang [2]. The symmetries of the partial Latin squares has been studied by Falcón [3]. Completing partial of the Latin squares has been discussed by Casselgren and Hăggkvist [4]. Hedayat and Seiden [5] clearly found that the Latin squares can be written in terms of orthogonal arrays in 1974. For completing Latin squares, technically speaking, it is actually about the constructive proof of the Evans conjecture, due to Smetaniuk [6]. The role of the Latin squares and sets of mutually orthogonal Latin squares in the design of experiments is obvious; see for example the book (Street and Street [7]). Dénes and Keedwell in their book [8] in 1991 that the Latin squears are very useful tool in theory and applications. With excellent properties Latin squares has a wide range of applications. Significantly, Latin squares can be used in statistics; see for example (Ryan and Morgan [9]), in quantum physics; see for example (Musto and

Received September 9, 2021. Revised November 24, 2021. Accepted November 26, 2021.

Vicary [10]) and in error correcting code of memory systems; see for example (Hsiao and Bossen [11]).

It is generally known that the multiplication table of finite group identified a Latin square, but it has very special properties, and it is hard for count the number of elements in Latin square of order n. The point is that BCL algebras were invented and studied by Yonghong [12] in 2011. (for other similar references and results see Yonghong [13]). In practice, the BCL algebras is a Cayley table, describes the structure of Latin squares by arranging all the possible products of all the group's elements in a square table reminiscent of an appropriate multiplication table.

On the general problem of determining the algebraic properties of a Latin square of standard form, has fascinated combinatorial designs theorists down the ages — so, in this paper, we will propose a standard form of Latin squares, which this is done when we define the symmetric BCL algebras matrix, along with a completely new method, a practice known as "hill climbing" used to construct it. We also will cover some new properties for Latin squares with BCL algebras.

2. Preliminaries

Definition 2.1. Let n be a positive integer, and let S be a set of n distinct elements that is we take S to be $\mathbb{Z}_n = \{0,\ 1,\ \cdots,\ n-1\}$, where we number the rows and the columns of Latin squares as $0,\ 1,\ \cdots,\ n-1$.

Definition 2.2. An n-by-n Latin square is a set of n^2 triples $(r,\ c,\ s)$, where r is the row, c is the column, and s is the symbol, and $1 \leq r,\ c,\ s \leq n$, such that all ordered pairs $(r,\ c)$ are distinct, all ordered pairs $(r,\ s)$ are distinct, and all ordered pairs $(c,\ s)$ are distinct. The sets of n^2 triples called the orthogonal array representation of the Latin squares.

Definition 2.3. An n-by-n array Latin L in which some positions are unoccupied and other positions are occupied by one of the integers $\{0, 1, \cdots,\ n-1\}$. Suppose that if an integer k occurs in L, then it occurs n times and no two k's belong to the same row or column. Then we call L a semi-Latin square. If m different integers defined occur in L, then we say L has index m. A semi-Latin square of order n and index m, will be denoted by L_n^m.

Theorem 2.4 (Brualdi [14, Theorem 10.4.2])**.** *Let n be a positive integer, and let r be a non-zero integer in $\mathbb{Z}_n$ such that the greatest common divisor (popularly abbreviated GCD) of r and n is* 1. *Let A be the $n \times n$ array whose entry a_{ij} in row i and column j is*

$$a_{ij} = r \times i + j \ \ (arithmeticr \ \ mod \ \ n), \tag{1}$$

for $i,\ j = 0,\ 1,\ \cdots,\ n-1$. Then A is a Latin square of order n based on $\mathbb{Z}_n$.

Remark 2.1. Using this idea of interchanging the positions occupied by the various elements 0, 1, $\cdots$, $n-1$ we can always bring a Latin square to standard form, that is row 0 the integers 0, 1, $\cdots$, $n-1$ occur in their natural order.

Remark 2.2. For all $n \geq 2$, we can obtain an $n \times n$ Latin square from the table of the group $(\mathbb{Z}_n, +)$ if we replace the occurrences of 0 by the value of n.

Definition 2.5. A nonrepresentational n-by-n conference matrix C_n is

$$C_n = \begin{pmatrix} 0 & & & \\ & & & \boldsymbol{H}' \\ & & \ddots & \\ & \boldsymbol{H} & & \\ & & & 0 \end{pmatrix}_{n\times n,} \tag{2}$$

where H and H' are 1 or -1. Then

$$C_n C_n^T = (n-1)I_n, \tag{3}$$

where I_n is unit matrix.

Theorem 2.6 (Smetaniuk [6])**.** *Any partial Latin square of order n with at most $n-1$ filled cells can be completed to a Latin square of the same order.*

3. Main results

As a generalization on conference matrix, the following definition is useful for Latin squares.

Definition 3.1. Let $L_n^{\#}$ be the $n \times n$ symmetric matrix, and let each main diagonal entry equal to 0.

$$L_n^{\#} = \begin{pmatrix} 0 & & & \\ & & & \boldsymbol{A}' \\ & & \ddots & \\ & \boldsymbol{A} & & \\ & & & 0 \end{pmatrix}_{n\times n} , \tag{4}$$

where $A = A'^T$, and row 0 the integers $0, 1, \cdots, n-1$. Than $L_n^{\#}$ is a standard form of Latin square of order n, such symmetric matrix $L_n^{\#}$ is also referred to as symmetric BCL algebra matrix.

The next example, we obtain a standard form of Latin square $L_n^{\#}$ from the arithmetic of this BCL algebra. On the other hand, the symmetric BCL algebras matrix in solving construct problems is very important in building a standard form Latin square.

Table 1. BCL operation

$\to$	0	1	i	$1+i$
0	0	1	i	$1+i$
1	1	0	$1+i$	i
i	i	$1+i$	0	1
$1+i$	$1+i$	i	1	0

Example 3.2. Let $X = \{0, 1, i, 1+i\}$. Define a binary operation $\to$ on X given by the following $\to$ multiplication Table 1 (Cayley table): Let $x, y, z \in X$ and write $x = 1, y = i, z = 1 + i$. Then $(X; \to, 0)$ is a BCL algebra, and the Table 1 is just a symmetric BCL algebra matrix by Definition 3.1, which is a standard form of Latin square, that is $L_4^{\#} = \{0, 1, i, 1+i\}$ of order 4. Using the arithmetic of this BCL algebras we obtain the following Latin square:

$$L_4^{\#} = \begin{pmatrix} 0 & 1 & i & 1+i \\ 1 & 0 & 1+i & i \\ i & 1+i & 0 & 1 \\ 1+i & i & 1 & 0 \end{pmatrix}. \tag{5}$$

Example 3.3. The 4-by-4 Latin squares in Example 3.2 in the orthogonal array representation of Latin squares there are the triples $(r,\ c,\ s)$ by Definition 2.2 that contain exactly 16 that is

$$\{(r,\ c,\ s)\} \to \{\cdots,\ (i,\ 1+i,\ 1),\ \cdots\}. \tag{6}$$

Definition 3.4. Let $L_n^{\#}$ be a standard form of Latin square of order n, if some positions are unoccupied in $L_n^{\#}$. Then $L_n^{\#}$ is a semi-Latin square, will be denoted by $L_n^{\#)}$.

Remark 3.1. Here is very similar to Definition 2.3, but not that we need to consider the index.

Definition 3.5. Let $L_n^{\#_1},\ L_n^{\#_2}, \cdots, L_n^{\#_{n-1}}$ be an arrays of $L_n^{\#}$ based on $\mathbb{Z}_n$ if each pair $(L_n^{\#_a}, L_n^{\#_b})$ $(a \neq b)$ is orthogonal. Then $L_n^{\#}$ is orthogonal, will be denoted by $L_n^{\#\perp}$.

Algebraically, a Latin square $L_n^{\#}$ is characterized as being the Cayley table of the BCL algebras.

Theorem 3.6. *Up to $n \times n$ symmetric matrix, the multiplication table of the BCL algebras is a standard form of Latin square $L_n^{\#}$ of order n.*

Proof. The Latin property of this array follows from the properties of multiplication in BCL algebras. By the definition of standard form $L_n^{\#}$, the multiplication table there is no repeated element in row or in column.

Let $L_n^{\#} = \{a_1,\ a_2,\ \cdots,\ a_n\}$ and let multiplication table $A = (a_{ij})$. Then $a_{ij} = a_i \to a_j$. For any i with $1 \le i \le n$, since $a_i \to x = b$, and this equation has a unique solution, therefore, x iterates over all of the element n in $L_n^{\#}$, and by this time, b, too. This shows that $a_{i1}, a_{i2}, \cdots, a_{in}$ is a permutation of $L_n^{\#}$. Similarly, we have $a_{ij},\ a_{2j},\ \cdots,\ a_{nj}$ for $1 \le j \le n$. So A is a Latin square of order n on $L_n^{\#}$.

Conversely, if $A = (a_{ij})$ is a Latin square of order n on $L_n^{\#}$, since $a_i \to a_j = a_{ij}$, for $1 \le i, j \le n$, by Definition 3.1, defined biary operation $\to$ on $L_n^{\#}$. Then $(L_n^{\#}; \to, 0)$ is a BCL algebra. □

Theorem 3.7. *Let n be a positive integer. Then there exists a set of $n-1L_n^{\#\perp'}s$.*

Proof. Let $L_n^{\#_1}, L_n^{\#_2}, \cdots, L_n^{\#_{n-1}}$ are arrays of $L_n^{\#\perp}$, without loss of generality, by Definition 3.5 the arrays $L_n^{\#_1}, L_n^{\#_2}, \cdots, L_n^{\#_{n-1}}$ are Latin squares of order n and contains one of the integers $1, 2, \cdots, n-1$ in the row 1, column 0 position, and no two of then contain the same integer in this position. Use pigeon-hole principle, the largest number is at most $n-1$. By Theorem 2.4 and let $\#_r$ and $\#_s$ be distinct nonzero integers, by reductio, we get $\#_r = \#_s$, it bound to mean row $i = k$ (same ordered pair), a contradiction. Thus, we find a set of $n - 1L_n^{\#\perp'}$s, as required. □

Theorem 3.8. *$L_n^{\#}$ is a BIBD of index parameter $\lambda = 1$.*

Proof. It suffices to observe that the pigeon-hole principle is used in the $L_n^{\#}$. □

Corollary 3.9. *Let n be a positive integer, let $\Lambda(n)$ denote the largest number of $L_n^{\#\perp}$. Then*

$$\Lambda(mk) \ge min\{\Lambda(m),\ \Lambda(k)\}, \tag{7}$$

where $n = mk$ (m and k are for orders).

Theorem 3.10. *Let A, $A' \in L_n^{\#}$ and let S be a symmetric BCL algebra matrix. Suppose C_n is a matrix*

$$C_n = \begin{pmatrix} 0 & A'^T \\ \mu A & S \end{pmatrix}. \tag{8}$$

where $\mu = 1$ or -1. Then C_n is a $L_n^{\#}$.

Proof. Application of Theorem 3.6 to C_n, and by Definition 3.1 let $S = L_n^{\#}$ and let $A = 1 = A'^T$, we have

$$L_n^{\#T} = \mu L_n^{\#} \quad \text{or} \tag{9}$$

$$\mu L_n^{\#T} = L_n^{\#}. \tag{10}$$

By Definition 2.5, C_n is conference matrix. We use C_n to construct a bigger $L_n^{\#}$, is that

$$L_n^{\#} = C_n \times I_n, \tag{11}$$

which proves that C_n is a $L_n^{\#}$. □

Remark 3.2. The Latin square of order n constructed in Theorem 3.10 is nothing but the $L_n^{\#}$ hill climbing to produce $L_n^{\#}$ of lager order, so to the structural form of the whole Latin square expanded continuously.

Theorem 3.11. *Let $L_n^{\#)}$ be a semi-Latin square, and let $n > 3$. Then $L_n^{\#)}$ has a completion.*

Proof. This is immediate from Theorem 2.6 and the fact that some positions are unoccupied in $L_n^{\#}$. By Definition 3.4, we use the inductive method, the cases $n \leq 2$ being trivial, we have a partial Latin square of order $n > 3$ with at most $n-1$ filled cells to get a complete Latin square. □

The $L_n^{\#}$ completion problem is a special case of the Latin square completion problem. We offer the following corollary.

Corollary 3.12. *Let A, $A' \in L_n^{\#}$ and let S be a symmetric BCL algebra matrix. Then $L_n^{\#}$ is NP-complete.*

Proposition 3.13. *Let order $n > 3$, the $L_n^{\#}$ can be constructed in polynomial-time.*

Remark 3.3. This Proposition 3.13, in fact, refer to Béjar *et al*. [15], the only difference here is that $L_n^{\#}$ has symmetry.

4. Conclusion

In this paper, the standard form Latin squares is presented. We've now packed Latin squares, via the basic representation of symmetric BCL algebras matrix, and their properties are discussed. One interesting consequence is that the Latin squares of $L_n^{\#}$-type is also NP-complete. In addition, we regard BCL algebras closely related to Balanced Incomplete Blockk Design (BIBD).

Acknowledgment : The author would like to thank the reviewers for their careful reading of this article and for valuable comments.

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Yonghong Liu did his M.S. at Huazhong University of Science and Technology, China. He is currently an Associate Professor at Wuhan University of Technology. He is a reviewer of dissertations for Academic Degrees (Ph.D. and M.S.) & Graduate Education Platform of MOE in China, and served as an editor at the two mathematics journals. His research interests are algebra, number theory, combinatorics and cosmology.

School of Automation, Wuhan University of Technology, Wuhan 430070, China.
`e-mail: hylinin@whut.edu.cn`